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On compositions of distributions

By KOU HUAIZHONG (Xixing City) and BRIAN FISHER (Leicester)

Abstract. Using a double neutrix limit, the paper investigates the compositions of one-dimensional distributions and obtains a universal definition of the compositions of distributions.

1. Introduction

Since 1950, much investigation has been carried out on the compositions of distributions, see [3], [6] and [7], but only in the case where the functions involved have been continuous or at most locally summable. No meaning could be given to expressions of the form $H(\delta^{(s)}(x))$, $\delta^{(r)}(\delta^{(s)}(x))$, $[\delta^{(r)}(x)]^s$ and the like. The purpose of this paper is to achieve a new universal definition of the compositions of distributions, where a double neutrix limit is used. Accordingly, the questions not resolved are worked out.

2. Preliminaries

2.1. Neutrix and Neutrix Limit

The following definition of a neutrix was given by van der Corput [1]:

Definition 2.1. Let N' be a set and let N be a commutative, additive group of functions mapping N' into a commutative, additive group N''. If N has the property that the only constant function in N is the zero function, then N is said to be a neutrix and the functions in N are said to be negligible.

Now suppose that N' is a subspace of a topological space X having a limit point y which is not contained in N'. Let N'' be the real (or complex) numbers and let N be a commutative, additive group of functions mapping N' into N'' with the property that if N contains a function $\nu(x)$ which converges to a finite limit c as x tends to y, then $c = 0$. N is a neutrix, since if f is in N and $f(x) = c$ for all x in N', then $f(x)$ converges to the finite limit c as x tends to y and so $c = 0$.

This leads us to a second definition of van der Corput [1].

Definition 2.2. Let $f(x)$ be a real (or complex) valued function defined on N' and suppose it is possible to find a constant c such that $f(x) - c$ is negligible in N. Then c is called the neutrix limit of $f(x)$ as x tends to y and we write

$$N\!-\lim_{x\to y} f(x) = c.$$

Note that if a neutrix limit exists, then it is unique, since if $f(x) - c$ and $f(x) - c'$ are in N, then the constant function $c - c'$ is also in N and so $c = c'$.

In the following we let N the neutrix having domain $N' = \{1, 2, \dots, n, \dots\}$, range the real numbers and $y = \infty$, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n: \quad \lambda > 0, \quad r = 1, 2 \dots$$

and all functions which converge to zero in the normal sense as n tends to infinity.

We will use n or m to denote a general term in N' so that if $\{a_n\}$ is a sequence of real numbers, then $N\!-\lim\limits_{n\to\infty} a_n$ means exactly the same thing as $N\!-\lim\limits_{m\to\infty} a_m$.

Note that if $\{a_n\}$ is a sequence of real numbers which converges to a in the normal sense as n tends to infinity, then the sequence $\{a_n\}$ converges to a in the neutrix sense as n tends to infinity and

$$\lim_{n\to\infty} a_n = N\!-\lim_{n\to\infty} a_n.$$

2.2. Convolution and regularity

We now let $\rho(x)$ be any infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,

(ii) $\rho(x) \geq 0$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\int\limits_{-1}^{1} \rho(x)dx = 1.$

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2 \ldots,$ it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Then if f is an arbitrary distribution in $\mathcal{D}$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x-t)\rangle$$

for $n = 1, 2, \ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differetiable functions converging to the distribution $f(x)$.

3. Compositions

Since 1983, the second author in [3], [4] and [5] has investigated the composition of distributions using neutrix limits. The following definition was given in [5] and is the most general.

Definition 3.1. Let F be a distributiuon in $\mathcal{D}'$ and let f be a locally summable function. We say that the distribution $F(f(x))$ exists and is equal to $h(x)$ on the interval (a, b) if

$$\operatorname*{N-lim}_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\phi(x)\, dx = \langle h(x), \phi(x)\rangle$$

for all test functions ϕ in $\mathcal{D}$ with support contained in the interval (a, b), where

$$F_n(x) = (F * \delta_n)(x)$$

for $n = 1, 2, \ldots$ and N is the neutrix given in the previous section.

We now give an alternative definition for the distribution $F(f)$.

