Publ. Math. Debrecen
51 / 3-4 (1997), 311–322

The problem of the extension of a parametric family of Diophantine triples

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Abstract. It is proven that if $k \geq 2$ is an integer and d is a positive integer such that the product of any two distinct elements of the set

$$\{k-1,\, k+1,\, 4k,\, d\}$$

increased by 1 is a perfect square, than d has to be $16k^3 - 4k$. This is a generalization of the well-known result of Baker and Davenport for $k = 2$.

1. Introduction

The Greek mathematician Diophantus of Alexandria noted that the set $\{1/16, 33/16, 17/4, 105/16\}$ has the following property: the product of any two of its distinct elements increased by 1 is a square of a rational number (see [5]). A set of positive integers $\{a_1, a_2, \dots, a_m\}$ is said to have *the property of Diophantus* if $a_i a_j + 1$ is a perfect square for all $1 \leq i < j \leq m$. Such a set is called *a Diophantine m-tuple*. Fermat first found an example of a Diophantine quadruple, and it was $\{1, 3, 8, 120\}$. In 1969, Baker and Davenport [2] proved that if d is a positive integer such that $\{1, 3, 8, d\}$ is a Diophantine quadruple, then d has to be 120.

There is a well-known generalization of the Fermat set: the set

$$\{k-1,\, k+1,\, 4k,\, 16k^3 - 4k\}$$

is a Diophantine quadruple for all integers $k \geq 2$ (see [6, 10]). For $k = 2$ we obtain the Fermat set. Thus we come to the following question:

Mathematics Subject Classification: Primary 11D09, 11D25; Secondary 11B37, 11J68, 11J86, 11Y50.

Let $k \geq 2$ be an integer, and let d be a positive integer such that the set $\{k-1, k+1, 4k, d\}$ has the property of Diophantus. Is then necessarily $d = 16k^3 - 4k$?

As we said before, for $k = 2$ an affirmative answer to the above question was given in [2] and also in [9,12,16], and for $k = 3$ in [18].

In the present paper we prove the following theorem which gives an affirmative answer to the above question for all integers $k \geq 2$.

Theorem 1. *Let $k \geq 2$ be an integer. If the set $\{k-1,\, k+1,\, 4k,\, d\}$ has the property of Diophantus, then d has to be $16k^3 - 4k$.*

2. A system of Pellian equations

Assume that the set $\{k-1,\, k+1,\, 4k,\, d\}$ has the property of Diophantus. It implies that there exist positive integers x, y and z such that the following holds:

$$(k-1)d + 1 = x^2, \quad (k+1)d + 1 = y^2, \quad 4kd + 1 = z^2.$$

Eliminating d, we obtain the following system of Pellian equations:

$$(k-1)y^2 - (k+1)x^2 = -2, \tag{1}$$

$$(k-1)z^2 - 4kx^2 = -3k - 1. \tag{2}$$

Since $k-1 < k+1 < 4(k-1)$ Theorem 8 from [11] implies that all solutions of (1) are given by $x = v_m$, $m \geq 0$, where (v_m) is the following recursive sequence:

$$v_0 = 1,\ v_1 = 2k-1,\ v_{m+2} = 2kv_{m+1} - v_m, \quad m \geq 0. \tag{3}$$

The theory of Pellian equations guarantees that all solutions of (2) are given by $x = w_n^{(i)}$, $n \geq 0$, where

$$\begin{gathered} w_0^{(i)} = x_0^{(i)},\ w_1^{(i)} = (2k-1)x_0^{(i)} + (k-1)z_0^{(i)}, \\ w_{n+2}^{(i)} = (4k-2)w_{n+1}^{(i)} - w_n^{(i)}, \end{gathered} \tag{4}$$

and $\sqrt{k-1}\, z_0^{(i)} + 2\sqrt{k}\, x_0^{(i)}$, $i = 1, \ldots, j$, are fundamental solutions of the equation (2) (see [13, 17]).

Thus our problem reduces to solving the equations

$$v_m = w_n^{(i)}, \tag{5}$$

$i = 1, \ldots, j$. From (3) and (4) it easily follows that $v_m \equiv 1 \pmod{(k-1)}$ for all $m \geq 0$, and $w_n^{(i)} \equiv x_0^{(i)} \pmod{(k-1)}$ for all $n \geq 0$. Hence, if the equation (5) has a solution in integers m and n, then we must have $x_0^{(i)} \equiv 1 \pmod{k-1}$. But from [13, Theorem 108a] we have:

$$0 < x_0^{(i)} \leq \frac{1}{\sqrt{2(2k-2)}}\sqrt{(3k+1)(k-1)} = \frac{1}{2}\sqrt{3k+1} < \sqrt{k}.$$

Therefore $x_0^{(i)} = 1$ and $z_0^{(i)} = \pm 1$.

