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Two remarks on Hosszú's functional inequality

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Hosszú's functional inequality

$$f(x+y-xy)+f(xy)\le f(x)+f(y) \tag{1}$$

was considered in [1] and the following results were proved:

(i) If $f:(0,1)\to R$ is a concave function, then f satisfies (1);

(ii) if $a\in[23/16,3/2)$ is fixed then the function f defined on $(0,1)$ by $f(x)=-x^4+2x^3-ax^2$ is a continuous solution of (1) and f is not concave.

Here, we shall give two generalizations of (i). We say $f:I\to R$ $(I=(0,1))$ is a Wright-convex function if for each $y>x$ and $\delta>0$ $(y+\delta,$ $x\in I)$,

$$f(x+\delta)-f(x)\le f(y+\delta)-f(y). \tag{2}$$

f is Wright-concave if the reverse inequality holds in (2). If C is a class of convex functions, and W a class of Wrigh-convex functions, then $C\subset W$, the inclusion being proper.

Using the substitutions: $x\to xy$, and $\delta\to x-xy$, we get:

Theorem 1. *If $f:(0,1)\to R$ is a Wright-concave function, then f satisfies (1).*

Note that the above proof os shorter than that from [1]. Now, we shall give a multidimensional generalization of Theorem 1.

Let R^k denote the k–dimensional vector lattice of points $X=(x_1,\dots,$ $x_k)$, x_i real for $i=1,\dots,k$, with the partial ordering

$$X=(x_1,\dots,x_k)\le Y=(y_1,\dots,y_k)$$

if and only if $x_i\le y_i$ for $i=1,\dots,k$. We shall write

$$aX+bY=(ax_1+by_1,\dots,ax_k+by_k),\ \text{where } a,b\in R,\ X,Y\in R^k,$$

and

$$XY = (x_1y_1, \dots, x_ky_k).$$

A real-valued function f on an interval $I = (0,1)^k$ will be said to have nondecreasing increments if

$$f(A+H) - f(A) \le f(B+H) - f(B) \tag{3}$$

whenever $A \in I,\ B + H \in I,\ 0 = (0, \dots, 0) \le H \in R^k,\ A \le B.$

For some properties of these functions see [2]. Of course, if the reverse inequality in (3) holds then f is a function with nonincreasing increments.

Using the substitutions: $A = XY,\ B = Y,\ H = X - XY$, we get

Theorem 2. *If $f : (0,1)^k \to R$ is a function with nonincreasing increments, then*

$$f(X+Y-XY) + f(XY) \le f(X) + f(Y). \tag{4}$$

In fact, using the Jensen inequality for these functions we have

$$f(X+Y-XY) + f(XY) \le f(X) + f(Y) \le 2f((X+Y)/2).$$

References

[1] GY. MAKSA and ZS. PÁLES, On Hosszú's functional inequality, *Publicationes Math.* **36** (1989), 187–189.

[2] H. D. BRUNK, Integral inequalities for functions with nondecreasing increments, *Pacific J. Math.* **14** (1964), 783–793.

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