Definition 3.2. Let F and f be distributions in $\mathcal{D}'$. We say that the distribution $F(f(x))$ exists and is equal to $h(x)$ on the interval (a, b) if

$$\operatorname*{N-lim}_{n\to\infty}\Big[\operatorname*{N-lim}_{m\to\infty} \int_{-\infty}^{\infty} F_n(f_m(x))\phi(x)\, dx\Big] = \langle h(x), \phi(x)\rangle$$

for all ϕ in $\mathcal{D}$ with support contained in the interval (a,b), where

$$F_n(x) = (F * \delta_n)(x), \quad f_m(x) = (f * \delta_m)(x)$$

for $m, n = 1, 2, \dots$.

In the following theorem, we show that Definition 3.2 generalizes Definition 3.1 for bounded, locally summable functions f.

Theorem 3.1. *Let F be a distribution in $\mathcal{D}$ and let f be a bounded, locally summable function. If the distribution $F(f(x))$ exists and equals $h(x)$ under Definition 3.1 on the interval (a,b), then $F(f(x))$ also exists under Definition 3.2 on the interval (a,b) and is equal to $h(x)$.*

Proof. Suppose that $F(f(x))$ exists and equals $h(x)$ under Definition 3.1 on the interval (a,b). Then

$$N\text{-}\lim_{n\to\infty}\langle F_n(f(x)), \phi(x)\rangle = \langle h(x), \phi(x)\rangle$$

for all ϕ in $\mathcal{D}(a,b)$. Now

$$\lim_{m\to\infty}\int_I |f_m(x) - f(x)|\,dx = 0,$$

for any bounded interval I, since f is a bounded, summable function. Further, since F_n is a continuously differentiable function and f, f_n are bounded, it follows that

$$|F_n(f_m(x)) - F_n(f(x))| \le K_n|f_m(x) - f(x)|,$$

for some K_n. We therefore have

$$\left|\int_a^b [F_n(f_m(x)) - F_n(f(x))]\phi(x)\,dx\right| \le MK_n\int_c^d |f_m(x) - f(x)|\,dx,$$

where

$$M = \sup\{|\phi(x)|\}$$

and $[c,d]$ is a bounded interval containing the support of ϕ and so

$$\lim_{m\to\infty}\int_a^b F_n(f_m(x))\phi(x)\,dx = \int_a^b F_n(f(x))\,dx,$$

or equivalently

$$\underset{m\to\infty}{N\text{-}\lim}\langle F_n(f_m(x)),\phi(x)\rangle = \langle F_n(f(x)),\phi(x)\rangle.$$

Thus

$$\begin{aligned}\underset{n\to\infty}{N\text{-}\lim}\Big[\underset{m\to\infty}{N\text{-}\lim}\langle F_n(f_m(x)),\phi(x)\rangle\Big] &= \underset{n\to\infty}{N\text{-}\lim}\langle F_n(f(x)),\phi(x)\rangle \\ &= \langle F(f(x)),\phi(x)\rangle \\ &= \langle h(x),\phi(x)\rangle,\end{aligned}$$

and it follows that $F(f(x))$ exists and equals $h(x)$ by Definition 3.2.

It is an open question as to whether Definition 3.2 is a generalization of Definition 3.1 for all locally summable functions.

From now on, the compositions that we will consider will be using Definition 3.2.

Theorem 3.2. *Let F be a bounded, continuous, summable function on the real line. Then the distribution $F(\delta^{(s)}(x))$ exists on the real line and*

$$F(\delta^{(s)}(x)) = F(0),$$

for $s = 0, 1, 2, \dots$.

Proof. We put

$$F_n(x) = (F * \delta_n)(x), \quad \delta_m^{(s)}(x) = (\delta^{(s)} * \delta_m)(x)$$

for $m, n = 1, 2, \dots$.

