Publ. Math. Debrecen
40 / 3–4 (1992), 261–271

Degree of approximation to functions in a normed space

By TIKAM SINGH (Ujjain)

1. Let $\{s_n\}$ be the sequence of partial sums of $\sum a_n$. Let $\Lambda = (\lambda_{nk})$ be a lower triangular infinite matrix, i.e. $\lambda_{nk} = 0$ for $k > n$ and let Λ–transform of the sequence $\{s_n\}$ be given by

$$T_n = \sum_{k=0}^{n} \lambda_{nk} s_k, \quad n = 0, 1, 2, \dots . \tag{1.1}$$

Let $C_{2\pi}$ be the class of all 2π–periodic continuous functions f on $[0, 2\pi]$ having Fourier series

$$f \sim a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \tag{1.2}$$

We define the space H_ω by

$$H_\omega = \{f \in C_{2\pi} : |f(x) - f(y)| \le K\omega(|x-y|)\} \tag{1.3}$$

and the norm $\|\,.\,\|_{\omega^*}$ by

$$\|f\|_{\omega^*} = \|f\|_C + \sup_{x,y}\{\Delta^{\omega^*} f(x,y)\}, \tag{1.4}$$

where

$$\|f\|_C = \sup_{0 \le x \le 2\pi} |f(x)|, \tag{1.5}$$

and

$$\Delta^{\omega^*} f(x,y) = \frac{|f(x) - f(y)|}{\omega^*(|x-y|)}, \quad x \ne y, \tag{1.6}$$

and $\Delta^0 f(x,y) = 0$, $\omega(t)$ and $\omega^*(t)$ being increasing functions of t. If $\omega(|x-y|) \le A|x-y|^\alpha$ and $\omega^*(|x-y|) \le K|x-y|^\beta$, $0 \le \beta < \alpha \le 1$, A

and K being positive constants, then the space

$$H_\alpha = \{f \in C_{2\pi} : |f(x) - f(y)| \le K|x-y|^\alpha,\ 0 < \alpha \le 1\} \tag{1.7}$$

is a Banach space ([4]) and the metric induced by the norm $\|\cdot\|_\alpha$ on H_α is said to be a Hölder metric. We write

$$\emptyset_x(t) = f(x+t) + f(x-t) - 2f(x). \tag{1.8}$$

Throughout the paper f will be taken to be periodic and $\omega(t)$ the modulus of continuity of $f \in C_{2\pi}$. $\{P_n\}\updownarrow$ and $\{p_n\}\updownarrow$ stands for non-decreasing and non-increasing sequences, respectively.

The following theorem is proved by CHANDRA ([2]) taking sup norm $\|\cdot\|$ on $0 \le x \le 2\pi$.

Theorem. *Let $\Lambda = (\lambda_{nk})$ satisfy the following conditions.*

$$\lambda_{nk} \ge 0 \qquad (n,k = 0,1,2,\dots),\ \sum_{k=0}^{n} \lambda_{nk} = 1, \tag{1.9}$$

and

$$\lambda_{nk} \le \lambda_{n,k+1} \qquad (k = 0,1,2,\dots,n-1;\ n = 0,1,2,\dots). \tag{1.10}$$

Let $\omega(t) \mid \int^{\pi} u^{-2}\omega(u)\,du = 0(H(t)),$

$$\int_t u^{-2}\omega(u)\,du = 0(H(t)), \quad H(t) \ge 0, \tag{1.11}$$

and

$$\int_0^t H(u)\,du = 0(tH(t)),\ \text{as } t \to 0+. \tag{1.12}$$

Then

$$\|T_n(f,x) - f\| = 0(\omega(\pi/n)) + 0(\lambda_{nn}H(\pi/n)). \tag{1.13}$$

If, in addition to (1.11), (1.12) *be satisfied, then*

$$\|T_n(f,x) - f\| = 0(\lambda_{nn}(H(\pi/n)). \tag{1.14}$$

2. The object of this paper is to widen the scope of the above theorem of CHANDRA under more general assumptions and to include a number of interesting results. For analogous conditions on the function, one may refer to MOHAPATRA and CHANDRA ([3]) and SINGH ([6]). Precisely, we prove

