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Inverse systems of quasi-compact spaces

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Abstract. In this paper we investigate the non-emptiness and the quasi-compactness of a limit of an inverse system $\mathbf{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ of the non-empty and quasi compact spaces X_α.

The main result of Section One is the following

1.9. THEOREM. *Let $\mathbf{X} = \{X_\alpha, f_{\alpha,\beta}, A\}$ be an inverse system of quasi-compact T_0 spaces X_α and* SWO*-mappings $f_{\alpha\beta}$ (almost closed mappings $f_{\alpha\beta}$, weakly closed mappings $f_{\alpha\beta}$). If the spaces X_α, $\alpha \in A$, are non-empty, then* $\lim \mathbf{X}$ *is non-empty.*

Section Two contains some theorems concerning the inverse systems $\mathbf{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ with Wallman extendible mappings. The main result of this Section is the following

2.13. THEOREM. *Let $\mathbf{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system with closed mappings $f_{\alpha\beta}$ and onto projections $f_\alpha : \lim \mathbf{X} \to X_\alpha$, $\alpha \in A$. Then the functor w is $\mathbf{X}$-continuous iff $\mathbf{X}$ is an S-system.*

0. Introduction

We denote inverse systems by $\mathbf{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ and their limits by $X = \lim \mathbf{X}$. For all basic properties of inverse systems we refer to R. ENGELKING [5].

By N is denoted the set of natural numbers. The set of all ordinal numbers of cardinality $\leq \aleph_m$ is denoted by W_m.

The symbol $cf(A)$ means the cofinality of a well-ordered set A i.e. the smallest ordinal number which is cofinal in A.

If $f : X \to Y$ is a mapping and if $A \subseteq X$, then $f^{\#}(A)$ denotes the set $\{y : f^{-1}(y) \subseteq A\}$.

The cardinality of a set A we denote by $|A|$.

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1. Non-emptiness of the limit space

We say that a mapping $f : X \to Y$ is an SWO–mapping if for each finite open cover $\mathcal{V} = \{V_1, \dots, V_n\}$ of Y, the cover $f^{-1}(\mathcal{V}) = \{f^{-1}(V_1), \dots, f^{-1}(V)\}$ has the property that $\mathrm{Cl}f(A) \subseteq V_j$ if A is closed and $A \subseteq f^{-1}(V_j)$.

Clearly, each closed mapping is an SWO–mapping.

1.1. Lemma. *Let $f : X \to Y$ be an SWO-mapping. If Y is T_1, then f is closed.*

PROOF. Let $F \subseteq X$ be closed. Suppose that there is a point $y \in \mathrm{Cl}f(F)\backslash f(F)$. For the point y we consider a cover $\mathcal{V} = \{Y\backslash\{y\}, V\}$, where V is open and $y \in V$. Clearly, $V \cap f(F) \neq \emptyset$. Since f is an SWO–mapping and $F \subseteq f^{-1}(Y\backslash\{y\})$, we have $\mathrm{Cl}f(F) \subseteq Y\backslash\{y\}$. This is impossible since $y \in \mathrm{Cl}f(F)$. The proof is complete.

We say that $F \subseteq X$ is almost closed if $y \in F$ for each closed point $y \in \mathrm{Cl}(F)$. A mapping $f : X \to Y$ is almost closed if $f(F)$ is almost closed for each closed $F \subseteq X$.

1.2. Lemma. *Let $f : X \to Y$ be an almost closed mapping. If Y is T_1 then f is closed.*

A mapping $f : X \to Y$ is called weakly closed if f/Y_x is closed for each set $Y_x = \cap\{\mathrm{Cl}U : U$ is a neighbourhood of $x \in X\}$.

1.3. Lemma. *If X is a Hausdorff space and Y is T_1, then each mapping $f : X \to Y$ is weakly closed.*

1.4. Lemma. *If X is a Hausdorff space and if $f : X \to Y$ is a weakly closed onto mapping, then Y is a T_1 space.*

We are now going to study the non-emptiness of the inverse limit space.

Let $\mathbf{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of non-empty spaces X_α. We say that $\mathbf{Y} = \{Y_\alpha, f_{\alpha\beta}/Y_\beta, A\}$ is a subsystem of $\mathbf{X}$ if $\emptyset \neq Y_\alpha \subseteq X_\alpha$ and $f_{\alpha\beta}(Y_\beta) \subseteq Y_\alpha$ for each pair $\alpha, \beta \in A$ such that $\alpha \leq \beta$.

A subsystem is closed if each $Y_\alpha \subseteq X_\alpha$ is closed.

1.5. Lemma. *Let $\mathbf{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of non-empty quasi-compact spaces X_α. There is a closed subsystem $\mathbf{Y} = \{Y_\alpha, f_{\alpha\beta}/Y_\beta, A\}$ such that $Y_\alpha = Clf_{\alpha\beta}(Y_\beta)$.*

PROOF. Let $\mathcal{N}$ be the set of all subsystems of $\mathbf{X}$. The set N is non-empty since $\mathbf{X} \in \mathcal{N}$. Let $\mathbf{Y} = \{Y_\alpha, f_{\alpha\beta}/Y_\beta, A\}$ and $\mathbf{Z} = \{Z_\alpha, f_{\alpha\beta}/Z_\beta, A\}$ be a pair of subsystems of $\mathbf{X}$. We write $\mathbf{Z} \leq \mathbf{Y}$ if $Z_\alpha \subseteq Y_\alpha$ for each $\alpha \in A$. Clearly, the set $(\mathcal{N}, \leq)$ is a partially ordered set. Moreover, if for each subsystem $\mathbf{Y} = \{Y_\alpha, f_{\alpha\beta}/Y_\beta, A\}$ we define $Y_\alpha^* = \cap\{\mathrm{Cl}f_{\alpha\beta}(Y_\beta),\ \beta \geq \alpha\}$, $\alpha \in A$, then $\mathbf{Y}^* = \{Y_\alpha^*, f_{\alpha\beta}/Y_\beta^*, A\}$ is a subsytem. From the quasi-compactness of X_α it follows that Y_α^* is non-empty since a family $\{f_{\alpha\beta}(Y_\beta),$

