Publ. Math. Debrecen
40 / 3–4 (1992), 291–295

I-compactness and prime ideals

By H. H. BRUNGS (Edmonton)† and G. TÖRNER (Duisburg)‡

Abstract. Let R be a right chain ring with $P \neq P^2$ for every completely prime ideal $P \neq R$, (0) of R. Then R is I-compact for every right ideal I of R if and only if R is P-compact for every completely prime ideal P of R.

1. Introduction

Let R be a ring with identity and M a right uniserial R-module, i.e. an R-module whose lattice of right submodules is totally ordered by inclusion. The module M is called (linearly) K-compact for a submodule $K \subseteq M$ if the natural mapping from M into the inverse limit of M/M_λ is onto for every indexed family of submodules $\{M_\lambda \mid \lambda \in \Lambda\}$, with $\bigcap_{\lambda\in\Lambda} M_\lambda = K$. A ring R which is right uniserial is called a right chain ring. The module M is called compact if it is K-compact for every submodule K of M. The following result is proved:

Theorem A. *Let $\mathcal{F}$ be the family of right ideals I of a right chain ring R such that the right module R/I is not compact. Then either $\mathcal{F} = \emptyset$ or there exists a completely prime ideal P of R with $\mathcal{F} = \{I \mid I \subset P\}$ or $\mathcal{F} = \{I \mid I \subseteq P\}$.*

This result (see Brandal [2] for the commutative valuation rings) is obtained by investigating the set of right ideals K' of R for which a K-compact right chain ring is also K'-compact. We obtain with the

---

Mathematics Subject Classification: Primary: 16W60, 16L30, 16D30. Secondary: 13A18, 13J10, 06F05.
Key words and phrases: Valuation ring, completion, I-compactness, prime ideal.
†Supported in part by NSERC
‡Supported in part by Deutsche Forschungsgemeinschaft

same methods, which however are very different from the ones used in the commutative case, an extension of a theorem about discrete commutative valuation rings by RIBENBOIM [8] and MACLANE [7].

As usual we say that an ideal $P \neq R$ of a ring R is *completely prime* if $ab \in P$ for $a, b \in R$ implies a or b in R.

A right chain ring R is called discrete if $P \neq P^2$ for every completely prime ideal $P \neq (0)$ of R.

Theorem B. *Let R be a discrete right chain ring. Then R is compact if and only if R is P-compact for every completely prime ideal P of R.*

2. Preliminaries

A right chain ring R is a ring with identity for which $aR \subseteq bR$ or $bR \subset aR$ for any $a, b \in R$. Compact commutative valuation domains were introduced by KRULL [6] and it was proved by KAPLANSKY [5] that these are exactly the maximal valuation domains that have no proper valuation overdomains with the same value group and residue field. A right chain domain R can be embedded into a skew field $D = Q(R)$ of quotients, has a unique maximal ideal $J(R)$ such that $R/J(R)$ is a skew field, but a group of values exists only if R is also a left chain ring and is invariant under all inner automorphisms of D (see SCHILLING [10]). A compact right chain domain is still maximal in the sense that no proper overdomain exists with the same lattice of principal right ideals and the same skew field of residues. However, not every maximal right chain domain is compact ([3]).

Let R be a right chain ring and $A, B(\neq R)$ right ideals of R. We say that A and B are related, $A \sim B$, if there exists $s, t \in R$, $s \notin A$, $t \notin B$ and $s^{-1}A = t^{-1}B$ where $s^{-1}A = \{r \in R \mid sr \in A\}$. This defines an equivalence relation on the set of all right ideals ($\neq R$) of R. One can prove (see [3]) in a straightforward way the following result:

Lemma 2.1. *Let A, B be related right ideals in a right chain ring R. Then R is A-compact if and only if R is B-compact.*

We will give proofs for the next two results even though they can be found in [1].

Lemma 2.2. *Let x, a be elements in a right chain ring R. Then $xa = uax'$ for a unit $u \in R$ and an element $x' \in R$.*

Proof. We can assume that x and a are nonzero nonunits in R. Either $xa = ax'$ for some $x' \in R$ or $xas = a$ for some nonunit $s \in R$. We obtain $xa(1+s) = xa + a = (x+1)a$ and $xa = (x+1)a(1+s)^{-1}$ for the units $1+x$, $1+s$ in R. □

In spite of Lemma 2.2 there exist right chain rings with $R \supset J(R) \supset (0)$ as the only two-sided ideals ([4]).