Theorem I. *Let $\Lambda = (\lambda_{nk})$ satisfy the conditions* (1.9) *and* (1.10). *Then for $f \in H_\omega$, $0 \le \beta < \eta \le 1$*

$$\|T_n(f,x) - f\|_{\omega^*} = 0[\{\omega(|x-y|)\}^{\beta/\eta}\{\omega^*(|x-y|)\}^{-1}\cdot \cdot (H(\pi/n))^{1-\beta/\eta}\lambda_{nn}\{n^{\beta/\eta} + \lambda_{nn}^{-\beta/\eta}\}] + 0(\lambda_{nn}H(\pi/n)), \tag{2.1}$$

if $\omega(t)$ satisfies (1.11) *and* (1.12), *and*

$$\|T_n(f,x) - f\|_{\omega^*} = 0[\{\omega(|x-y|)\}^{\beta/\eta}\{\omega^*(|x-y|)\}^{-1}\cdot \cdot \{(\omega(\pi/n))^{1-\beta/\eta} + \lambda_{nn}n^{\beta/\eta}(H(\pi/n))^{1-\beta/\eta}\}] + + 0(\omega(\pi/n)) + 0(\lambda_{nn}H(\pi/n)), \tag{2.2}$$

if $\omega(t)$ satisfies (1.11).

Theorem II. *Let $\Lambda = (\lambda_{nk})$ satisfy the conditions* (1.9) *and*

$$\lambda_{nk} \ge \lambda_{n,k+1} \ (k = 0,1,2,\dots,n-1;\ n = 0,1,2,\dots). \tag{2.3}$$

Let $\omega(t)$ be such that the conditions (1.11) *and* (1.12) *be satisfied, then for $f \in H_\omega$, $0 \le \beta < \eta \le 1$*

$$\|T_n(f,x) - f\|_{\omega^*} = 0[\{\omega(|x-y|)\}^{\beta/\eta}\{\omega^*(|x-y|)\}^{-1}\cdot \cdot \lambda_{n0}(H(\lambda_{n0}))^{1-\beta/\eta}\{n^{\beta/\eta} + \lambda_{n0}^{-\beta/\eta}\}] + 0(\lambda_{n0}H(\lambda_{n0})). \tag{2.4}$$

3. We shall require the following lemma in the proof of the theorems.

Lemma. *Let $\omega(t)$ satisfy* (1.11) *and* (1.12), *then*

$$\int_0^u t^{-1}\omega(t)\,dt = 0(uH(u)), \quad u \to 0+. \tag{3.1}$$

For the proof of the lemma see CHANDRA ([2]).

4. Proof of Theorem I. It is to be noted that

$$|\emptyset_x(t) - \emptyset_y(t)| \le 4K\omega(|t|), \tag{4.1}$$

and also

$$|\emptyset_x(t) - \emptyset_y(t)| \le 4A\omega(|x-y|). \tag{4.2}$$

We have

$$T_n(f,x) = \sum_{k=0}^{n} \lambda_{n,k}\ S_k(x).$$

Setting

$$E_n(x) = T_n(f,x) - f(x) = \frac{1}{2\pi}\int_0^{\pi} \frac{\emptyset_x(t)}{\sin(t/2)} \sum_{k=0}^{n} \lambda_{nk} \sin(k+1/2)t\ dt$$

and

$$\begin{aligned} E_n(x,y) &= E_n(x) - E_n(y) = \\ &= \frac{1}{2\pi}\int_0^{\pi} \frac{\emptyset_x(t) - \emptyset_y(t)}{\sin(t/2)} \sum_{k=0}^{n} \lambda_{nk} \sin(k+1/2)t\ dt, \end{aligned}$$