$\beta \geq \alpha\}$ is a centred family. It is easy to prove that $\mathbf{Y}^* \leq \mathbf{Y}$. Let us prove that $(\mathcal{N}, \leq)$ has a minimal member. It suffices to prove that for each chain $\mathbf{Y}_1 \geq \mathbf{Y}_2 \geq \ldots \geq \mathbf{Y}_\mu \geq \ldots$, $\mu \in M$, there is $\mathbf{Y}$ such that $\mathbf{Y}_\mu \geq \mathbf{Y}$ for each $\mu \in M$. Since $\mathbf{Y}_\mu = \{Y_\alpha^\mu, f_{\alpha\beta}/Y_\beta^\mu, A\}$ we have a non-empty set $Y_\alpha = \cap\{Y_\alpha^\mu : \mu \in M\}$ (X_α is quasi-compact). Clearly, $\mathbf{Y} = \{Y_\alpha, f_{\alpha\beta}/Y_\beta, A\}$ is a subsystem since $f_{\alpha\beta}(\cap\{Y_\beta^\mu : \mu \in M\}) \subseteq \cap\{f_{\alpha\beta}(Y_\beta^\mu : \mu \in M\}) \subseteq \cap\{Y_\alpha^\mu : \mu \in M\} = Y_\alpha$. Thus, $(\mathcal{N}, \leq)$ has a non-empty subset $\mathcal{N}'$ of minimal elements. Let $\mathbf{Y} = \{Y_\alpha, f_{\alpha\beta}/Y_\beta, A\}$ be any member of $\mathcal{N}'$. Suppose that there is a pair $\alpha, \beta \in A$ such that $\mathrm{Cl} f_{\alpha\beta}(Y_\beta) \subset Y_\alpha$. Then $\mathbf{Y}^* \leq \mathbf{Y}$. On the other hand we have $\mathbf{Y} \geq \mathbf{Y}^*$ since $\mathbf{Y} \in \mathcal{N}'$. Thus $\mathbf{Y} = \mathbf{Y}^*$. This means that $\mathrm{Cl} f_{\alpha\beta}(Y_\beta) = Y_\alpha$ for each $\beta \geq \alpha$. The proof is completed.

1.5.1. *Remark.* A space X is called C–closed if each quasi-compact subset $A \subseteq X$ is closed. If the X_α, $\alpha \in A$ in Lemma 1.5. are C–closed, then we have $Y_\alpha = f_{\alpha\beta}(Y_\beta)$.

1.5.2. *Remark.* In fact, from the proof of Lemma 1.5. it follows that each closed subsystem $\mathbf{Z}$ contains some minimal closed subsystem $\mathbf{Y}$.

1.6. Lemma. *Let $\mathbf{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of non-empty quasi-compact spaces X_α. Each minimal subsystem $\mathbf{Y} = \{Y_\alpha, f_{\alpha\beta}/Y_\beta, A\}$ has the property that $Y_\alpha \subseteq Y_{x(\alpha)}$ for some point $x(\alpha) \in Y_\alpha$, $\alpha \in A$.*

PROOF. Let $x(\alpha)$ be any point of Y_α. From the relation $\mathrm{Cl} f_{\alpha\beta}(Y_\beta) = Y_\alpha$, $\beta \geq \alpha$ (Lemma 1.5.) it follows that $U \cap f_{\alpha\beta}(Y_\beta) \neq \emptyset$ for each $\beta \geq \alpha$ and each open neighbourhood of $x(\alpha)$. This means that $f_{\alpha\beta}^{-1}(U) \cap Y_\beta \neq \emptyset$, $\beta \geq \alpha$. A family $\{\mathrm{Cl} f_{\alpha\beta}^{-1}(U) \cap Y_\beta : U$ is a neighbourhood of $x(\alpha)\}$ is centred and $\cap\{\mathrm{Cl} f_{\alpha\beta}^{-1}(U) \cap Y_\beta : U$ is a neighbourhood of $x(\alpha)\} = Y'_\beta$ is non-empty. Clearly $f_{\alpha\beta}(Y'_\beta) \subseteq Y_\alpha \cap Y_{x(\alpha)}$, where $Y_{x(\alpha)} = \cap\{\mathrm{Cl} U : U$ is open and $x(\alpha) \in U\}$. If we suppose that $Y_\alpha \not\subset Y_{x(\alpha)}$, then $Y_\alpha \cap Y_{x(\alpha)} = Z_\alpha$. Now we define $Z_\beta = Y'_\beta$ for each $\beta \geq \alpha$. For all other $\gamma \in A$ let Z_γ be the set $Y_\gamma \in \mathbf{Y}$. From the relation $f_{\alpha\beta}(Y'_\beta) \subseteq Y_\alpha \cap Y_{x(\alpha)} = Z_\alpha$ we infer that $\mathbf{Z} = \{Z_\alpha, f_{\alpha\beta}/Z_\beta, A\}$ is a subsystem such that $\mathbf{Z} \leq \mathbf{Y}$. This is impossible since $\mathbf{Y}$ is minimal. The proof is complete.

In the sequel we use the following lemmas:

1.7. Lemma. *Each closed subset F of a T_0 quasi-compact space X contains a closed point.*

PROOF. See [21].

1.8. Lemma. *Let $f : X \to Y$ be an SWO-mapping and let Y be T_0. Then for each closed $F \subseteq X$ a set $f(F)$ contains each closed point of $Clf(F)$.*

PROOF. Suppose that $y \in \mathrm{Cl}f(F)\backslash f(F)$ is a closed point. Consider a cover $\mathcal{V} = \{Y\backslash\{y\}, V\}$, where V is any neighbourhood of y. Then $f^{-1}(\mathcal{V}) = \{f^{-1}(Y\backslash\{y\}), f^{-1}(V)\}$ is a cover of X and $F \subseteq f^{-1}(Y\backslash\{y\})$. By virtue of the definition of SWO–mappings we have $\mathrm{Cl}f(F) \subseteq Y\backslash\{y\}$ i.e. $y \notin \mathrm{Cl}f(F)$. This is impossible since $y \in \mathrm{Cl}f(F)$. The proof is complete.

Now we prove the following theorem concerning the non-emptiness of the inverse limit.