We have thus proved the following lemma.

Lemma 1. *Let x, y, z be positive integer solutions of the system of Pellian equations* (1) *and* (2). *Then there exist integers $m \geq 0$ and n such that*

$$x = v_m = w_n, \tag{6}$$

where the sequence (v_m) is given by (3), *and the two-sided sequence (w_n) is given by the following recursive formula:*

$$w_0 = 1,\ w_1 = 3k-2,\ w_{n+2} = (4k-2)w_{n+1} - w_n, \quad n \in \mathbb{Z}. \tag{7}$$

3. Application of a result of Rickert

In this section we will use a result of Rickert [15] on simultaneous rational approximations to the numbers $\sqrt{(k-1)/k}$ and $\sqrt{(k+1)/k}$ and we will prove the statement of Theorem 1 for $k \geq 29$. For the convenience of the reader, we recall Rickert's result.

Theorem 2. *For an integer $k \geq 2$ the numbers*

$$\theta_1 = \sqrt{(k-1)/k}, \quad \theta_2 = \sqrt{(k+1)/k}$$

satisfy

$$\max\bigl(|\theta_1 - p_1/q|, |\theta_2 - p_2/q|\bigr) > (271k)^{-1}q^{-1-\lambda}$$

for all integers p_1, p_2, q with $q > 0$, where

$$\lambda = \lambda(k) = \frac{\log\,(12k\sqrt{3}+24)}{\log\,[27(k^2-1)/32]}.$$

From (1) and (2) it follows

$$(k+1)z^2 - 4ky^2 = -3k+1, \tag{8}$$

and the system of Pellian equations (1) and (2) is equivalent to the system (2) and (8).

Lemma 2. *Let $k \geq 2$ and $\theta_1 = \sqrt{(k-1)/k}$, $\theta_2 = \sqrt{(k+1)/k}$. Then all positive integer solutions x, y, z of the simultaneous Pellian equations* (2) *and* (8) *satisfy*

$$\max\left(\left|\theta_1 - \frac{2x}{z}\right|, \left|\theta_2 - \frac{2y}{z}\right|\right) < 2.475 z^{-2}.$$

Proof. We have:

$$\left|\sqrt{\frac{k-1}{k}} - \frac{2x}{z}\right| = \left|\frac{k-1}{k} - \frac{4x^2}{z^2}\right| \cdot \left|\sqrt{\frac{k-1}{k}} + \frac{2x}{z}\right|^{-1}$$
$$< \frac{1}{kz^2}|(k-1)z^2 - 4kx^2| \cdot \frac{1}{\sqrt{2}} = \frac{3k+1}{k\sqrt{2}} z^{-2} < 2.475 z^{-2}$$

and

$$\left|\sqrt{\frac{k+1}{k}} - \frac{2y}{z}\right| = \left|\frac{k+1}{k} - \frac{4y^2}{z^2}\right| \cdot \left|\sqrt{\frac{k+1}{k}} + \frac{2y}{z}\right|^{-1}$$
$$< \frac{1}{kz^2}|(k+1)z^2 - 4ky^2| \cdot \frac{1}{2} = \frac{3k-1}{2k} z^{-2} \leq 1.5 z^{-2}.$$

□

Lemma 3. *Let m and n be integers such that $v_m = w_n$. Then $n \equiv 0$ or $-2 \pmod{4k}$.*

Proof. Let us consider the sequences

$$(v_m \mod (2k-1))_{m \geq 0} = (1, 0, -1, -1, 0, 1, 1, 0, \dots) \text{ and}$$
$$(w_n \mod (2k-1))_{n \geq 0} = (1, -k, -1, k, 1, -k, \dots).$$

We conclude that $v_m = w_n$ implies that n is even. Set $n = 2l$.

