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Characterization of additive functions with values in a compact Abelian group

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1. Let G be an additively written, metrically compact Abelian topological group, the one-dimensional torus. Let $\mathbf{N}$ denote the set of all positive integers. A function $\phi : \mathbf{N} \to G$ is called completely additive, if

$$\phi(nm) = \phi(n) + \phi(m)$$

holds for each couple $n, m \in \mathbf{N}$. Let A_G^* be the class of completely additive functions.

Let $A > 0, B$ be fixed integers. We shall say that an infinite sequence $\{x_\nu\}_{\nu=1}^\infty$ in G is of property $D[A, B]$ if for any convergent subsequence $\{x_{\nu_n}\}_{n=1}^\infty$ the sequence $\{x_{A\nu_n+B}\}_{n=1}^\infty$ has a limit, too. We say that it is of property $E[A, B]$ if for any convergent subsequence $\{x_{A\nu_n+B}\}_{n=1}^\infty$ the sequence $\{x_{\nu_n}\}_{n=1}^\infty$ is convergent. We shall say that an infinite sequence $\{x_\nu\}_{\nu=1}^\infty$ in G is of property $\Delta[A, B]$ if $\{x_{A\nu+B} - x_\nu\}_{\nu=1}^\infty$ is convergent.

Let $A_G^*(D[A, B])$, $A_G^*(E[A, B])$ and $A_G^*(\Delta[A, B])$ be the classes of those $\phi \in A_G^*$ for which $\{x_\nu = \phi(\nu)\}_{\nu=1}^\infty$ is of property $D[A, B]$, $E[A, B]$ and $\Delta[A, B]$, respectively.

It is obvious that

$$A_G^*(\Delta[A, B]) \subseteq A_G^*(D[A, B])$$

and

$$A_G^*(\Delta[A, B]) \subseteq A_G^*(E[A, B]).$$

Z. Daróczy and I. Kátai [1] proved in the case $A = B = 1$ that

$$A_G^*(\Delta[1, 1]) = A_G^*(D[1, 1]).$$

By using an unpublished result due to E. Wirsing [7] which asserts that $\phi \in A_T^*(\Delta[1, 1])$ if and only if

$$\phi(n) \equiv \tau \log n \ (\mathrm{mod} 1) \qquad (\forall n \in \mathbf{N})$$

for some real number τ, Z. DARÓCZY and I. KÁTAI [2] deduced the following assertion: If $\phi \in A_G^*(\Delta[1,1]) = A_G^*(D[1,1])$, then there exists a continuous homomorphism $\psi : \mathbf{R}_x \to G$, where $\mathbf{R}_x$ denotes the multiplicative group of the positive reals, such that ϕ is a restriction of ψ on the set $\mathbf{N}$, i.e.

$$\phi(n) = \psi(n) \qquad (\forall n \in \mathbf{N}).$$

For the case $A = 2$ and $B = -1$ the complete characterization of $A_G^*(D[2,-1])$ and $A_G^*(\Delta[2,-1])$ has been given by Z. DARÓCZY and I. KÁTAI [3], [4]. The basic idea of their proof is to reduce the condition $\phi \in A_G^*(D[2,-1])$ to the relation

$$\phi(2n+1) - \phi(2n-1) \to 0 \quad \text{as} \quad n \to \infty$$

and apply a modification of Wirsing's theorem.

Our main purpose in this paper is to give a complete determination of $A_G^*(E[A,B])$ and of $A_G^*(\Delta[A,B])$. We shall prove the following

Theorem 1. *For any fixed integers $A > 0$ and $B \neq 0$, we have $A_G^*(E[A < B]) = A_G^*(\Delta[A,B])$.*

Theorem 2. *Let $A > 0,\ B \neq 0$ be fixed integers. If*

$$\phi \in A_G^*(E[A,B]) = A_G^*(\Delta[A,B])$$

then there exists a continuous homomorphism $\psi : \mathbf{R}_x \to G$, where $\mathbf{R}_x$ denotes the multiplicative group of the positive reals, such that ϕ is a restriction of ψ on the set $\mathbf{N}$, i.e.

$$\phi(n) = \psi(n)$$

for all $n \in \mathbf{N}$.