1.9. Theorem. *Let $\mathbf{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of quasi-compact T_0 spaces X_α and SWO–mappings $f_{\alpha\beta}$ (almost closed mappings $f_{\alpha\beta}$, weakly closed mappings $f_{\alpha\beta}$). If the spaces X_α, $\alpha \in A$, are non-empty, then $\lim \mathbf{X}$ is non-empty.*

PROOF. Firstly we consider the inverse systems with SWO–mappings. Let $\mathbf{Y} = \{Y_\alpha, f_{\alpha\beta}/Y_\beta, A\}$ be a minimal subsystem of $\mathbf{X}$. Now we prove that for each $\alpha \in A$ the set Y_α has only one point. By virtue of Lemma 1.7. there is a closed point $y_\alpha \in Y_\alpha$. From Lemma 1.8. it follows that $y_\alpha \in f_{\alpha\beta}(Y_B)$ for each $\beta \geq \alpha$. (This is true also if the mappings $f_{\alpha\beta}$ are almost closed). This means that the sets $Z_\beta = f_{\alpha\beta}^{-1}(y_\alpha) \cap Y_\beta$ are non-empty. For each $\beta < \alpha$ let $Z_\beta = f_{\alpha\beta}(y_\alpha)$. For all other $\gamma \in A$ we define $Z_\gamma = Y_\gamma$. Now we obtain a subsystem $\mathbf{Z} = \{Z_\alpha, f_{\alpha\beta}/Z_\beta, A\}$. Clearly, $\mathbf{Z} \leq \mathbf{Y}$. On the other hand we have $\mathbf{Y} \leq \mathbf{Z}$ since $\mathbf{Y}$ is a minimal subsytem. Thus, $\mathbf{Y} = \mathbf{Z}$ i.e. $Y_\alpha = y_\alpha$. Since this is true for each $\alpha \in A$ we have a subsystem $\mathbf{Y} = \{\{y_\alpha\}, f_{\alpha\beta}/\{y_\alpha\}, A\}$. Clearly, $\mathbf{Y}$ is a point of $\lim \mathbf{X}$ i.e. $\lim \mathbf{X}$ is non-empty. In order to complete the proof it suffices to prove that $\lim \mathbf{X}$ is non-empty if the mappings $f_{\alpha\beta}$ are weakly closed. Let us note that closed mappings are SWO–mappings. From the preceding part of this proof it follows that Theorem 1.9. is true for closed mappings $f_{\alpha\beta}$. Finally, if the mappings $f_{\alpha\beta}$ are weakly closed, then by Lemma 1.6. we infer that each minimal inverse subsystem $\mathbf{Y} = \{Y_\alpha, f_{\alpha\beta}/Y_\beta, A\}$ has the closed bonding mappings $f_{\alpha\beta}/Y_\beta$. Thus, $\lim \mathbf{Y} \subseteq \lim \mathbf{X}$ is non-empty i.e. $\lim \mathbf{X}$ is non-empty. The proof is complete.

1.10. *Remark.* Theorem 1.9. is a generalization of Stone's well–known theorem [21] since closed mappings are SWO–mappings (almost closed and weakly closed mappings).

Now we prove the quasi-compactness of the limit space.

1.11. Theorem. *Let $\mathbf{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be as in Theorem 1.9. Then $\lim \mathbf{X}$ is quasi-compact.*

PROOF. Let $\mathcal{U} = \{U_\mu : \mu \in M\}$ be any open cover of $\lim \mathbf{X}$. By virtue of the definition of a base in $\lim \mathbf{X}$ there is an open $U_{\mu,\alpha} \subseteq X_\alpha$, for each $\alpha \in A$ and $\mu \in M$, such that $U_\mu = \cup\{U_{\mu,\alpha} : \alpha \in A\}$, $f_\alpha^{-1}(U_{\mu,\alpha}) \subseteq U_\mu$ and $U_{\mu,\alpha}$ is a maximal set with respect to property $f^{-1}(U_{\mu,\alpha}) \subseteq U_\mu$. Let $\mathcal{U}_\alpha$ be a family $\{U_{\mu,\alpha} : \alpha \in A\}$. If $\mathcal{U}_\alpha$ is the cover of X_α then $f_\alpha^{-1}(\mathcal{U}_a)$ is a cover of $\lim \mathbf{X}$ which refines $\mathcal{U}$. This means that $\mathcal{U}$ has a finite subcover since $\mathcal{U}_a$ has a finite subcover. Now we prove that there exists an $\alpha \in A$ such that $\mathcal{U}_\alpha$ is a cover of X_α. In the opposite case the set $Z_\alpha = X_\alpha \backslash (\cup\{U_{\alpha,\mu} : \mu \in M\})$ is non-empty for each $\alpha \in A$. Now we obtain a closed subsystem $\mathbf{Z} = \{Z_\alpha, f_{\alpha\beta}/Z_\beta, A\}$. By virtue of 1.5.2. it follows that there is a closed subsystem $\mathbf{Y} \leq \mathbf{Z}$ such that $\mathbf{Y}$ is minimal. From the proof of Theorem 1.9. it follows that $\lim \mathbf{Y}$ is non-empty. This means that $\lim \mathbf{Z} \neq \emptyset$. Let z be any point of $\lim \mathbf{Z}$. It is easy to prove that $z \notin \cup\{f_\alpha^{-1}(U_{\mu,\alpha}) : \alpha \in A,\ \mu \in M\}$. This is impossible since $\mathcal{U} = \{U_\mu : \mu \in M\}$ is the cover of $\lim \mathbf{X}$. Thus, there is an $\alpha \in A$ such that $\mathcal{U}_\alpha$ is a cover of X_α. The proof is complete.

We close this Section with two theorems concerning the week closedness of the projections.