we get

$$(4.3) \qquad |E_n(x,y)| \le \int_0^{\pi/n} + \int_{\pi/n}^{\pi} = I_1 + I_2,$$

say, where

$$I_1 = \frac{1}{2\pi}\int_0^{\pi/n} \frac{|\emptyset_x(t) - \emptyset_y(t)|}{\sin(t/2)} \Big| \sum_{k=0}^{n} \lambda_{nk} \sin(k+1/2)t \Big|\, dt$$

and

$$I_2 = \frac{1}{2\pi}\int_{\pi/n}^{\pi} \frac{|\emptyset_x(t) - \emptyset_y(t)|}{\sin(t/2)} \Big| \sum_{k=0}^{n} \lambda_{nk} \sin(k+1/2)t \Big|\, dt.$$

Now using (1.9), (4.1) and the lemma and taking $\sin(k+1/2)t = 0(1)$, we get

$$(4.4) \qquad I_1 = 0(1)\int_0^{\pi/n} t^{-1}\omega(t)\ dt = 0\left(\frac{\pi}{n} H(\pi/n)\right) = 0(\lambda_{nn} H(\pi/n)),$$

since by (1.10), $\lambda_{nk} \le \lambda_{n,k+1}$ and, therefore, $(n+1)\lambda_{nn} \ge \sum_{k=0}^{n} \lambda_{nk} = 1$ gives $n^{-1} = 0(\lambda_{nn})$. Further, by (1.11) and (1.12) and integration by parts

$$\text{(4.5)}\qquad I_1 = 0(1)\int_0^{\pi/n} t^{-1}\omega(t)\sum_{k=0}^{n}\lambda_{nk}(k+1/2)t\,dt = 0(n)\int_0^{\pi/n}\omega(t)dt$$

$$= 0(n)\left[-t^2\int_t^{\pi} u^{-2}\omega(u)du\Big|_0^{\pi/n} + \int_0^{\pi/n} 2t\,dt\int_t^{\pi} u^{-2}\omega(u)du\right]$$

$$\text{(4.6)}\qquad = 0\left[n^{-1}H(\pi/n)\right] = 0[\lambda_{nn}H(\pi/n)].$$

But from (4.5) and (1.12)

$$\text{(4.7)}\qquad I_1 = 0(n)\,\omega(\pi/n)\cdot\frac{\pi}{n} = 0(\omega(\pi/n)).$$

By (1.10) and Abel's lemma, we see that

$$\text{(4.8)}\qquad \sum_{k=0}^{n}\lambda_{nk}\sin(k+1/2)t = 0(\lambda_{nn}t^{-1})$$

and, therefore, by (1.11)

$$\text{(4.9)}\qquad I_2 = 0(\lambda_{nn})\int_{\pi/n}^{\pi} t^{-2}\omega(t)\,dt = 0(\lambda_{nn}H(\pi/n)).$$

Again by (1.9) and (4.2)

$$\begin{aligned} I_1 &= 0[\omega(|x-y|)]\int_0^{\pi/n} t^{-1}\sum_{k=0}^{n}\lambda_{nk}|\sin(k+1/2)t|\,dt \\ \text{(4.10)}\qquad &= 0[\omega(|x-y|)]\int_0^{\pi/n}\sum_{k=0}^{n}\lambda_{nk}(k+1/2)\,dt \\ &= 0[\omega(|x-y|)], \end{aligned}$$

and by (4.2) and (4.8)

$$\text{(4.11)}\qquad I_2 = 0[\omega(|x-y|)]\int_{\pi/n}^{\pi} t^{-2}\lambda_{nn}\,dt = 0[\omega(|x-y|)\,n\,\lambda_{nn}].$$