Let us now consider the sequences $(v_m \mod 4k(k-1))$ and $(w_{2l} \mod 4k(k-1))$. We have:

$$(v_m \mod 4k(k-1))_{m \geq 0} = (1, 2k-1, 2k-1, 1, 1, 2k-1, \dots),$$
$$(w_{2l} \mod 4k(k-1))_{l \geq 0} = (1, -2k+3, -4k+5, -6k+5, \dots).$$

It follows easily by induction that $w_{2l} \equiv -2lk + (2l+1) \pmod{4k(k-1)}$, for all $l \in \mathbb{Z}$. Hence, if $v_m = w_{2l}$, then we have two possibilities:

1) $-2lk + (2l+1) \equiv 1 \pmod{4k(k-1)}$

This implies $2l(k-1) \equiv 0 \pmod{4k(k-1)}$, and $n = 2l \equiv 0 \pmod{4k}$.

2) $-2lk+(2l+1)\equiv 2k-1 \pmod{4k(k-1)}$

This implies $2(l+1)(k-1)\equiv 0 \pmod{4k(k-1)}$, and $n=2l\equiv -2 \pmod{4k}$. □

Lemma 4. *Let x, y, z be positive integer solutions of the system of Pellian equations* (1) *and* (2) *such that $z\notin\{1,8k^2-1\}$. Then*

$$\log z \geq (4k-2)\log(4k-3).$$

Proof. If z satisfies the conditions of the lemma then from the results of Section 2 it follows that there exists an integer n such that $z=s_n$, where

$$s_0=1,\ s_1=6k-1,\ s_{n+2}=(4k-2)s_{n+1}-s_n,\quad n\in\mathbb{Z}.$$

Let $\varphi=2k-1+2\sqrt{k^2-k}$. Now it follows easily by induction that for $n>0$ we have $s_n\geq\varphi^n$, and for $n<0$ we have $s_n\geq\frac{1}{2}\varphi^{|n|}$.

If $n>0$, then Lemma 3 implies $n\geq 4k-2$, and so $z\geq\varphi^{4k-2}$. If $n<0$, then Lemma 3 implies $|n|\geq 4k$, and so $z\geq\frac{1}{2}\varphi^{4k}\geq\varphi^{4k-2}$. Hence,

$$\log z\geq(4k-2)\log\varphi\geq(4k-2)\log(4k-3).$$ □

Proposition 1. *If $k\geq 29$ and if the set $\{k-1,k+1,4k,d\}$ has the property of Diophantus, then d has to be $16k^3-4k$.*

Proof. Let z be a positive integer such that $4kd+1=z^2$. Suppose that $d\neq 16k^3-4k$. Then Lemma 4 implies

$$\log z\geq(4k-2)\log(4k-3). \tag{9}$$

On the other hand, Theorem 2 and Lemma 2 imply

$$(271k)^{-1}z^{-1-\lambda}<2.475z^{-2}.$$

It follows that

$$z^{1-\lambda}<671k$$

and

$$\log z<\frac{\log(671k)}{1-\lambda}. \tag{10}$$

Since $k\geq 29$, we have

$$\frac{1}{1-\lambda}=\frac{\log[27(k^2-1)/32]}{\log\left[\frac{27(k^2-1)}{32(12k\sqrt{3}+24)}\right]}<\frac{2\log(0.9186k)}{\log(0.03899k)}.$$

Combining (9) and (10) we obtain

$$4k-2<\frac{2\log(671k)\log(0.9186k)}{\log(4k-3)\log(0.03899k)}. \tag{11}$$

Since the function on the right side of (11) is decreasing, it follows that $4k-2<112$. This contradicts our assumption that $k\geq 29$. $\square$

4. Linear forms in three logarithms and the Grinstead method

In the proof of the statement of Theorem 1 for $k\leq 28$ we will use the Grinstead method (see [9, 4, 14]). In this section we assume that $2\leq k\leq 28$.

Let $x=v_m=w_n$, where $m,n\geq 0$. Then

$$\begin{aligned}2\sqrt{k+1}x=&\left(\sqrt{k-1}+\sqrt{k+1}\right)\left(k+\sqrt{k^2-1}\right)^m\\&-\left(\sqrt{k-1}-\sqrt{k+1}\right)\left(k-\sqrt{k^2-1}\right)^m,\end{aligned} \tag{12}$$

and

$$\begin{aligned}4\sqrt{k}x=&\left(\sqrt{k-1}+2\sqrt{k}\right)\left(2k-1+2\sqrt{k^2-k}\right)^n\\&-\left(\sqrt{k-1}-2\sqrt{k}\right)\left(2k-1-2\sqrt{k^2-k}\right)^n.\end{aligned} \tag{13}$$