Conversely, let $\psi : \mathbf{R}_x \to G$ be an arbitrary continuous homomorphism. Then the function

$$\phi(n) := \psi(n) \qquad (n = 1, 2, \dots)$$

belongs to $A_G^*(E[A,B]) = A_G^*(\Delta[A,B])$.

2. PROOF OF THEOREM 1.

Assume that $A > 0$ and $B \neq 0$ are integers.

Let $\phi \in A_G^*(E[A,B])$. Let X denote the set of limit points of $\{\phi(n) \mid n \in \mathbf{N}\}$, i.e. $g \in X$ if there exists a sequence

$$n_1 < \dots < n_\nu < \dots \qquad (n_\nu \in \mathbf{N}),$$

for which $\phi(n_\nu) \to g$. Let X_1 ($\subseteq X$) be the set of limit points of $\{\phi(An+1) \mid n \in \mathbf{N}\}$. Since $\mathbf{N}$ and the natural numbers $m \equiv 1(\mathrm{mod} A)$ form semigroups, therefore X and X_1 are semigroups as well. Thus, X and X_1 are closed semigroups in G, so by a known theorem (see [6], Theorem (9.16)) they are compact groups. Since $0 \in X_1 \subseteq X$, we have

$$\phi(n) \in X \text{ and } \phi(An+1) \in X_1 \text{ for each } n \in \mathbf{N}. \tag{2.1}$$

Let X_B denote the set of limit points of $\{\phi(An+B) \mid n \in \mathbf{N}\}$. If $g \in X_B$, then there is a sequence $\{n_\nu\}_{\nu=1}^{\infty}$ such that $\phi(An_\nu+B) \to g$. Since $\phi \in A_G^*(E[A,B])$, the sequence $\{\phi(n_\nu)\}_{\nu=1}^{\infty}$ is convergent. Let $\phi(n_\nu) \to g'$. It is obvious that g' is determined by g, and so the correspondence $F : g \to g'$ is a function, furthermore $F(X_B) = X$. For the proof of these simple assertions see [1].

Lemma 1. *We have*

$$F(g) = g - \phi(A)$$

for every $g \in X_B$.

PROOF. Since X_1 is a subgroup in G, we have $0 \in X_1$, and so there exists a sequence

$$N_1 < \ldots < N_\nu < \ldots \qquad (N_\nu \in \mathbf{N})$$

for which $\phi(AN_\nu + 1) \to 0$. Since G is sequentially compact, therefore $\{\phi(N_\nu)\}_{\nu=1}^{\infty}$ contains at least one limit point. Let

$$\phi(N_{\nu_k}) \to \tau \qquad (\tau \in X). \tag{2.2}$$

Let $g \in X_B$ be an arbitrary element. By using (2.1) we have $\phi(A) \in X$. Since $g \in X_B \subseteq X$ and X is a group, we have $g - \phi(A) \in X$. Thus, it follows from $F(X_B) = X$ that there exists an element $h \in X_B$, for which

$$F(h) = g - \phi(A). \tag{2.3}$$

From the definition of X_B it is clear that there exists a sequence

$$M_1 < \ldots < M_\nu < \ldots \qquad (M_\nu \in \mathbf{N})$$

for which $\phi(AM_\nu + B) \to h$.

Let us consider the sequence

$$\{\phi(A^2 M_{\nu_k} N_{\nu_k} + B)\}_{k=1}^{\infty},$$

where ν_k is determined in (2.2). Since G is sequentially compact, therefore the above sequence contains at least one limit point. Let

$$\phi(A^2 M_{\nu_{k_j}} N_{\nu_{k_j}} + B) \to h' \qquad (\in X_B). \tag{2.4}$$

From the definition of F it follows by (2.2) and (2.3) that

$$F(h') = g + \tau. \tag{2.5}$$

Applying the following relation

$$(A^2mn + B)(An + 1) = An[Am(An + 1) + B] + B$$

with $m = M_{\nu_{k_j}}, n = N_{\nu_{k_j}}$ and using the definition of F and (2.4), we have