Now noting that

$$\text{(4.12)}\qquad I_r = I_r^{1-\beta/\eta}\,I_r^{\beta/\eta},\quad r = 1, 2\ ,$$

we have, from (4.4) or (4.6) and (4.10)

$$
\begin{aligned}
I_1 &= [0\{\lambda_{nn} H(\pi/n)\}]^{1-\beta/\eta}\, [0\{\omega(|x-y|)\}^{\beta/\eta}] \\
&= 0[\{\omega(|x-y|)\}^{\beta/\eta}\, \{\lambda_{nn} H(\pi/n)\}^{1-\beta/\eta}],
\end{aligned}
\tag{4.13}
$$

and from (4.9) and (4.11)

$$
I_2 = 0[\{\omega(|x-y|)\}^{\beta/\eta}\, \lambda_{nn}\, n^{\beta/\eta}\{H(\pi/n)\}^{1-\beta/\eta}].
\tag{4.14}
$$

On the other hand, from (4.7) and (4.10)

$$
I_1 = 0[\{\omega(|x-y|)\}^{\beta/\eta}\, \{\omega(\pi/n)\}^{1-\beta/\eta}],
\tag{4.15}
$$

Thus from (4.13) and (4.14)

$$
\begin{aligned}
\sup_{x,y} |\Delta^{\omega^*} E_n(x,y)| &= \sup_{x,y} \frac{|E_n(x) - E_n(y)|}{\omega^*(|x-y|)} \\
&= 0[\{\omega^*(|x-y|)\}^{-1}\{\omega(|x-y|)\}^{\beta/\eta}\cdot \\
&\quad \cdot \left\{H\left(\frac{\pi}{n}\right)\right\}^{1-\beta/\eta} \lambda_{nn}\{\lambda_{nn}^{-\beta/\eta} + n^{\beta/\eta}\}],
\end{aligned}
\tag{4.16}
$$

and from (4.14) and (4.15)

$$
\begin{aligned}
&= 0[\{\omega^*(|x-y|)\}^{-1}\{\omega(|x-y|)\}^{\beta/\eta}\cdot \\
&\quad \{(\omega(\pi/n))^{1-\beta/\eta} + \lambda_{nn} n^{\beta/\eta}\cdot \\
&\quad \cdot (H(\pi/n))^{1-\beta/\eta}\}].
\end{aligned}
\tag{4.17}
$$

It is to be noted that

$$
\begin{aligned}
\|E_n(x)\|_C &= \max_{0 \le x \le 2\pi} |T_n(f,x) - f| \\
&= \begin{cases} 0(\lambda_{nn} H(\pi/n)), \text{ from (4.6) and (4.9)} \\ 0(\omega(\pi/n)) + 0(\lambda_{nn} H(\pi/n)), \text{ from (4.7) and (4.9).} \end{cases}
\end{aligned}
\tag{4.18}
$$

Combining (4.16) with the first part of (4.18) and (4.17) with the second part of (4.18), the required results are established.

5. Proof of Theorem II. From the proof of Theorem I, we have

$$
|E_n(x,y)| \le \int_0^{\lambda_{n0}} + \int_{\lambda_{n0}}^{\pi} = J_1 + J_2.
\tag{5.1}
$$

By the lemma

$$(5.2)\qquad J_1 \leq \int_0^{\lambda_{n0}} t^{-1}\omega(t) \sum_{k=0}^{n} \lambda_{nk} \, |\sin(k+1/2)t| \, dt = 0(\lambda_{n0} H(\lambda_{n0})),$$

and by (1.9), Abel's lemma and (2.3), we have

$$(5.3)\qquad J_2 = 0(\lambda_{n0}) \int_{\lambda_{n0}}^{\pi} t^{-2}\omega(t)\, dt = 0(\lambda_{n0} H(\lambda_{n0})).$$

Again, using (4.2) and (1.9), we get

$$(5.4)\qquad J_1 = 0(n\ \omega(|x-y|) \sum_{k=0}^{n} \lambda_{nk} \int_0^{\lambda_{n0}} dt = 0(n\ \omega(|x-y|)\lambda_{n0}),$$

and

$$(5.5)\qquad J_2 = 0(\omega(|x-y|) \int_{\lambda_{n0}}^{\pi} \lambda_{n0}\ t^{-2} dt = 0(\omega(|x-y|).$$