If we put

$$P=\frac{\sqrt{k-1}+\sqrt{k+1}}{\sqrt{k+1}}\left(k+\sqrt{k^2-1}\right)^m, \tag{14}$$

$$Q=\frac{\sqrt{k-1}+2\sqrt{k}}{2\sqrt{k}}\left(2k-1+2\sqrt{k^2-k}\right)^n, \tag{15}$$

the relations (12) and (13) give

$$P+\frac{2}{k+1}P^{-1}=Q+\frac{3k+1}{4k}Q^{-1}. \tag{16}$$

It is clear that $P>1$ and $Q>1$, and from

$$P-Q>\frac{2}{k+1}Q^{-1}-\frac{2}{k+1}P^{-1}=\frac{2}{k+1}(P-Q)P^{-1}Q^{-1}$$

we see that $Q < P$. As we may assume that $m \geq 1$, we have

$$P \geq \frac{(2k+1)\sqrt{k-1}+(2k-1)\sqrt{k+1}}{\sqrt{k+1}} > \sqrt{k^2-1}+(2k-1) > 2k.$$

Furthermore, (16) implies

$$Q > P - \frac{3k+1}{4k}Q^{-1} > P - \frac{3k+1}{4k}.$$

Hence,

$$\begin{aligned} P - Q &= \frac{3k+1}{4k}Q^{-1} - \frac{2}{k+1}P^{-1} \\ &< \frac{3k+1}{4k}\left(P - \frac{3k+1}{4k}\right)^{-1} - \frac{2}{k+1}P^{-1} < \frac{3}{4}P^{-1} \end{aligned}$$

and finally

$$0 < \log\frac{P}{Q} = -\log\left(1 - \frac{P-Q}{P}\right) < \frac{3}{4}P^{-2} + \left(\frac{3}{4}P^{-2}\right)^2 < \frac{4}{5}P^{-2}$$

(since $-\log(1-x) < x + x^2$, for $x \in \langle 0, \frac{1}{2}\rangle$). Now from (14) and (15) we obtain the following inequality:

$$\begin{aligned} 0 &< m\log\left(k+\sqrt{k^2-1}\right) - n\log\left(2k-1+2\sqrt{k^2-k}\right) \\ &\quad + \log\frac{2\left(\sqrt{k-1}+\sqrt{k+1}\right)\sqrt{k}}{\left(\sqrt{k-1}+2\sqrt{k}\right)\sqrt{k+1}} \qquad (17) \\ &< \frac{0.8}{\left(k+\sqrt{k^2-1}\right)^{2m}} < e^{-2m\log(2k-1)}. \end{aligned}$$

Now we will apply the following theorem of Baker and Wüstholz [3]:

Theorem 3. *For a linear form $\Lambda \neq 0$ in logarithms of l algebraic numbers $\alpha_1, \dots, \alpha_l$ with rational integer coefficients $b_1, \dots, b_l$ we have*

$$\log|\Lambda| \geq -18(l+1)!\, l^{l+1}(32d)^{l+2}h'(\alpha_1)\cdots h'(\alpha_l)\log(2nd)\log B,$$

where $B = \max(|b_1|, \dots, |b_l|)$, and where d is the degree of the number field generated by $\alpha_1, \dots, \alpha_l$.

Here

$$h'(\alpha) = \frac{1}{d}\max(h(\alpha), |\log\alpha|, 1),$$

and $h(\alpha)$ denotes the standard logarithmic Weil height of α.

In the present situation we have $l=3$, $d=4$, $B=m$, and

$$\alpha_1 = k+\sqrt{k^2-1}, \quad \alpha_2 = 2k-1+2\sqrt{k^2-k},$$

$$\alpha_3 = \frac{2\left(\sqrt{k-1}+\sqrt{k+1}\right)\sqrt{k}}{\left(\sqrt{k-1}+2\sqrt{k}\right)\sqrt{k+1}},$$

with corresponding minimal polynomials

$$\alpha_1^2 - 2k\alpha_1 + 1 = 0, \quad \alpha_2^2 - 2(2k-1)\alpha_2 + 1 = 0,$$

$$(9k^4+24k^3+22k^2+8k+1)\alpha_3^4 - 16k(3k^3+7k^2+5k+1)\alpha_3^3$$
$$+48k^2(k^2+4k+3)\alpha_3^2 - 128k^2(k+1)\alpha_3 + 64k^2 = 0.$$

If $x=v_m=w_n$, $m\ge 0$ and $n\le 0$, then we obtain an identical result, since

$$\alpha_3' = \frac{2\left(\sqrt{k-1}+\sqrt{k+1}\right)\sqrt{k}}{\left(-\sqrt{k-1}+2\sqrt{k}\right)\sqrt{k+1}}$$

has the same minimal polynomial as α_3.