1.12. Theorem. *Let $\mathbf{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of quasi-compact T_1 spaces. The projections $f_\alpha : \lim \mathbf{X} \to X_a$, $\alpha \in A$, are weakly closed if the mappings $f_{\alpha\beta}$ are weakly closed.*

PROOF. Let x be any point of $\lim \mathbf{X}$. We have the inverse subsystem $\mathbf{Y} = \{Y_{x_\alpha} : x_\alpha = f_\alpha(x),\ \alpha \in A\}$, where $Y_{x_\alpha} = \cap\{\mathrm{Cl}U_\alpha : U_\alpha$ is open and $x_\alpha \in U_\alpha\}$. Let F be a closed subset of Y_x. Let $\alpha \in A$ be fixed. Now we prove that $f_\alpha(F)$ is closed. Suppose that $Z_\alpha = \mathrm{Cl}f_\alpha(F)\backslash f_\alpha(F)$ is non-empty. Then $Z_\beta = \mathrm{Cl}f_\beta(F)\backslash f_\beta(F)$ is non-empty for each $\beta \geq \alpha$ since $f_\beta(F) \subseteq Y_{x_\beta}$ and $f_{\alpha\beta}/Y_{x_\beta}$ is closed. From the closedness of $f_{\alpha\beta}/Y_{x_\beta}$ it follows that for each $z_\alpha \in Z_\alpha$ and each $\beta \geq \alpha$ the set $W_\beta = f^{-1}(z_\alpha) \cap Z_\beta$ is closed and non-empty. From Theorem 1.9. it follows that the inverse system $\mathbf{W} = \{W_\beta, f_{\beta\gamma}/W_\gamma,\ \alpha \leq \beta \leq \gamma\}$ has a non-empty limit W. Clearly, $W \subseteq F$ since $F = \lim\{\mathrm{Cl}f_\alpha(F),\ f_{\alpha\beta}/\mathrm{Cl}f_\beta(F), A\}$. On the other hand, for any $w \in W$ we have $f_\alpha(w) = z_\alpha \in \mathrm{Cl}f_\alpha(F)\backslash f_a(F)$. This is impossible since $W \subseteq F$. The proof is complete.

If A is the set N of natural numbers, then the quasi-compactness of X_α can be omitted since the point w can be constructed by total induction. Thus we have

1.13. Theorem. *Let $\mathbf{X} = \{X_n, f_{nm}, N\}$ be an inverse sequence of T_1 spaces X_n with weakly closed mappings f_{nm}. Then the projections $f_n : \lim \mathbf{X} \to X_n$, $n \in N$, are weakly closed.*

2. Inverse systems with Wallman extendible bonding mappings

Let X be a topological T_1 space and let $\mathcal{J} = \{A_\mu : \mu \in M\}$ be a centred family of closed subsets $A_\mu \subseteq X$. We say that $\mathcal{J}$ is fixed (free) if $\cap\mathcal{J} = \cap\{F : F \in \mathcal{J}\}$ is non-empty (empty). By Zorn's lemma each centred family is contained in some maximal centred family.

The Wallman extension wX of a space X is a set $wX = X \cup F_0(X)$, where $F_0(X)$ is a set of all free maximal centred families of closed subsets of X, with topology whose base is the family of all sets $U^* = U \cup \{\mathcal{J} \in F_0(X) : F \subseteq U$ for some $F \in \mathcal{J}\}, U$ is open in X [5].

We say that a continuous mapping $f : X \to Y$ is Wallman extendible if there is a continuous mapping $wf : wX \to wY$ such that $f = wf/X$.

A category $\mathcal{C}$ of T_1 spaces and continuous mappings is said to be a W–category if each morphism of $\mathcal{C}$ has a unique Wallman extension.

2.1. Lemma. *If $\mathcal{C}$ is any W–category, then $w : X \to wX$ is a covariant functor in a category* Qcpt *of T_1 quasi-compact spaces and continuous mappings. Moreover, if $\mathbf{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ is an inverse system, then wX, defined to be $\{wX_\alpha, wf_{\alpha\beta}, A\}$, is an inverse system.*

PROOF. Trivial.

2.2. *Definition.* The functor w is called $\mathbf{X}$*–continuous* if there is a homeomorphism $h : w(\lim \mathbf{X}) \to \lim w\mathbf{X}$ such that $h(x) = x,\ x \in \lim \mathbf{X}$. In this case we write $w(\lim \mathbf{X}) \approx \lim w\mathbf{X}$. We say that w is $\mathcal{C}$*–continuous* if w is $\mathbf{X}$–continuous for each $\mathbf{X}$ in pro-$\mathcal{C}$, the category with the inverse systems in $\mathcal{C}$ as the objects and the mappings of the inverse systems as the morphisms.

2.3. *Remark.* The functor w is not Top–continuous since there exists an inverse system with empty limit.

2.4. *Definition.* An inverse system $\mathbf{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ is called an S–system if for each pair F, G of disjoint closed subsets of $\lim \mathbf{X}$ there is an $\alpha \in A$ such that $\mathrm{Cl} f_a(F) \cap \mathrm{Cl} f_\alpha(G) = \emptyset$, where $f_\alpha : \lim \mathbf{X} \to X_\alpha$ is a projection.

2.5. *Examples.* a) Each inverse system of quasi-compact spaces and closed bonding mappings is an S–system. This follows from [21].

b) If $\mathbf{X} = \{X_n, f_{nm}, N\}$ is an inverse sequence of countably compact spaces X_n and closed mappings f_{nm}, then $\mathbf{X}$ is an S–system [14].

c) Let $\mathbf{X}$ be an inverse sequence of sequentially compact (strongly countably compact, D–compact) spaces. Then $\mathbf{X}$ is an S–system.

d) Let $\mathbf{X} = \{X_\alpha, f_{\alpha\beta}, W_m\}$ be an inverse system of $\aleph_m$–compact spaces X_α and closed mappings $f_{\alpha\beta}$. Then $\mathbf{X}$ is an S–system [14].

e) Let $\mathbf{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be a well-ordered inverse system with weight $w(X_\alpha) < \tau$ and $cf(A) > \tau$. If X is continuous or $f_{\alpha\beta}$ are perfect (open) mappings, then $\mathbf{X}$ is an S–system. This follows from [23: Theorem 2.2.] since now the weight $w(\lim \mathbf{X}) < \tau$ and each closed $F \subseteq \lim \mathbf{X}$ is $f_\alpha^{-1}(F_\alpha)$ for some closed $F_\alpha \subseteq X_\alpha$.

f) Similarly, each well-ordered inverse system $\mathbf{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ with $hl(X_\alpha) < \tau$ and $cf(A) > \tau$ is an S–system [15].