Choosing an arbitrary $\varepsilon > 0$, there exists an M such that $m\varepsilon > 1$ for $m > M$. Then with $m > M$, we have

$$\delta_m^{(s)}(x) = m^{s+1}\rho(mx) = 0$$

for $|x| > \varepsilon$ and so

$$F_n(\delta_m^{(s)}(x)) = F_n(0)$$

for $|x| > \varepsilon$. Thus, for arbitrary ϕ in $\mathcal{D}$ with support contained in the interval (a, b), which we may suppose contains the origin,

$$\begin{aligned}\langle F_n(\delta_m^{(s)}(x),\phi(x)\rangle &= \int_a^b F_n(\delta_m^{(s)}(x))\phi(x)\,dx \\ &= F_n(0)\int_a^{-\varepsilon}\phi(x)\,dx + F_n(0)\int_\varepsilon^b \phi(x)\,dx + \int_{-\varepsilon}^{\varepsilon} F_n(\delta_m^{(s)}(x))\phi(x)\,dx,\end{aligned}$$

for $m > M$. Thus

$$\begin{aligned}\left|\langle F_n(\delta_m^{(s)}(x) - F_n(0), \phi(x)\rangle\right| &= \left|\int_{-\varepsilon}^{\varepsilon} \left[F_n(\delta_m^{(s)}(x)) - F_n(0)\right]\phi(x)\,dx\right| \\ &\leq 2\varepsilon K\end{aligned}$$

for $m > M$, where

$$K = \sup\left\{\left|\left[F_n(\delta_m^{(s)}(x)) - F_n(0)\right]\phi(x)\right| : m, n = 1, 2, \ldots;\ x \in \mathbf{R}\right\} < \infty,$$

since F_n and ϕ are bounded functions. It follows that

$$\lim_{m\to\infty}\langle F_n(\delta_m^{(s)}(x)), \phi(x)\rangle = \langle F_n(0), \phi(x)\rangle$$

and so

$$\begin{aligned}N\!-\!\lim_{n\to\infty}\left[N\!-\!\lim_{m\to\infty}\langle F_n(\delta_m^{(s)}(x)), \phi(x)\rangle\right] &= \lim_{n\to\infty}\langle F_n(0), \phi(x)\rangle \\ &= \langle F(0), \phi(x)\rangle.\end{aligned}$$

This proves that $F(\delta^{(s)}(x))$ exists and is equal to $F(0)$ for $s = 0, 1, 2, \ldots$.

Theorem 3.3. *The distribution $H(\delta^{(s)}(x))$ exists on the real line and*

$$H(\delta^{(s)}(x)) = \frac{1}{2}\ ,$$

for $s = 0, 1, 2, \ldots$, where H denotes Heaviside's function.

Proof. We put

$$H_n(x) = (H * \delta_n)(x)$$

for $n = 1, 2, \ldots$, so that

$$H_n(x) = \begin{cases} 1, & x > 1/n, \\ \int_{-1/n}^{x} \delta_n(x)\,dt, & |x| \leq 1/n, \\ 0, & x < -1/n, \end{cases}$$

$$0 \leq H_n(x) \leq 1, \quad H_n(0) = \frac{1}{2},$$

for $n = 1, 2, \ldots$.

Choosing arbitrary $\varepsilon > 0$, there exists M such that $m\varepsilon > 1$ for $m > M$. It then follows as above that

$$H_n(\delta_m^{(s)}(x)) = H_n(0) = \frac{1}{2},$$

for $|x| > \varepsilon$ and $m > M$ and so

$$\Big|\Big\langle H_n(\delta_m^{(s)}(x)) - \frac{1}{2}, \phi(x)\Big\rangle\Big| = \Big|\int_{-\varepsilon}^{\varepsilon} \Big[H_n(\delta_m^{(s)}(x)) - \frac{1}{2}\Big]\phi(x)\,dx\Big| \le \int_{-\varepsilon}^{\varepsilon} |\phi(x)|\,dx$$

for $m > M$ and arbitrary ϕ in $\mathcal{D}$. The result of the theorem follows as above.

Theorem 3.4. *Let F be a bounded, summable function on the real line which is continuous everywhere except for a simple discontinuity at the origin. Then the distribution $F(\delta^{(s)}(x))$ exists on the real line and*

$$F(\delta^{(s)}(x)) = \frac{1}{2}[F(0+) + F(0-)]$$

for $s = 0, 1, 2, \dots$.