We get

$$h'(\alpha_1) = \frac{1}{2}\log\alpha_1 < \frac{1}{2}\log(2k),$$
$$h'(\alpha_2) = \frac{1}{2}\log\alpha_2 < \frac{1}{2}\log(4k-2),$$
$$h'(\alpha_3) = h'(\alpha_3') = \frac{1}{4}\left[2\log(3k^2+4k+1) + \log\alpha_3 + \log\alpha_3'\right] < \frac{1}{4}\log(147k^4).$$

From (17) and Theorem 3 we obtain

$$\frac{m}{\log m} < 1.1941\cdot 10^{14}\cdot\log(4k-2)\log\left(147k^4\right). \tag{18}$$

Since $k\le 28$ we have

$$\frac{m}{\log m} < 1.044\cdot 10^{16},$$

and so

$$m < 5\cdot 10^{17}.$$

Now we adopt GRINSTEAD's strategy [9] in order to show that $v_0 = w_0 = 1$ and $v_2 = w_{-2} = 4k^2 - 2k - 1$ are the only solutions of the equation $v_m = w_n$, $m \geq 0$ for $2 \leq k \leq 28$. These solutions correspond to $d = 0$ and $d = 16k^3 - 4k$.

We will prove that from $v_m = w_{4l}$ (resp. $v_m = w_{4l-2}$) it follows that $l = 0$. Since $|n| < m < 5 \cdot 10^{17}$, it is sufficient to show that

$$l \equiv 0 \pmod{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47}.$$

Let $b_l = w_{4l}$, resp. $b_l = w_{4l-2}$. We define $L(q)$ to be the length of the period of the sequence ($b_l \mod q$). Let p be a prime. If $p = 2$, we choose an integer q such that $L(q)$ is even and the sequences ($b_{2l+1} \mod q$) and ($v_m \bmod q$) have empty intersection. Thus we conclude that $l \equiv 0 \pmod 2$. In the same manner we prove $l \equiv 0 \pmod 3$ and $l \equiv 0 \pmod 5$. Let $5 < p \leq 47$ and assume that for all primes $r < p$, it has been shown that $l \equiv 0 \pmod r$. We follow [9] in proving that $l \equiv 0 \pmod p$ by considering ($v_m \mod q$) and ($b_l \mod q$), where q is a prime with the property that $L(q)$ is divisible only by primes not exceeding p, is power-free and is divisible by p (see [9, 4] for details). It is useful to observe that if $\left(\frac{k(k-1)}{q}\right) = 1$ then $L(q) | q-1$, and if $\left(\frac{k(k-1)}{q}\right) = -1$ then $L(q) | q+1$.

We will illustrate this method with an example. We will show that $l \equiv 0 \pmod{19}$ in the case $k = 4$ and $b_l = w_{4l}$. The two values of q we will use are $q = 113$ and $q = 151$. We have $L(113) = 57$ and $L(151) = 19$. First, let $q = 113$. We have:

$$\begin{aligned}
&(w_{4l} \mod 113)_{l\geq 0} = \\
&\quad (1, 71, 15, 4, 5, 21, 100, 27, 35, 35, 27, 100, 21, 5, 4, 15, 71, 1, 47, 8, 106, \\
&\quad 70, 18, 20, 82, 51, 60, 23, 55, 26, 75, 10, 88, 91, 28, 49, 104, 19, 104, 49, \\
&\quad 28, 91, 88, 10, 75, 26, 55, 23, 60, 51, 82, 20, 18, 70, 106, 8, 47, 1, 71, \dots),
\end{aligned}$$

$$\begin{aligned}
&(v_m \mod 113)_{m\geq 0} = \\
&\quad (1, 7, 55, 94, 19, 58, 106, 112, 112, 106, 58, 19, 94, 55, 7, 1, 1, 7, \dots).
\end{aligned}$$

We assume that $l \equiv 0 \pmod 3$, which can be proved by considering ($w_{4l} \mod 68$) and ($v_m \mod 68$). By comparing sequences, we see that $w_{4l} \equiv 1$ or $106 \pmod{113}$ and $l \equiv 0$ or $16 \pmod{19}$.