Proceeding as in the proof of theorem I, we have

$$(5.6a)\qquad J_1 = 0\left[\lambda_{n0}\{n\omega(|x-y|\}^{\beta/\eta}\{H(\lambda_{n0})\}^{1-\beta/\eta}\right],$$

and

$$(5.6b)\qquad J_2 = 0\left[\{(\omega(|x-y|\}^{\beta/\eta}\{H(\lambda_{n0})\lambda_{n0}\}^{1-\beta/\eta}\right].$$

Thus

$$(5.7)\qquad \begin{aligned} \operatorname*{Sup}_{x,y} |\Delta^{\omega^*} E_n(x,y)| &= 0\Big[\{\omega(|x-y|)\}^{\beta/\eta}\{\omega^*(|x-y|)\}^{-1}\cdot \\ &\cdot \lambda_{n0}\{H(\lambda_{n0})\}^{1-\beta/\eta}\{n^{\beta/\eta} + \lambda_{n0}^{-\beta/\eta}\}\Big]. \end{aligned}$$

It is obvious from (5.2) and (5.3) that

$$(5.8)\qquad \|E_n(x,y)\|_C = 0(\lambda_{n0} H(\lambda_{n0})).$$

Combine (5.7) and (5.8) for the proof of the Theorem II.

6. Corollaries. The following are the corollaries based on Theorem I.

Put $\omega^*(|x-y|) \le K|x-y|^\beta$, $\omega(|x-y|) \le A|x-y|^\alpha$, $H(t) = \log(\pi/t)$ for $\alpha = 1$ and $H(t) = t^{\alpha-1}$ for $0 < \alpha < 1$ and replace η by α to get

Corollary 1. *Let* $\Lambda = (\lambda_{nk})$ *satisfy* (1.9) *and* (1.10) *and let* $f \in H_\alpha$, $0 \le \beta < \alpha \le 1$, *then*

$$\|T_n(f,x) - f\|_\beta = \begin{cases} 0(\lambda_{nn}\log n) + 0\left[n^\beta\lambda_{nn}(\log n)^{1-\beta} + (\lambda_{nn}\log n)^{1-\beta}\right], \\ \qquad\qquad \alpha = 1 \\ 0\left[n^{1-\alpha+\beta}\lambda_{nn}\right] + 0\left[(\lambda_{nn}n^{1-\alpha})^{1-\beta/\alpha}\right], \ 0 < \alpha < 1, \end{cases}$$

and

$$\|T_n(f,x) - f\|_\beta = \begin{cases} 0(n^{\beta-1}) + 0(\lambda_{nn}\ n^\beta(\log n)^{1-\beta}), & \alpha = 1, \\ 0(n^{\beta-\alpha}) + 0(\lambda_{nn}\ n^{1-\alpha+\beta}), & 0 < \alpha < 1. \end{cases}$$

The later result is the theorem 2 due to MOHAPATRA and CHANDRA ([3]).

If we put $\beta = 0$, so that $\omega^*(|x-y|) \le K$ and the norm $\|\,.\,\|$ stands for the sup norm on $C_{2\pi}$ in Theorem I, then it reduces to the CHANDRA's Theorem (loc. cit.).