$$\phi\{AN_{\nu_{k_j}}[AM_{\nu_{k_j}}(AN_{\nu_{k_j}} + 1) + B] + B\} \to h',$$
$$\phi[AM_{\nu_{k_j}}(AN_{\nu_{k_j}} + 1) + B] \to F(h') - \tau,$$

and so

$$F(h) = F[F(h') - \tau]. \tag{2.6}$$

Finally, from (2.3), (2.5) and (2.6) we get that

$$F(g) = g - \phi(A).$$

So we have proved Lemma 1.

We now prove Theorem 1.

Let $\phi \in A_G^*(E[A, B])$. Let S denote the set of limit points of $\{\phi(An + B) - \phi(n) - \phi(A) \mid n \in \mathbf{N}\}$. It is obvious that $S \neq \emptyset$. We shall prove that $S = \{0\}$.

Let $\delta \in S$. Then there exists a sequence $\{n_\nu\}_{\nu=1}^{\infty}$ for which

$$\phi(An_\nu + B) - \phi(n_\nu) - \phi(A) \to \delta. \tag{2.7}$$

Since G is sequentially compact, therefore we can choose a suitable convergent subsequence of $\phi(An_\nu + B)$. Let

$$\phi(An_{\nu i} + B) \to g \qquad (\in X_B). \tag{2.8}$$

From (2.7) and (2.8) we have

$$g - F(g) - \phi(A) = \delta,$$

which, using Lemma 1, implies that $\delta = 0$. Thus, we have proved that $S = \{0\}$, and so

$$\phi(An + B) - \phi(n) - \phi(A) \to 0 \text{ as } n \to \infty.$$

This shows that $\phi \in A_G^*(\Delta[A, B])$, consequently

$$A_G^*(E[A, B]) = A_G^*(\Delta[A, B]).$$

The proof of Theorem 1 is finished.

3. Proof of Theorem 2. Let $A > 0,\ B > 0$ be fixed integers.

Assume that $\phi \in A^*_G(\Delta[A,B])$, i.e. there is an element $E \in G$ such that

$$\phi(An+B) - \phi(n) - E \to 0 \text{ as } n \to \infty. \tag{3.1}$$

Let $\chi : G \to T$ be any continuous character, where T denotes the unit circle, i.e. the set of all complex–numbers of modulus 1. Let

$$V(n) := \chi(\phi(n)) \quad \text{for all} \quad n \in \mathbf{N}$$

and

$$C := \chi(E) \qquad (\in T).$$

Then, by (3.1), we have

$$V(An+B)(CV(n))^{-1} = \chi[\phi(An+B) - \phi(n) - E] \to \chi(0) = 1. \tag{3.2}$$

In [5] (Theorem 3) we have proved that if $V : \mathbf{N} \to T$ is a completely multiplicative function and it satisfies the relation

$$V(An+B)(CV(n))^{-1} \to 1$$

for some positive integers A, B and a non-zero complex-number C, then there exists a real-number τ such that $V(n) = n^{i\tau}$ for all $n \in \mathbf{N}$.

From (3.2) and by using this result we get immediately

$$V(n) = \chi(\phi(n)) = n^{i\tau}$$

for some real τ. Thus, by using an argument based on the proof of Theorem 1 of Z. Daróczy and I. Kátai [2], we deduce immediately that there exists a continuous homomorphism $\Psi : \mathbf{R}_x \to G$, such that $\Psi(n) = \phi(n)$ for all $n \in \mathbf{N}$.

Assume now that $A > 0$ and $B < 0$. In this case our Theorem 2 also holds, since it is easily seen that

$$A^*_G(\Delta[A,B]) \subseteq A^*_G(\Delta[A,-1])$$

and

$$A^*_G(\Delta[A,-1]) \subseteq A^*_G(\Delta[A,1]).$$

So we have proved the first assertion of Theorem 2. The proof of the converse assertion is obvious. Thus completes the proof of Theorem 2.

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