Publ. Math. Debrecen
40 / 3–4 (1992), 303–311

On the theory of multiplier operators in a Banach-module

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1. Introduction

If A is an algebra and W is an A-module, then a (left) multiplier from A into W is a linear operator T with the property

$$(*) \qquad Tab = aTb \quad a, b \in A;$$

i.e. a (left) multiplier is a linear operator commuting with left multiplications. Similarly, a *right multiplier* is a linear operator commuting with right multiplication. The (left) multipliers from A into W from a Banach space $\mathcal{M}(A,W)$ (in the case of $W = A$ they form a Banach algebra $\mathcal{M}(A)$).

Every $w \in W$ represents a multiplier operator by

$$T_w a := aw \quad a \in A$$

called *inner multiplier* since, according to the module axioms, $(ba)w = b(aw)$ for $b, a \in A$ and $w \in W$. One of our main objects in this paper is to find conditions for every multiplier to be inner.

$\mathcal{M}(A,W)$ resp. $\mathcal{M}(A)$ is a fine module resp. algebra extension of W resp. A. The property $(*)$ can be expressed also as

$$TR_b a = R_{Tb} a \quad b, a \in A$$

where the indexed R means the right multiplication operator. Hence $\mathcal{M}(A)$ is the left idealizer of $\{R_a;\ a \in A\}$ and $\mathcal{M}(A,W)$ sends $\{R_a;\ a \in A\}$ into $\{T_w;\ w \in W\}$ by the left multiplication.

To find the multiplier extension of a Banach-module resp. Banach algebra is called *multiplier problem*.

Moreover, $\mathcal{M}(A)$ is included in $\{R_a;\ a \in A\}''$, the second commutant of $\{R_a;\ a \in A\}$. In fact $\mathcal{M}(A) = \{L_a;\ a \in A\}'$ by definition, where the indexed L means the left multiplication operator, and if $\hat{T}$ is a right multiplier, then

$$\hat{T}Tab = \hat{T}[aTb] = \hat{T}a.Tb$$

and

$$T\hat{T}ab = T[\hat{T}a]b = \hat{T}a.Tb \quad a, b \in A.$$

Hence, if $\{ab;\ a, b \in A\}$ span the Banach algebra A, then every right multiplier belongs to the second commutant $\{L_a;\ a \in A\}''$. Similarly $\mathcal{M}(A) \subseteq \{R_a;\ a \in A\}''$.

A similar observation is valid for $\mathcal{M}(A, W)$.

The most general representation theorems for multiplier operators are in Mate [4], [5] and Rieffel [7] p. 461–77. After having analysed the main ingredient of these tensor product like representations in section 2, in section 3 we obtain conditions for multiplier operators to be represented by elements of a dual Banach space possibly different from the tensor product like representations.

Finally, in section 4, we find conditions for every multiplier being inner.

2. Preliminary results

If A has an identity i.e. A is a unital algebra, then there is no multiplier problem.

Proposition 1. *If A has a (right) identity e, then every (left) multiplier is inner.*

Proof. In this case

$$(*) \qquad Tb = Tbe = bTe$$

for every $b \in A$.

For the case of non-unital A the usual extension to a unital algebra A_e does not help. In fact, if we can define $Te \in W$ so that $(*)$ is satisfied, then

$$T[b + \lambda e][a + \mu e] = [b + \lambda e]T[a + \mu e] \quad \lambda, \mu \in \Omega$$

and hence

$$Tb = bTe \quad b \in A$$

by an easy calculation and so T is inner. Otherwise there is no multiplier extension of T onto A_e. In this latter case, to extend W by elements $\{Te;\ T \in \mathcal{M}(A, W)\}$ is not an adequate solution.

Example. Let $A = W = C_0(-\infty, +\infty)$ be the usual Banach space of continuous functions tending to zero at infinity. C_0 is also a Banach algebra with pointwise multiplication and the multiplication by any bounded continuous function is a multiplier operator. E.g.

$$Ty := y(t)\sin t \quad y \in C_0$$

is a multiplier operator which cannot be extended onto the linear space $\{y + \lambda 1;\ y \in C_0,\ \lambda \in \Omega$ and 1 is the $y = 1$ function$\}$. In fact

$$\sin t \neq 1 + g(t) \quad g \in C_0$$

and hence T is not inner.