If $\beta = 0$ and $f \in \operatorname{Lip}\alpha$, $0 < \alpha \le 1$, so that $\omega(t) = 0(t^\alpha)$ and

$$H(t) = \begin{cases} \log(\pi/t), & \alpha = 1, \\ t^{\alpha-1}, & 0 < \alpha < 1, \end{cases} \tag{6.1}$$

we have from Corollary 1

Corollary 2. *Let* $\Lambda = (\lambda_{nk})$ *satisfy* (1.9), (1.10) *and* $f \in \operatorname{Lip}\alpha$, $0 < \alpha \le 1$, *then*

$$\|T_n(f,x) - f\| = \begin{cases} 0(\lambda_{nn}\ \log n), & \alpha = 1, \\ 0(n^{1-\alpha}\lambda_{nn}), & 0 < \alpha < 1, \end{cases}$$

and

$$\|T_n(f,x) - f\| = \begin{cases} 0(n^{-1}) + 0(\lambda_{nn}\ \log n), & \alpha = 1, \\ 0(n^{-\alpha}) + (n^{1-\alpha}\lambda_{nn}), & 0 < \alpha < 1. \end{cases}$$

If we put $\omega^*(|x-y|) \le K|x-y|^\beta$, $\omega(|x-y|) \le A|x-y|^\alpha$ and replace η by α in Theorem II, we have by (6.1).

Corollary 3. *Let* $\Lambda = (\lambda_{nk})$ *satisfy* (1.9) *and* (2.3) *and let* $f \in H_\alpha$, $0 \le \beta < \alpha \le 1$, *then*

$$\|T_n(f,x) - f\|_\beta = \begin{cases} 0\left[\left(\lambda_{n0}\log\frac{\pi}{\lambda_{n0}}\right)\right] + 0\left[n^\beta\lambda_{n0}\left(\log\frac{\pi}{\lambda_{n0}}\right)^{1-\beta}\right] + \\ \qquad +0\left[\left(\lambda_{n0}\ \log\frac{\pi}{\lambda_{n0}}\right)^{1-\beta}\right], & \alpha = 1, \\ 0(\lambda_{n0}^\alpha) + 0(n^{\beta/\alpha}\lambda_{n0}^{\alpha-\beta+\beta/\alpha}) + 0(\lambda_{n0}^{\alpha-\beta}), & 0 < \alpha < 1. \end{cases}$$

If we put $\beta = 0$ in the above corollary, then

Corollary 4. *Let* $\Lambda = (\lambda_{nk})$ *satisfy* (1.9) *and* (2.3) *and* $f \in \operatorname{Lip}\alpha$, $0 < \alpha \le 1$, *then*

$$\|T_n(f,x) - f\| = \begin{cases} 0\left(\lambda_{n0}\log\frac{\pi}{\lambda_{n0}}\right), & \alpha = 1, \\ 0(\lambda_{n0}^\alpha), & 0 < \alpha < 1. \end{cases}$$

Now we specialize the matrix $\Lambda = (\lambda_{nk})$. If we put $\lambda_{nk} = \frac{p_k}{P_n}$ and $\lambda_{nk} = \frac{p_{n-k}}{P_n}$ in the transform $T_n(f,x)$, we have, respectively

$$\overline{N}_n(f,x) = \frac{1}{P_n}\sum_{k=0}^{n} p_k S_k(x),$$

and

$$N_n(f,x) = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k} S_k(x),$$

transforms, where

$$P_n = \sum_{k=0}^{n} p_k, \quad p_n > 0.$$

If we put $\omega^*(|x-y|) \le K|x-y|^\beta$, $\omega(|x-y|) \le A|x-y|^\alpha$, and replace η by α in Theorems I and II, and if we put $\lambda_{nn} = p_n/P_n$, $\{p_n\}\updownarrow$ in Theorem I and $\lambda_{n0} = p_n/P_n$, $\{P_n\}\updownarrow$, in Theorem II, then the following corollaries are obtained, respectively, keeping in view (6.1)