Next, let $q = 151$. We have:

$$(w_{4l} \mod 151)_{l\geq 0} = \\ (1, 87, 24, 149, 57, 34, 76, 59, 26, 96, 12, 22, 3, 83, 33, 15, 39, 142, 99, \\ 1, 87, \dots),$$

$$(v_m \mod 151)_{m\geq 0} = \\ (1, 7, 55, 131, 87, 112, 54, 18, 90, 98, 90, 18, 54, 112, 87, 131, 55, 7, 1, \\ 1, 7, \dots).$$

Since the number 39 is in the position 16 (mod 19) in the first sequence, and it does not occur in the second sequence, we have $l \equiv 0 \pmod{19}$.

We list the values of q used in the proof of Theorem 1 for $k = 4$ and $k = 5$:

p	q for $k = 4$	q for $k = 5$
2	8	23
3	68^*, 380^{**}	51
5	29^{**}, 55^*	35
7	41, 71, 139, 337^{**}, 421^{**}	13, 29, 71
11	23, 43, 307, 439^*	43, 89, 197, 199, 263, 307^{**}, 331^{**}, 661^{**}
13	103, 131	79, 103, 131
17	67, 101, 239, 271^{**}	67, 239, 373
19	113, 151	37, 113, 191, 227^*
23	47, 137, 277, 367, 599^*	137, 139, 461, 599, 643, 691^{**}, 827^{**}
29	59, 173, 349, 463	59, 173, 347
31	311, 373, 619, 683	311, 433, 557^{**}, 743^{**}
37	739, 1109, 1259	73, 149, 443, 887
41	83, 163, 1229	163, 739, 821, 983^*
43	257, 431, 859^{**}, 947^{**}, 1033^{**}	257, 431, 773, 1117
47	281, 659, 751, 1129^*	563, 659

The numbers with *, resp. **, are used in the case $b_l = w_{4l}$, resp. $b_l = w_{4l-2}$ only. In the actual running of this algorithm for all cases $2 \leq k \leq 28$, no prime p required more than eight values of q, and the greatest value of q which appeared was 3011. The computer program was developed in FORTRAN and the computation time was about 50 seconds on a HP 9000 workstation.

5. Final remarks

We can prove Theorem 1 for $k \leq 28$ using the reduction method based on the Baker–Davenport lemma ([2], see also [8, Lemma 2]). Let $\kappa = \log\left(k + \sqrt{k^2-1}\right) / \log\left(2k - 1 + 2\sqrt{k^2-k}\right)$ and $\mu_{1,2} = \log \frac{2\left(\sqrt{k-1}+\sqrt{k+1}\right)\sqrt{k}}{\left(\pm\sqrt{k-1}+2\sqrt{k}\right)\sqrt{k+1}} \;/\; \log\left(2k-1+2\sqrt{k^2-k}\right)$. Assume that $m < M$. Let p/q be the convergent of the continued fraction expansion of κ such that $q > 3M$ and let $\varepsilon = \|\mu q\| - M \cdot \|\kappa q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then

$$m < \frac{1}{2\log(2k-1)} \log \frac{q}{\varepsilon \log\left(2k-1+2\sqrt{k^2-k}\right)}.$$

Starting with $M = 5 \cdot 10^{17}$ we obtain after reduction that $m \leq 14$ (for all $3 \leq k \leq 28$), and the next step of the reduction gives $m \leq 0$ for μ_1 and $m \leq 2$ for μ_2, which completes the proof.

We can combine Lemma 3 and inequality (18) to prove the statement of Theorem 1 for k sufficiently large, without using Rickert's result. The bound obtained in this way ($k \leq 2 \cdot 10^{19}$) can be slightly improved by considering the sequences (v_m) and (w_n) mod $(2k-1)^2$, but it will be still much weaker than the bound ($k \leq 28$) obtained in Proposition 1.

From Theorem 1 it follows that for $k \geq 2$ the Diophantine quadruple $\{k-1, k+1, 4k, 16k^3-4k\}$ cannot be extended to a Diophantine quintuple. However, the rational number

$$\frac{4k(2k-1)(2k+1)(4k^2-2k-1)(4k^2+2k-1)(8k^2-1)}{(64k^6-80k^4+16k^2-1)^2}$$

has the property that its product with any of the elements of the above set increased by 1 is the square of a rational number (see [1, 7]). This is a special case of the more general fact that for every Diophantine quadruple $\{a_1, a_2, a_3, a_4\}$ there exists a positive rational number a_5 such that $a_i a_5 + 1$ is the square of a rational number for $i = 1, 2, 3, 4$ (see [7, Corollary 1]).

Acknowledgements. The author would like to thank Professor Attila Pethő for many helpful comments and improvements on an earlier draft of the manuscript.

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(Received January 17, 1997, revised March 17, 1997)