However, there is a more powerful extension than A_e. Namely, this is the second dual A^{**} with the Arens product.

For $a \in A$ and $a^* \in A^*$ define $aa^* \in A^*$ as

$$(a^*a \mid c) := (a^* \mid ac) \quad c \in A;$$

then A^* is a (right) Banach A-module and similarly A^{**} is a (left) Banach A-module with the bidual of the left multiplication operator in A. Moreover, we have the same definition for w^*a ($w^* \in W^*$, $a \in A$) resp. aw^{**} ($w^{**} \in W$, $a \in A$). In this case the dual operator T^* of a left multiplier T is a right multiplier. In fact, for every $c \in A$

$$\begin{aligned}(T^*w^*a \mid c) = (w^*a \mid Tc) = (w^* \mid aTc) = (w^* \mid Tac) = \\ = (T^*w^* \mid ac) = ([T^*w^*]a \mid c)\end{aligned}$$

and hence

$$T^*w^*a = [T^*w^*]a \qquad w^* \in W^* \quad a \in A.$$

We conclude

Proposition 2. *If T is a multiplier operator, then T^{**} is a multiplier extension of T. More precisely, if $T \in \mathcal{M}(A, W)$ then $T^{**} \in \mathcal{M}(A^{**}, W^{**})$ so that the restriction of T^{**} to A is T.*

Remark. Let $\hat{a}$ be the copy of $a \in A$ in the natural embedding of A into A^{**}, then it can be verified that

$$aa^{**} = \hat{a}a^{**} \qquad a \in A, \quad a^{**} \in A^{**}$$

where the left hand side product is the bidual of the left multiplication in A by $a \in A$ and the product on the right side is the Arens product (see e.g. [2] p. 224).

It may happen, that A^{**} has an identity however A is nonunital:

Theorem 1. *([1], [2] p. 224) Let A be a Banach algebra. Then A has bounded (right) approximate identity if and only if the Banach algebra A^{**} has a (right) identity.*

Proposition 1 and 2 and Theorem 1 together imply

Proposition 3. *Let A be a Banach algebra with bounded approximate identity. Then $\mathcal{M}(A,W) \subseteq W^{**}$ in the sense that for every $T \in \mathcal{M}(A,W)$ there exists $w^{**} \in W^{**}$ so that*

$$Ta = aw^{**} \quad a \in A. \tag{$*$}$$

Moreover, based on [3] VI.4.2, $\mathcal{M}(A,W) = W$ in the sense of $(*)$ if and only if

$$w \Rightarrow aw$$

is a weakly compact operator for every $a \in A$.

Corollary. If A has bounded approximate identity and W is a reflexive Banach space, then every $T \in \mathcal{M}(A,W)$ is inner.

Example. If $A = L^1(G)$, $W = L^p(G)$ for $1 < p < \infty$ where G is a locally compact group and the module operation is the convolution, then $\mathcal{M}(L^1, L^p) = L^p$ in the sense of $(*)$.

3. The fundamental model

Now we turn to the case when there is no identity in A^{**} i.e. there is no bounded approximate identity in A by Theorem 1.

We begin with two simple observations:

I. A characteristic property of the identity e in A^{**} is

$$(e \mid aa^*) = (a^* \mid a).$$

In fact

$$(e \mid a^*a) = (ae \mid a^*) = (\hat{a} \mid a^*) = (a^* \mid a)$$

and conversely, if $e' \in A^{**}$ with the property $(e' \mid a^*a) = (a^* \mid a)$ then $ae' = \hat{a}$ for $a \in A$ and this is sufficient for the representation of $\mathcal{M}(A,W)$ by W^{**} via Proposition 3.