Corollary 6. *For* $f \in H_\alpha$, $0 \le \beta < \alpha \le 1$, $\{p_n\}\updownarrow$, *we have*

$$\|\overline{N}_n(f,x) - f\|_\beta = \begin{cases} 0\left(\frac{p_n}{P_n}n^{1-\alpha+\beta}\right) + 0\left(n^{1-\alpha}\frac{p_n}{P_n}\right)^{1-\beta/\alpha}, & 0 < \alpha < 1, \\ 0\left(n^\beta\frac{p_n}{P_n}(\log n)^{1-\beta}\right), & \alpha = 1. \end{cases}$$

and

$$\|\overline{N}_n(f,x)-f\|_\beta = \begin{cases} 0(n^{\beta-\alpha}) + 0\left(\frac{p_n}{P_n} n^{1-\alpha+\beta}\right), & 0<\alpha<1, \\ 0(n^{\beta-1}) + 0\left(\frac{p_n}{P_n} n^\beta \log^{1-\beta} n\right), & \alpha = 1. \end{cases}$$

Corollary 7. *For* $f \in H_\alpha,\ 0 \le \beta < \alpha \le 1,\ \{p_n\}\updownarrow$, *we have*

$$\|N_n(f,x)-f\|_\beta = \begin{cases} 0\left[n^\beta \frac{p_n}{P_n}\left(\log \frac{P_n}{p_n}\right)^{1-\beta}\right] + 0\left[\left(\frac{p_n}{P_n}\log\frac{P_n}{p_n}\right)^{1-\beta}\right], \\ \qquad \alpha = 1 \\ 0\left(\left(\frac{p_n}{P_n}\right)^{\alpha-\beta}\right) + 0\left[n^{\beta/\alpha}\left(\frac{p_n}{P_n}\right)^{\alpha-\beta+\beta/\alpha}\right], \\ \qquad 0<\alpha<1. \end{cases}$$

Putting $\beta = 0$ in the Corollary 6, we have special results for $\overline{N}_n(f,x)$ transformations and from Corollary 7, we have the following theorem due to SINGH ([5]).

Corollary 8. *If* $f \in \operatorname{Lip}\alpha,\ 0<\alpha\le 1$ *and* $\{p_n\}\updownarrow$, *then*

$$\|N_n(f,x)-f\| = \begin{cases} 0\left[\frac{p_n}{P_n}\log\frac{P_n}{p_n}\right), & \alpha = 1, \\ 0\left[\left(\frac{p_n}{P_n}\right)^\alpha\right], & 0<\alpha<1. \end{cases}$$

If we put $\omega^*(|x-y|) \le K|x-y|^\beta,\ \omega(|x-y|) \le A|x-y|^\alpha,\ \lambda_{nn} = 1/n$ and replace η by α in the second part of Theorem I, we have

Corollary 9. *For* $f \in H_\alpha,\ 0 \le \beta < \alpha \le 1$

$$\|\sigma_n(f,x)-f\|_\beta = \begin{cases} 0(n^{\beta-\alpha}), & 0<\alpha<1, \\ 0[n^{\beta-1}(1+\log n)^{1-\beta}], & \alpha = 1. \end{cases}$$

This is result of PRÖSSDORF ([4]), where $\sigma_n(f,x)$ is the Fejér operator. Put $\beta = 0$ in the above corollary to get one of the results of ALEXITS ([1]).

References

[1] G. ALEXITS, Convergence Problems of Orthogonal Series, *Pergamon Press*, 1961.

[2] P. CHANDRA, On the degree of approximation of a class of functions by means of Fourier series, *Acta Math. Hung.* **52** (1988), 199–205.

[3] R. N. MOHAPATRA and P. CHANDRA, Degree of approximation of functions in the Hölder metric, *Acta Math. Hung.* **41** (1983), 67–76.

[4] S. PRÖSSDORF, Zur Konvergenz der Fourierreihen Hölderstetiger Funktionen, *Math. Nachr.* **69** (1975), 7–14.

[5] T. SINGH, Approximation to a function by Nörlund operators, *Indian J. Math.* **22** (1980), 173–176.

[6] T. SINGH, Degree of approximation to functions in the Banach space, *Math. Student.* ((*to appear*)

TIKAM SINGH
DEPARTMENT OF MATHEMATICS
GOVERNMENT ENGINEERING COLLEGE
UJJAIN–456010, INDIA

(Received June 21, 1990)