The importance of S–systems is shown by the following

2.6. Theorem. *Let* $\mathbf{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ *be an inverse system. If* $\lim wX \approx w(\lim \mathbf{X})$, *then* $\mathbf{X}$ *is an* S*–system.*

PROOF. Let F, G be disjoint closed subsets of $\lim \mathbf{X}$. It is known [4] that the closures of F, G in $\lim w\mathbf{X}$ are the sets $F_* = F \cup \{\mathcal{J} \in F_0(\lim w\mathbf{X}) : F \in \mathbf{J}\}$, $G_* = G \cup \{\mathcal{J} \in F_0(\lim w\mathbf{X}) : G \in \mathcal{J}\}$ and that $(F \cap G)_* = F_* \cap G_*$. This means that $F_* \cap G_* = \emptyset$. Since $w\mathbf{X}$ is an S–system (see Examples 2.5.) and since $F_* \supseteq F$, $G_* \supseteq G$ are disjoint closed sets in $\lim w\mathbf{X}$, we infer that there is an $\alpha \in A$ such that $\mathrm{Cl} f'_\alpha(F_*) \cap \mathrm{Cl} f'_\alpha(G_*) = \emptyset$, where f'_α is a projection. Since $f_\alpha(F) \subseteq f'_a(F_*)$ and $f_\alpha(G) \subseteq f'_\alpha(G)$ we infer that $\mathrm{Cl} f_\alpha(F) \cap \mathrm{Cl} f_\alpha(G) = \emptyset$. Thus $\mathbf{X}$ is an S–system and the proof is complete.

2.7. Lemma. *If* $\mathbf{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ *is an inverse* S*–system with closed mappings* $f_{\alpha\beta}$ *and onto projections* $f_\alpha : \lim \mathbf{X} \to X_\alpha$, $\alpha \in A$, *then the projections* f_α, $\alpha \in A$, *are closed.*

PROOF. Let F be any closed subset of $\lim \mathbf{X}$. In order to prove that f_α is closed it suffices to prove that $f_\alpha(F)$ is closed. For each $x_\alpha \notin f_\alpha(F)$ we have $f_\alpha^{-1}(x_\alpha) \cap F = \emptyset$. There is a $\beta \in A$, $\beta > \alpha$, with $f_\beta f_\alpha^{-1}(x_\alpha) \cap f_\beta(F) = \emptyset$ since $\mathbf{X}$ is an S–system. From the closedness of $f_{\alpha\beta}$ it follows that $\mathrm{Cl} f_\alpha(F) = f_{\alpha\beta}(\mathrm{Cl} f_\beta(F))$. We now have that $\{x_\alpha\} \cap f_{\alpha\beta}(\mathrm{Cl} f_\beta(F)) = \emptyset$ i.e. $x_\alpha \notin \mathrm{Cl} f_\alpha(F)$. Thus, $x_\alpha \notin f_\alpha(F)$ implies $x_\alpha \notin \mathrm{Cl} f_\alpha(F)$. This means that $f_\alpha(F)$ is closed. The proof is complete.

2.8. *Remark.* We say that a mapping $f : X \to Y$ is hereditarily quotient [5, Exercise 2.4.F.] if for each $y \in Y$ and any open $U \supseteq f^{-1}(y)$ we have $y \in \mathrm{Int} f(U)$.

By the same method as in the proof of Lemma 2.7. we have the following

2.9. Lemma. *If* $\mathbf{X}$ *is an* S*–system with quotient (hereditarily quotient) mappings* $f_{\alpha\beta}$ *and onto projections, then the projections are quotient (hereditarily quotient).*

2.10. *Remark.* We say that a topological property $\mathcal{P}$ is relatively continuous with respect to $\mathbf{X}$ if $\lim \mathbf{X}$ has $\mathcal{P}$ when the spaces $X_\alpha \in \mathbf{X}$ have $\mathcal{P}$. Let us note that if $\mathbf{X}$ is an S–system, then $\mathcal{P} =$ "normal" ("connected") is relatively continuous with respect to $\mathbf{X}$.

2.11. *Definition.* A mapping $f : X \to Y$ is called a WC–mapping if f has a unique closed extension $wf : wX \to wY$.

A class of WC–mappings was introduced by D. Harris [9].

2.12. Lemma. [20]. *Every closed onto mapping* $f : X \to Y$ *has a closed onto extension* $wf : wX \to wY$.

PROOF. In [20] the proof for multi–valued mappings was given. We now give an alternate proof. The proof is broken up into several steps.

Step 1. A w–mapping $f : X \to Y$ is a wc–mapping iff $wf(F_*)$ is closed in wY for each closed subset $F \subseteq X$.

PROOF. *Necessity.* For each closed $F \subseteq X$ we have that $\mathrm{Cl}_{wX}F = F_*$. If $f : X \to Y$ is a wc–mapping, then $wf : wX \to wY$ is closed. Thus, $wf(F_*)$ is closed in wY.

Sufficiency. Suppose that each $wf(F_*)$ is closed and let us prove that wf is closed. Let A be a closed subset of wX. There is a family $\{F_\mu : F_\mu \text{ is closed in } X, \mu \in M\}$ such that $a = \cap\{F_{\mu*},\ \mu \in M\}$. Clearly, $wf(A) \subseteq \cap\{wf(F_{\mu*}) : \mu \in M\}$. Let us prove that $wf(A) \supseteq \cap\{wf(F_{\mu*}) : \mu \in M\}$. For each $y \in \cap\{wf(F_{\mu*}) : \mu \in M\}$ we infer that $(wf)^{-1}(y) \cap F_{\mu*}$ is non-empty. Since $(wf)^{-1}(y)$ is quasi-compact, we have that the intersection $\cap\{(wf)^{-1}(y) \cap F_{\mu*} : \mu \in M\}$ is non-empty. Thus, there is a point $x \in (wf)^{-1}(y)$ such that $y \in \cap\{F_{\mu*} : \mu \in M\} = A$. This means that $y \in wf(A)$. Finally, we have that $wf(A) = \cap\{wf(F_{\mu*}) : \mu \in M\}$. Since each $wf(F_{\mu*})$ is closed, we infer that $wf(A)$ is closed. The proof is complete.