II. If $\operatorname{ess} W^*$ is the linear space spanned by

$$\{w^*a; \quad w \in W^*, \quad a \in A\}$$

and, viewing A^* as a (right) Banach A-module with the dual action, essA^* is the linear space spanned by

$$\{a^*a; \quad a^* \in A^*, \quad a \in A\}$$

then, by the considerations preceding Proposition 2, T^*, the dual operator of $T \in \mathcal{M}(A, W)$, maps $\operatorname{ess} W^*$ into $\operatorname{ess} A^*$.

Based on the above observations I. and II. we suppose that there is a norm $\| \, \|_\tau$ resp. $\| \, \|_{\tau_A}$ in $\operatorname{ess} W^*$ resp. $\operatorname{ess} A^*$, *possibly different from the original ones*, so that

$$T^* : \quad \operatorname{ess} W^*, \tau) \Rightarrow (\operatorname{ess} A^*, \tau_A)$$

is continuous and there exists $I \in (\operatorname{ess} A^*, \tau_A)^*$ so that

$$(I \mid a^*a) = (a^* \mid a).$$

Our main object in this section is a description of $\mathcal{M}(A, W)$, under the above conditions, if W is the dual of a (right) Banach A-module M.

Let $\operatorname{ess} M$ be the linear space spanned by

$$\{ma; \quad m \in M, \quad a \in A\}$$

and let us consider the linear spaces $\operatorname{ess} M$ resp. $\operatorname{ess} A^*$ with norm $\| \,.\, \|_\tau$ resp. $\| \,.\, \|_{\tau_A}$ possibly different from the original norms but also

$$\|ma\|_\tau \leq \|m\| \, \|a\| \qquad m \in M, \quad a \in A$$
$$\|a^*a\|_{\tau_A} \leq \|a^*\| \, \|a\| \qquad a^* \in A^*, \quad a \in A$$

For $F \in [\operatorname{ess} M; \tau]^*$, let

$$(F \mid ma) \qquad m \in M, \quad a \in A$$

be the value of F at ma. Then the formula $(F \mid ma)$ can be viewed both as

$$m \Rightarrow (F \mid ma) \quad \text{and as} \quad a \Rightarrow (F \mid ma)$$

and both are continuous linear functionals.

Theorem 2. *[6] Let $F \in [\operatorname{ess} M, \tau]^*$ and we define the operator $F\diamond$ from M into A^* as*

$$(F \diamond m \mid a) = (F \mid ma)$$

and the operator $\diamond F$ from A into W as

$$(a \diamond F \mid m) = (F \mid ma).$$

Then $F\diamond$ is a right multiplier, $\diamond F$ is a left multipler, both are continuous, and

$$F\diamond = (\diamond F)^* \quad \textit{restricted to} \quad M.$$

Proof.

$$(ca \diamond F \mid m) = (F \mid m[ca]) = (F \mid [mc]a) = \\ = (a \diamond F \mid mc) = (c[a \diamond F] \mid m)$$

for every $m \in M$ and hence $\diamond F$ is a left multiplier.

For the continuity of $\diamond F$ we have

$$|(a \diamond F \mid m) = |(F \mid ma)| \leq \|F\| \, \|ma\|_\tau \leq \|F\| \, \|m\| \, \|a\|$$

and hence

$$\|a \diamond F\| \leq \|F\| \, \|a\|.$$

The connection between $\diamond F$ and $F\diamond$ follows from the definition and hence, the remaining part of the theorem follows from the considerations preceding Proposition 2., namely that the dual of a left multiplier is a right multiplier.