For every right ideal $A \neq R$ of the ring R we define the (right) associated prime ideal $P_r(A) = P(A) = \{p \in R \mid Ap^{-1} \supset A\}$.

Lemma 2.3. *Let R be a right chain ring and $A \neq R$ a right ideal in R. Then $P(A)$ is a completely prime ideal of R.*

Proof. That $pr \in P(A)$ for $p \in P(A)$, $r \in R$, is obvious. To prove $rp \in P(A)$ for $r \in R$ and $p \in P(A)$ we observe that there exists $t \in R \setminus A$ with $tp \in A$. By Lemma 2.2 there exist elements $u, r' \in R$, u a unit, with $rp = upr'$. Hence, $(tu^{-1})rp = tu^{-1}upr' = tpr' \in A$ and $tu^{-1} \notin A$ implies $rp \in P(A)$. If p_1, p_2 are in $P(A)$ we can assume $p_1 = p_2 b$ for some $b \in R$ and $p_1 + p_2 = p_2(b+1) \in P(A)$ follows. Finally, if $s_1, s_2 \in R \setminus P(A), t \in R$, then $ts_1 s_2 \in A$ implies $ts_1 \in A$, $t \in A$ and $s_1 s_2 \notin P(A)$. □

3. The main results

We consider the associated prime ideals for related right ideals.

Lemma 3.1. *Let R be a right chain ring, A and B related right ideals in R. Then $P(A) = P(B)$.*

Proof. Assume $s^{-1}A = t^{-1}B$ for $s \in R \setminus A$, $t \in R \setminus B$. Take $p \in R \setminus P(A)$ and assume $xp \in B$ for some $x \in R$. To show that $p \in R \setminus P(B)$ suppose first that $tz_1 = x$ for some $z_1 \in R$. Then $xp = tz_1 p \in B$, hence $z_1 p \in t^{-1}B = s^{-1}A$ and $sz_1 p \in A, sz_1 \in A, z_i \in s^{-1}A$ and $tz_1 = x \in B$ follows, i.e. $p \in R \setminus P(B)$. If $t = xz_2$ and $z_2 = pz'$ for some $z_2, z' \in R$ we obtain $t = xpz' \in B$ – a contradition. Hence, we are left with the case $t = xz_2$ and $z_2 z' = p$ and $xp = xz_2 z' = tz' \in B$ and $sz' \in A$ for some $z' \in R$. This implies $s \in A$, since $z_2 z' = p \in R \setminus P(A)$ and hence $z' \in R \setminus P(A)$. The other containment $R \setminus P(B) \subseteq R \setminus P(A)$ follows by symmetry. □

The next result gives some indication about the range of a class of related right ideals in the lattice of all right ideals of R.

Lemma 3.2. *Let $A \neq R$ be a right ideal in the right chain ring R and let $\mathcal{R}(A) = \{I \mid I$ is a right ideal of R related to $I\}$. Then $P = P(A) = \bigcup \mathcal{R}(A)$.*

Here and in the following we write $\bigcup \mathcal{S} = \bigcup_{I \in S} I$ for a set $\mathcal{S}$ of right ideals of R.

Proof. We write $B = \bigcup \mathcal{R}(A)$. For $I \in \mathcal{R}(A)$ we have $I \subseteq P$ and $B \subseteq P$ follows. Conversely, pick $z \in P$ and let I be in $\mathcal{R}(A)$. There exists $t \notin I$ with $z \in t^{-1}I$, since $P(I) = P$ be the previous lemma. Similarly we have $P(t^{-1}I) = P(I)$ and $z \in t^{-1}I \subseteq P(t^{-1}I) = P$ follows which proves $B = P$. $\square$

Proof (of Theorem A). Let R be a right chain ring and $\mathcal{F}$ be the set of right ideals I in R such that R/I is not compact. Then for any I in $\mathcal{F}$ there exists a right ideal K of R with $I \subseteq K$ and $K \in \mathcal{F}_0 = \{L \mid R$ not L-compact $\} \subseteq \mathcal{F}$. Therefore: $\bigcup \mathcal{F} = \bigcup \mathcal{F}_0$.