Proof. Let $F(0+) - F(0-) = c$. Then the function G defined by

$$G(x) = F(x) - cH(x)$$

satisfies the conditions of Theorem 3.2. Thus

$$G(\delta^{(s)}(x)) = G(0) = F(0-),$$

and so

$$\begin{aligned} G(\delta^{(s)}(x)) + cH(\delta^{(s)}(x)) &= F(0-) + \frac{1}{2}[F(0+) - F(0-)] \\ &= \frac{1}{2}[F(0+) + F(0-)] \end{aligned}$$

for $s = 0, 1, 2, \dots$. The result of the theorem follows.

Theorem 3.5. *The distribution $\delta^{(r)}(\delta^{(s)}(x))$ exists on the real line and*

$$\delta^{(r)}(\delta^{(s)}(x)) = 0$$

for $r, s = 0, 1, 2, \dots$.

Proof. Choosing arbitrary $\varepsilon > 0$, there exists M such that $m\varepsilon > 1$ for $m > M$. Then $m\varepsilon > 1$ and $|x| > \varepsilon$ implies that $\rho^{(s)}(mx) = 0$ and so

$$\begin{aligned} \delta_n^{(r)}(\delta_m^{(s)}(x)) &= n^{r+1}\rho^{(r)}(nm^{s+1}\rho^{(s)}(mx)) \\ &= n^{r+1}\rho^{(r)}(0) \end{aligned}$$

for $|x| > \varepsilon$ and $m > M$.

Thus, for $m > M$ and all ϕ in $\mathcal{D}$, we have

$$\langle\delta_n^{(r)}(\delta_m^{(s)}(x)), \phi(x)\rangle = n^{r+1}\rho^{(r)}(0) \int\limits_{|x|>1/m} \phi(x)\,dx +$$

$$+\, n^{r+1} \int\limits_{|x|<1/m} \rho^{(r)}(nm^{s+1}(mx))\phi(x)\,dx \to n^{r+1}\rho^{(r)}(0) \int\limits_{-\infty}^{\infty} \phi(x)\,dx$$

as m tends to infinity. It follows that

$$\lim_{m\to\infty} \langle\delta_n^{(r)}(\delta_m^{(s)}(x)), \phi(x)\rangle = n^{r+1}\langle\rho^{(r)}(0), \phi(x)\rangle$$

and so

$$\begin{aligned} N\!-\!\lim_{n\to\infty}\Big[N\!-\!\lim_{m\to\infty}\langle\delta_n^{(r)}(\delta_m^{(s)}(x)), \phi(x)\rangle\Big] &= N\!-\!\lim_{n\to\infty} n^{r+1}\langle\rho^{(r)}(0), \phi(x)\rangle \\ &= 0. \end{aligned}$$

The result of the theorem follows. This completes the proof of the theorem.

Theorem 3.6. *The distribution $\left[\delta^{(r)}(x)\right]^s$ exists on the real line and*

$$\left[\delta^{(r)}(x)\right]^s = \frac{(-1)^{rs+s-1}c(\rho, r, s)}{(rs+s-1)!}\delta^{(rs+s-1)}(x) \tag{1}$$

for $r = 0, 1, 2, \ldots$ and $s = 2, 3, \ldots$, where

$$c(\rho, r, s) = \int\limits_{-1}^{1} [\rho^{(r)}(y)]^s y^{rs+s-1}\,dy.$$

In particular

$$\left[\delta^{(r)}(x)\right]^s = 0 \tag{2}$$

for even s.

Proof. Put

$$\begin{aligned} (x^s)_n = x^s * \delta_n(x) &= \int\limits_{-1/n}^{1/n} (x-t)^s \delta_n(t)\,dt \\ &= \sum_{i=0}^{s} (-1)^{s-i}\binom{s}{i} x^i \int\limits_{-1/n}^{1/n} t^{s-i}\delta_n(t)\,dt \end{aligned}$$

for $n = 1, 2, \ldots,$ where

$$\binom{s}{i} = \frac{s!}{i!(s-i)!}.$$

Then

$$\left[(\delta_m^{(r)}(x))^s\right]_n = \sum_{i=0}^{s} (-1)^{s-i} \binom{s}{i} \left[\delta_m^{(r)}(x)\right]^i \int_{-1/n}^{1/n} t^{s-i} \delta_n(t)\, dt, \tag{3}$$