Step 2. In the sequel we use the following relations. The continuity of wf implies

$$(1)\qquad wf(F_*) \subseteq \mathrm{Cl}_{wY} wf(F) = \mathrm{Cl}_{wY} f(F),\ F \text{ is closed in } X.$$

On the other hand we have

$$(2)\qquad \mathrm{Cl}_Y f(F) = \mathrm{Cl}_{wY} f(F) \cap Y \subseteq \mathrm{Cl}_{wY} f(F).$$

The inclusion $f(F) \subseteq \mathrm{Cl}_Y f(F)$ gives

$$(3)\qquad \mathrm{Cl}_{wY} f(F) \subseteq \mathrm{Cl}_{wY}(\mathrm{Cl}_Y f(F)).$$

Similarly from (2) we obtain

$$(4)\qquad \mathrm{Cl}_{wY}(\mathrm{Cl}_Y f(F)) \subsetneq \mathrm{Cl}_{wY} f(F).$$

Finally we have

$$(5)\qquad \mathrm{Cl}_{wY} f(F) = \mathrm{Cl}_{wY}(\mathrm{Cl}_Y f(F)).$$

From (1) and the last relation it follows

$$(6)\qquad wf(F_*) \subseteq \mathrm{Cl}_{wY}(\mathrm{Cl}_Y f(F)),\ F \text{ is closed in } X.$$

Step 3. A w–mapping $f : X \to Y$ is a wc–mapping iff for each closed set $F \subseteq X$ it follows $wf(F_*) = \mathrm{Cl}_{wY}(\mathrm{Cl}_Y f(F)) = (\mathrm{Cl}_Y f(F))_*$

PROOF. Apply Step 1. and the relations (1)–(6).

Step 3. If $f : X \to Y$ is closed then wf is a wc–mapping.

PROOF. It is suffiecient to prove that $wf(F_*) = (f(F))_*$ for each closed $F \subseteq X$. From (1) it follows that $wf(F_*) \subseteq (f(F))_*$. Clearly, $f(F) \subseteq wf(F_*) \subseteq (f(F))_*$. We now use the condition (KC).

(KC) If A is a closed subset of Y and $K \subseteq wY$ is quasi-compact with $A \subseteq K \subseteq \mathrm{Cl}_{wY} A$, then K is closed.

If we prove that wY satisfies condition (KC) then Step 3. is proved since $wf(F_*)$ is quasi-compact.

Step 4. The Wallman compactification wX of a T_1 space X satisfies condition (KC).

PROOF. Suppose that we have a closed subset of X and a quasi-compact subset K such that $A \subseteq K \subseteq \mathrm{Cl}_{wY} A$. If we suppose that K is not closed then there exists a point $y \in \mathrm{Cl}_{wX} A \backslash K$. For each point $k \in K$ there is an open set U_k^* [2:232] such that $k \in U_k^*$ and $y \notin U_k^*$. From the compactness of K it follows that there is a finite subfamily $\{U_{k_1}^*, \dots, U_{k_n}^*\}$ which covers K. Since $(U_{k_1} \cup \dots \cup U_{k_n})^* = (U_{k_1}^* \cup \dots \cup U_{k_n}^*)$ [5] we infer that $A \subseteq U_{k_1} \cup \dots \cup U_{k_n}$. This means that $\mathrm{Cl}_{wX} A \subseteq (U_{k_1} \cup \dots U_{k_n})^*$. This is impossible since $y \notin (U_{k_1} \cup \dots \cup Uk_n)^*$. The proof of Lemma 2.12. is complete.

The main result of this Section is the following

2.13. Theorem. *Let* $\mathbf{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ *be an inverse system with closed mappings* $f_{\alpha\beta}$ *and onto projections* $f_\alpha : \lim \mathbf{X} \to X_\alpha$, $\alpha \in A$. *Then the functor* w *is* $\mathbf{X}$*–continuous iff* $\mathbf{X}$ *is an* S*–system.*

PROOF. Let the functor w be $\mathbf{X}$–continuous. Then by Theorem 2.5. $\mathbf{X}$ is an S–system. Conversely, if $\mathbf{X}$ is an S–system we consider the inverse system $w\mathbf{X} = \{wX_\alpha, wf_{\alpha\beta}, A\}$. This system exists since $wf_{\alpha\beta}$ are the unique extensions of $f_{\alpha\beta}$. Since $f_\alpha : \lim \mathbf{X} \to X_\alpha$ is onto and closed (Lemma 2.7.) we have a closed extension $wf_\alpha : w(\lim \mathbf{X}) \to wX_\alpha$, $\alpha \in A$. The mappings wf_α, $\alpha \in A$, induce a mapping $H : w(\lim \mathbf{X}) \to \lim w\mathbf{X}$ such that $f'H = wf_\alpha$, where $f'_\alpha : \lim w\mathbf{X} \to wX_\alpha$, $\alpha \in A$, are projections. Now we prove that H is onto and a 1–1 mapping. If x is a $\lim w\mathbf{X}$, then $f'_\alpha(x) \in wX_\alpha$, $\alpha \in A$, and $(wf_\alpha)^{-1} f'_\alpha(x)$ is a non-empty subset of $w(\lim \mathbf{X})$. Since $w(\lim \mathbf{X})$ is quasi-compact and since $\{(wf_\alpha)^{-1} f'_\alpha(x) : \alpha \in A\}$ is a centred family of closed sets, there is a point $y \in \cap\{(wf_\alpha)^{-1} f'_\alpha(x) : \alpha \in A\}$. Clearly, $wf_\alpha(y) = f'_\alpha(x)$ i.e. $H(y) = x$. Thus, H is onto. Let us prove that H is 1–1. Let y, z be a pair of distinct points in $w(\lim \mathbf{X})$. This means that there is a pair of disjoint closed subsets F, G of $\lim \mathbf{X}$ $F \in y$, $G \in z$. Since $\mathbf{X}$ is an S–system we have some $\alpha \in A$ such that $f_\alpha(F)$ and $f_\alpha(G)$ are disjoint (f_α is closed !). This means that $wf_\alpha(y) \neq wf_\alpha(z)$ and, consequently, $H(y) \neq H(z)$. In order to prove that H is a homeomorphism it remains to prove that H is closed. If $F \subseteq w(\lim \mathbf{X})$ is closed, then each $wf_\alpha(F)$, $\alpha \in A$, is closed (Lemma 2.12.). The set