By Lemma 2.1 it follows that the set $\mathcal{F}_0$ is a union of classes of related right ideals and hence, by Lemma 3.2, that $\mathcal{F} = \emptyset$ or that $\bigcup \mathcal{F}_0$ is the union of completely prime ideals of the form $\bigcup \mathcal{R}(L_\lambda) = P_\lambda \in \mathcal{F}_0$, and hence equal to a completely prime ideal P, i.e. $\bigcup \mathcal{F} = \bigcup \mathcal{F}_0 = P$. Either $P \in \mathcal{F}$ and then $\mathcal{F} = \{I \mid I \subseteq P\}$ or $P \notin \mathcal{F}$ and $\mathcal{F} = \{I \mid I \subset P\}$ where I indicates right ideal in R. $\square$

Examples exist, even in the commutative case (see [2]), that show that none of these alternatives is void.

We saw earlier that $P(A)$ is the union of right ideals related to A. The next results gives a condition under which $P = P(A)$ itself is related to A.

Proposition 3.3. *Assume that $A \neq R$ is a right ideal of the right chain ring R with $P(A) = P \neq P^2$. Then A is related to P.*

Proof. By Lemma 3.2 we have $P = \bigcup A_\lambda$ for the set $\{A_\lambda \mid \lambda \in \Lambda\}$, of right ideals of R related to A. Hence, there exists a right ideal B, related to A, with $P \supseteq B \supset P^2$. We have $P(B) = P$ by Lemma 3.1. To prove $P = B$ assume that there exists $x \in B \setminus P$ and choose $a \in B \setminus P^2$. Then there exists $s \notin P$ with $xs = a$ (otherwise $a \in P^2$) and $x \in B$ follows (by the definition of $P(B)$) – a contradiction. $\square$

We can apply this result to discrete right chan rings R.

Proof (of Theorem B). Let A be a right ideal $\neq R$ in a discrete right chain ring R which is P-compact for all completely prime ideals P.

We want to show that R is A-compact. This follows immediately from Lemma 2.1 and the above Proposition if $A \neq (0)$. If $A = (0)$ then either A is completely prime and R is A-compact or $ab = 0$ for some elements $a \neq 0 \neq b$ in R. Then $B = a^{-1}(0) \ni b \neq 0$ is a nonzero right ideal related to (0) and it follows that R is B as well as (0)-compact. $\square$

Since non-completely prime prime ideals in a right chain R are paired with idempotent completely prime ideals ([1], Theorem 3.12), the assumption of Theorem B guarantees that all prime ideals in a discrete right chain ring are completely prime.

References

[1] C. Bessenrodt, H.H. Brungs and G. Törner, Right chain rings. Part 1, *Preprint Fachbereich Mathematik, Universität Duisburg* No. **181**, *Duisburg*, 1990.

[2] W. Brandal, Almost maximal integral domains and finitely generated modules, *Trans. Amer. Math. Soc.* **183** (1973), 203–222.

[3] H.H. Brungs and G. Törner, Maximal immediate extensions are not necessarily maximally complete, *J. Australian Math. Soc.* (Series A) **49** (1990), 196–221.

[4] N.I. Dubrovin, Chain domains, *Moscov Univ. Math. Bull. Ser. I* **35** (1980), 2, 51–54.

[5] I. Kaplansky, Maximal fields with valuations, *Duke Math. J.* **9** (1942), 303–321.

[6] W. Krull, Allgemeine Bewertungstheorie, *J. Reine Angew. Math.* **167** (1932), 160–196.

[7] S. MacLane, The uniqueness of power series representations of certain fields with valuations, *Annals of Math.* **39** (1938), 370–382.

[8] P. Ribenboim, Corps maximaux et complets par des valuations de Krull, *Math. Z.* **69** (1958), 466–479.

[9] B. Roux, Sur les anneaux de Köthe, *An. Acad. Brasil Cienc.* **48** (1976), 13–28.

[10] O. Schilling, The theory of valuations, *Math. Surveys, Amer. Math. Soc.*, *Providence R.I.*, 1950.

HANS HEINRICH BRUNGS
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ALBERTA
EDMONTON
CANADA

GÜNTER TÖRNER
FACHBEREICH MATHEMATIK
UNIVERSITÄT DUISBURG
DUISBURG
WEST GERMANY

(Received July 24, 1990)