where

$$\delta_m^{(r)}(x) = m^{r+1} \rho^{(r)}(mx),$$

the support of $\delta_m^{(r)}$ being contained in the interval $[-1/m, 1/m]$. Making the substitution $y = mx$ we have

$$\int_{-1/m}^{1/m} \left[\delta_m^{(r)}(x)\right]^i x^j\, dx = m^{ri+i-j-1} \int_{-1}^{1} \left[\rho^{(r)}(y)\right]^i y^j\, dy.$$

It follows that

$$N\!-\!\lim_{m \to \infty} \int_{-1/m}^{1/m} \left[\delta_m^{(r)}(x)\right]^i x^j\, dx = 0, \tag{4}$$

for $i = 0, 1, \ldots, s$, $j = 0, 1, 2, \ldots$ and $j \neq ri + i - 1$.

In the particular case $j = ri + i - 1$ we have

$$\int_{-\infty}^{\infty} \left[\delta_m^{(r)}(x)\right]^i x^{ri+i-1}\, dx = \int_{-1}^{1} \left[\rho^{(r)}(y)\right]^i y^{ri+i-1}\, dy = c(\rho, r, i). \tag{5}$$

Now let ϕ be an arbitrary function in $\mathcal{D}$. Then by Taylor's theorem we have

$$\phi(x) = \sum_{j=0}^{rs+s-1} \frac{\phi^{(j)}(0)}{j!} x^j + \frac{\phi^{(rs+s)}(\xi x)}{(rs+s)!} x^{rs+s},$$

where $0 \leq \xi \leq 1$. Thus

$$\int_{-1/m}^{1/m} \left[(\delta_m^{(r)}(x)^i\right]_n \phi(x)\,dx = \sum_{j=0}^{rs+s-1} \frac{\phi^{(j)}(0)}{j!} \int_{-1/m}^{1/m} \left[\delta_m^{(r)}(x)\right]^i x^j\,dx + \int_{-1/m}^{1/m} \frac{\phi^{(rs+s)}(\xi x)}{(rs+s)!} \left[\delta_m^{(r)}(x)\right]^i x^{rs+s}\,dx,$$

where

$$\left| \int_{-1/m}^{1/m} \frac{\phi^{(rs+s)}(\xi x)}{(rs+s)!} \left[\delta_m^{(r)}(x)\right]^i x^{rs+s}\,dx \right| \leq \leq \frac{2m^{(r+1)(i-s)-1}}{(rs+s)!} \sup\{|\phi^{(rs+s)}(x)|\} \cdot \sup\{|\rho^{(r)}(x)|\} \to 0$$

as m tends to infinity for $i = 0, 1, \ldots, s$.

Using equations (4) and (5), it follows that

$$N\text{-}\lim_{m\to\infty} \int_{-1/m}^{1/m} \left[(\delta_m^{(r)})^i\right]_n \phi(x)\,dx = \begin{cases} 0, & i = 0, \\ \dfrac{c(\rho, r, i)\phi^{(ri+i-1)}(0)}{(ri+i-1)!}, & i = 1, \ldots, s. \end{cases}$$

It now follows from equation (3) that

$$\left\langle \left[(\delta_m^{(r)}(x))^s\right]_n, \phi(x) \right\rangle = = \sum_{i=0}^{s} (-1)^i \binom{s}{i} \int_{-1/m}^{1/m} \left[(\delta_m^{(r)}(x))^{s-i}\right]_n \phi(x)\,dx. \int_{-1/n}^{1/n} t^{s-i}\delta_n(t)\,dt$$

and it follows from what we have just proved that

$$N\text{-}\lim_{m\to\infty} \left\langle \left[(\delta_m^{(r)}(x))^s\right]_n, \phi(x) \right\rangle = = \sum_{i=1}^{s} (-1)^{s-i} \binom{s}{i} \frac{c(\rho, r, i)\phi^{(ri+i-1)}(0)}{(ri+i-1)!} \int_{-1/n}^{1/n} t^{s-i}\delta_n(t)\,dt,$$

where

$$\int_{-1/n}^{1/n} t^{s-i}\delta_n(t)\,dt = \begin{cases} n^{i-s}\int_{-1}^{1} u^{s-i}\rho(u)\,du, & i = 1,\dots,s-1 \\ 1, & i = s, \end{cases}$$