$Y = \cap\{(f'_\alpha)^{-1} w f_\alpha(F) : \alpha \in A\}$ is closed and $H(F) \subseteq Y$. We prove that $Y = H(F)$. Suppose that $y \in Y \backslash H(F)$. We have $f'_\alpha(x) \in w f_\alpha(F)$ and $(w f_\alpha)^{-1} f'_\alpha(x) \cap F \neq \emptyset$. Since F is quasi-compact the intersection $Z = \cap\{(w f_\alpha)^{-1} f'_\alpha(x) \cap F : \alpha \in A\}$ is non-empty. For each $z \in Z$ we have $w f_\alpha(z) = f'_\alpha(x)$, $\alpha \in A$. This means that $H(z) = x$. On the other hand we have $z \in F$ and $H(z) \in H(F)$. A contradiction $H(z) = x \in Y \backslash H(F)$ and $H(z) \in H(F)$ completes the proof of the closedness of H. The proof of Theorem 2.13. is complete.

If the spaces X_α, $\alpha \in A$, are normal then $\lim \mathbf{X}$ is normal if $\mathbf{X}$ is an S–system (see Remark 2.10.). Moreover, $wX_a = \beta X_\alpha$ and $w(\lim \mathbf{X}) \approx \beta(\lim \mathbf{X})$ [5]. Thus, from Theorem 2.13. follows

2.14. Theorem. *Let $\mathbf{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of normal spaces X_α, $\alpha \in A$, with onto projections $f_\alpha : \lim \mathbf{X} \to X_\alpha$, $\alpha \in A$. Then $\beta(\lim \mathbf{X}) \approx \lim \beta\mathbf{X}$ iff $\mathbf{X}$ is an S–system.*

PROOF. Now, $f_{\alpha\beta}$ and f_α, $\alpha \in A$, are WC–mappings since $wX_\alpha = \beta X_\alpha$. Apply Theorem 2.3.

Applying the Examples 2.5. we obtain the following corollaries of Theorem 2.13.

2.15. Corollary. *Let $\mathbf{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of T_1 quasi-compact spaces X_α and closed onto mappings $f_{\alpha\beta}$. Then $\lim \mathbf{X}$ is T_1 and quasi-compact.*

PROOF. Now, $wX_\alpha = X_\alpha$ and $w\mathbf{X} = \mathbf{X}$. By Theorem 2.13. $w(\lim \mathbf{X}) \approx \lim w\mathbf{X} = \lim \mathbf{X}$. The proof is complete.

Let us recall that the proof of Corollary 2.15. is an alternative proof of Stone's theorem [21].

We say that a space X is a C–space if each countably compact subspace $Y \subseteq X$ is closed in X. It is readily seen that each first-countable regular space is a C–space. Moreover, if $f : X \to Y$ is a mapping of a countably compact X onto a C–space Y, then f is closed. From these facts and from Example 2.5.b) follows the

2.16. Corollary. *Let $\mathbf{X} = \{X_n, f_{nm}, N\}$ be an inverse sequence of countably compact spaces X_n and closed onto mappings f_{nm} or countably compact C–spaces (regular first-countable spaces) X_n and onto mappings f_{nm}. Then $w(\lim \mathbf{X}) \approx \lim w\mathbf{X}$.*

By virtue of Examples 2.5.d) – 2.5.f) and Theorem 2.13. we obtain

2.17. Corollary. *Let $\mathbf{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system from examples 2.5.d) – 2.5.f). If the mappings $f_{\alpha\beta}$ are closed and the projections $f_\alpha : \lim \mathbf{X} \to X_\alpha$ are onto mappings, then $\lim w\mathbf{X} \approx w(\lim \mathbf{X})$.*

2.18. *Remark.* If $\mathbf{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ is an inverse system of normal spaces X_α, $\alpha \in A$, then Corollaries 2.16. and 2.17. are corresponding theorems for the continuity of the Stone–Čech functor β.

We say that a mapping $f : X \to Y$ is fully closed [6] if for each $y \in Y$ and each open cover $\{U_1, \dots, U_n\}$ of a set $f^{-1}(y)$ the set $\{y\} \cup f^{\#}(U_1) \cup \dots \cup f^{\#}(U_n)$ is open.

2.19. Theorem. *Let $\mathbf{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system such that the $f_{\alpha\beta}$ are perfect fully closed. If the spaces X_α, $\alpha \in A$, are countably compact, then $w(\lim \mathbf{X}) \approx \lim w\mathbf{X}$.*

PROOF. The projections $f_\alpha : \lim \mathbf{X} \to X_\alpha$, $\alpha \in A$, are perfect fully closed [6]. If F, G are disjoint closed subsets of $\lim \mathbf{X}$, then $Y_\alpha = f_\alpha(F) \cap f_\alpha(G)$, $\alpha \in A$ is discrete. By countable compactness of X_α it follows that Y_α is finite. This means that $\mathbf{Y} = \{Y_\alpha, f_{\alpha\beta}/Y_\beta, A\}$ has a non-empty limit $Y \subseteq F \cap G$. Since this is impossible we infer that there is an $\alpha \in A$ such that $Y_\alpha = \emptyset$. This means that $\mathbf{X}$ is an S–system. Theorem 2.13. completes the proof.

The space $\lim \mathbf{X}$ in Theorem 2.19. is countably compact as shown by the following

2.20. Lemma. *Let $\mathbf{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse S–system with closed mappings $f_{\alpha\beta}$ and onto projections f_α. A space $X = \lim \mathbf{X}$ is countably compact if and only if the spaces X_α, $\alpha \in A$, are countably compact.*

PROOF. The "only if" part follows from the fact that a continuous image of a countably compact space is countably compact [5: Theorem 3.10.5].

The "if" part: Let F be a countably closed subset of X. Then $f_\alpha(F)$, $\alpha \in A$, is a countably closed subset of X_a since f_α, $\alpha \in A$, is closed (Lemma 2.7.). By the countable compactness of X_α $f_\alpha(F)$ is compact [5: Exercise 3.10.a)]. We have a system $\mathbf{Y} = \{f_a(F),\ f_{\alpha\beta}/f_\beta(F), A\}$ whose limit Y is compact [5]. Since $Y = F$ [5: Proposition 2.5.6.] we infer that F is compact. The proof is complete.

2.21. Theorem. *Let $\mathbf{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system in a W–category $\mathcal{C}$ (i.e. $\mathbf{X}$ is an object in pro-$\mathcal{C}$). Then there exists a continuous mapping $H : w(\lim \mathbf{X}) \to \lim w\mathbf{X}$. If $\mathbf{X}$ is an S–system, then H is 1–1.*

PROOF. A straightforward modification of the proof of Theorem 2.3.

2.22. Theorem. *Let $\mathbf{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system in a W–category $\mathcal{C}$ of quasi-compact T_1 spaces X_α. Then $\lim \mathbf{X}$ is a quasi-compact T_1 space.*

PROOF. Now we have $w\mathbf{X} = \{wX_\alpha, wf_{\alpha\beta}, A\} = \{X_\alpha, f_{\alpha\beta}, A\}$. By Theorem 2.21. we have a continuous mapping $H : w(\lim \mathbf{X}) \to \lim w\mathbf{X}$.

Since $w(\lim \mathbf{X})$ is T_1 and quasi-compact it follows that $\lim w\mathbf{X} = \lim \mathbf{X}$ is quasi-compact. The proof is complete.

A mapping $f : X \to Y$ is said to be a WO–mapping if for each finite open cover $\mathcal{U} = \{U_1, \dots, U_n\}$ of Y there exists a finite open cover $\mathcal{V} = \{V_1, \dots, V_m\}$ of X with the following property [9]:

(WO) If $A \subseteq X$ is closed and $A \subseteq V_j \in \mathcal{V}$, then there is $U_1 \in \mathcal{U}$ such that $\mathrm{Cl} f(A) \subseteq U_i$.

If $\mathcal{U}$ and $\mathcal{V}$ are as in the last definition, then we write $\mathcal{V} <_f \mathcal{U}$. The importance of WO–mappings lies in the following.

2.23. Theorem. [9: Theorem A.]. *Every* WO*–mapping has a unique* w*–extension, and this extension is also a* WO*–mapping.*

2.24. Theorem. *Let* $\mathbf{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ *be an inverse system of quasi-compact* T_1 *spaces* $X_\alpha, \alpha \in A$, *and* WO*–mappings* $f_{\alpha\beta}$ *such that the projections* $f_a : \lim \mathbf{X} \to X_\alpha$, $\alpha \in A$, *are onto* WO*–mappings. Then* $\lim \mathbf{X}$ *is a quasi-compact* T_1 *space.*

PROOF. Let us observe that from the assumption of Theorem it follows that $\mathbf{X}$ is an object in pro-$\mathcal{C}$, where $\mathcal{C}$ is the category of quasi-compact T_1 spaces and WO–mappings. Thus, from Theorem 2.21. it follows that there exists a continuous mapping $H : w(\lim \mathbf{X}) \to \lim w\mathbf{X}$. The proof is complete.

A filter $\mathcal{J}$ in the lattice of closed subsets of a T_1 space will be called indicative provided that $\cap\{C(A) : A \in \mathcal{J}\}$ is a singleton in wX, where $C(A)$ is the family of all ultrafilters in wX which contain A. A continuous mapping $f : X \to Y$ from a T_1 space X to a T_1 space Y will be called a WI–mapping provided that: i) f has a continuous Wallman extension,
ii) for every indicative filter $\mathcal{J}$ in the lattice of closed subsets of X, $\{B \subseteq Y :$ B is closed in Y and $f(A) \subseteq B$ for some $A \in \mathcal{J}\}$ is indicative [10].

The category of all T_1 spaces and all WI–mappings is larger than the category of all T_1 spaces and all WO–mappings [10].

2.25. Lemma. [10: Proposition 4.] *If* $f : X \to Y$ *is a* WI*–mapping, then the continuous Wallman extension* $wf : wX \to wY$ *is unique.*

2.26. Theorem. *Let* $\mathbf{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ *be an inverse system of* T_1 *quasi-compact spaces* X_α, $\alpha \in A$, *and* WI*–mappings* $f_{\alpha\beta}$ *such that the projections* $f_\alpha : \lim \mathbf{X} \to X_\alpha$, $\alpha \in A$, *are onto* WI*–mappings. Then* $\lim \mathbf{X}$ *is a quasi-compact* T_1 *space.*

PROOF. A straightforward modification of the proof of Theorem 2.24.

At the end of this Section we consider a WC–category i.e. the category of T_1 spaces and WC–mappings (not necessarily closed) (see Definition 2.11.).

2.27. Theorem. *Let* $\mathbf{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ *be an inverse* S*–system of* T_1 *spaces* X_α *and* WC*–mappings* $f_{\alpha\beta}$ *which is onto. Then* $w(\lim \mathbf{X}) \approx \lim w\mathbf{X}$ *iff the projections* $f_\alpha : \lim \mathbf{X} \to X_\alpha$, $\alpha \in A$, *are onto* W*–mappings.*

PROOF. The "if" part: If the projections f_α, $\alpha \in A$, are W–mappings, then there exist the mappings $wf_\alpha : w(\lim \mathbf{X}) \to \lim w\mathbf{X}$ which are onto. As in the proof of Theorem 2.13. we obtain a mapping $H : w(\lim \mathbf{X}) \to \lim w\mathbf{X}$ which is onto and 1–1 (see the proof of Theorem 2.13.). Similarly, as in the proof of Theorem 2.13. it follows that H is closed. Thus, H is a homeomorphism.

The "only if" part: If a homeomorphism $H : w(\lim \mathbf{X}) \to \lim w\mathbf{X}$ exists such that $H(x) = x$ for each $x \in \lim \mathbf{X}$, then the mappings $Hp_\alpha : w(\lim \mathbf{X}) \to wX_\alpha$, $\alpha \in A$, are extensions of the projections $f_\alpha : \lim \mathbf{X} \to X_\alpha$, $\alpha \in A$, onto $w(\lim \mathbf{X})$, where $p_\alpha : \lim w\mathbf{X} \to wX_\alpha$, $\alpha \in A$, are the projections. The proof is complete.

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