Thus

$$\begin{aligned} \underset{n\to\infty}{N\text{--}\lim}\Big[\underset{m\to\infty}{N\text{--}\lim}\big\langle \big[(\delta_m^{(r)}(x))^s\big]_n, \phi(x)\big\rangle\Big] &= \frac{c(\rho,r,s)\phi^{(rs+s-1)}(0)}{(rs+s-1)!} \\ &= \frac{(-1)^{rs+s-1}c(\rho,r,s)}{(rs+s-1)!}\big\langle \delta^{(rs+s-1)}(x), \phi(x)\big\rangle \end{aligned}$$

and equation (1) follows. Equation (2) follows on noticing that

$$\big[\rho^{(r)}(y)\big]^s y^{rs+s-1}$$

is an odd function for even s and so $c(\rho,r,s) = 0$ for even s. This completes the proof of the theorem.

The next definition for the product of two distributions was given in [2].

Definition 3.3. Let f and g be distributions in D' and let

$$f_n(x) = (f * \delta_n)(x), \quad g_n(x) = (g * \delta_n)(x).$$

Then the product $f.g$ is defined to exist and be equal to the distribution h on the interval (a,b) if

$$\underset{n\to\infty}{N\text{--}\lim}\langle f_n(x)g_n(x), \phi(x)\rangle = \langle h(x), \phi(x)\rangle$$

for all test functions ϕ in $\mathcal{D}$ with support contained in the interval (a,b).

We note that with this definition of the product of two distributions, the definition of the distribution f^2 as the composition of the function x^2 and the distribution f if it exists, is distinct from the definition of the product $f.f$ if it exists. However, the following theorem holds:

Theorem 3.7. *Let f be a distribution in D'. Then the distribution f^2 exists on the interval (a,b) if and only if the distribution $f.f$ exists on the interval (a,b) and then*

$$f^2 = f \cdot f$$

on the interval (a,b).

PROOF. It follows as in the proof of Theorem 3.6 that

$$\big[(f_m(x))^2\big]_n = \int_{-1/n}^{1/n} t^2\delta_n(t)\,dt - 2f_m(x)\int_{-1/n}^{1/n} t\delta_n(t)\,dt + \big[f_m(x)\big]^2\int_{-1/n}^{1/n} \delta_n(t)\,dt,$$

where

$$\lim_{n\to\infty}\int_{-1/n}^{1/n} t^2\delta_n(t)\,dt = \lim_{n\to\infty}\int_{-1/n}^{1/n} t\delta_n(t)\,dt = 0,\quad \int_{-1/n}^{1/n}\delta_n(t)\,dt = 1.$$

Then it follows that f^2 exists on the interval (a,b), if and only if

$$\operatorname*{N-lim}_{n\to\infty}\Big[\operatorname*{N-lim}_{m\to\infty}\big\langle \big[f_m(x))^2\big]_n, \phi(x)\big\rangle\Big]$$

exists and is equal to

$$\operatorname*{N-lim}_{m\to\infty}\big\langle \big[f_m(x)\big]_n^2, \phi(x)\big\rangle = \operatorname*{N-lim}_{n\to\infty}\langle f_n(x)f_n(x), \phi(x)\rangle,$$

for all ϕ in $\mathcal{D}$ with support contained in the interval (a,b). That is, if and only if $f.f$ exists on the interval (a,b).

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KOU HUAI ZHONG
DEPARTMENT OF MATHEMATICS
HENAN NORMAL UNIVERSITY
XIXING CITY, HENAN PROVINCE
PEOPLE'S REPUBLIC OF CHINA

BRIAN FISHER
DEPARTMENT OF MATHEMATICS
THE UNIVERSITY, LEICESTER
LE1 7RH
ENGLAND

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