We can prove similarly

Theorem 2∼. *Let $G \in [\operatorname{ess} A^*, \tau_A]^*$ and define the operator $G\diamond$ of A^* as*

$$(G \diamond a^* \mid a) = (G \mid a^* a)$$

and the operator $\diamond G$ from A into A^{**} as

$$(a \diamond G \mid a^*) = (G \mid a^* a).$$

Then $G\diamond$ is a right multiplier, $\diamond G$ is a left multiplier, both are continuous, and

$$\diamond G = (G\diamond)^* \quad \textit{restricted to} \quad A.$$

Now, let T be a continuous (left) multiplier from A into W. Then T^* maps $\operatorname{ess} M \subseteq \operatorname{ess} W^*$ into $\operatorname{ess} A^*$ and *if T^* is continuous also in the τ-topologies* (i.e. as $[\operatorname{ess} M, \tau] \Rightarrow [\operatorname{ess} A^*, \tau_A]$ operator) then we can define $T^{\star\backslash}$ from $[\operatorname{ess} A^*, \tau_A]^*$ into $[\operatorname{ess} M, \tau]^*$ as the dual of T^*. Then we have

Theorem 3. *If there exists a τ-topology in* $\operatorname{ess} M$ *and a τ_A-topology in* $\operatorname{ess} A^*$ *so that*

I. T^* *is continuous;*

II. There exists $I \in [\operatorname{ess} A^*, \tau_A]^*$ *defined by*

$$(I \mid a^* a) = (a^* \mid a);$$

Then there is a unique $F \in [\operatorname{ess} M, \tau]^*$ *so that*

$$T^* m = F \diamond m \quad \text{and} \quad Ta = a \diamond F$$

where $F = T^{\star\setminus} I$.

Proof. The dual $T^{\star\setminus}$ of T^* restricted to $\operatorname{ess} M$ maps $[\operatorname{ess} A^*, \tau_A]^*$ into $[\operatorname{ess} M, \tau]^*$ since T^* maps $\operatorname{ess} M$ into $\operatorname{ess} A^*$ and T^* is continuous. In particular, for $I \in [\operatorname{ess} A^*, \tau_A]^*$ we have

$$(T^{\star\setminus} I \mid ma) = (I \mid T^* ma) = (I \mid [T^* m] a) = (T^* m \mid a) = (m \mid Ta)$$
$$a \in A, \quad m \in M.$$

On the other hand

$$(T^{\star\setminus} I \mid ma) := (a \diamond [T^{*/} I] \mid m) \quad a \in A, \quad m \in M$$

and hence $Ta = a \diamond [T^{\star\setminus} I]$ for every $a \in A$.

We have established till now, that the operator $\diamond F$ is a multiplier for every $F \in (\operatorname{ess} M, \tau)^*$ and if $T \in \mathcal{M}(A, W)$ satisfies the conditions of Theorem 3. then $T = \diamond F$ with $F = T^{\star\setminus} I$. However, the following problems remain open:

a) When are the conditions of Theorem 3. satisfied for the inner multipliers ?

b) When are the conditions of Theorem 3. satisfied for every multiplier ?

Obviously, if $\| \ \|_\tau$ and $\| \ \|_{\tau_A}$ are equivalent to the original norm then the answere is "yes" to both questions. Moreover, this is the case also when for $h \in \operatorname{ess} M$ resp. $\operatorname{ess} A^*$

$$\|h\|_\tau := \inf \left\{ \sum_{k:=1}^{n} \|m_k\| \, \|a_k\| \ : \ h = \sum_{k:=1}^{n} m_k a_k \right\}$$

resp.

$$\|h\|_{\tau_A} := \inf \left\{ \sum_{k:=1}^{n} \|a_k^*\| \, \|a_k\| \ : \ h = \sum_{k:=1}^{n} a_k^* \right\}$$

which is similar to the tensor product type norms in [4] and [7] but not the same.

3. When is every multiplier inner ?

Renember that "every multiplier from A into W is inner" means that every $T \in \mathcal{M}(A, W)$ has the form

$$T_w a = aw \quad a \in A.$$

$\mathcal{M}(A, W)$ is a closed linear subspace of $\mathcal{B}(A, W)$, the Banach space of bounded linear operators from A into W, and obviously $\|T_w\| \le \|w\|$. Hence, it follows from the Banach Homomorphism Theorem that if every multiplier is inner then the original and the operator norm are equivalent in W, i.e. there exists $c > 0$ such that

$$(*) \qquad c\|w\| \le \|T_w\| \le \|w\|.$$

We shall show that, under the conditions of Theorem 3, $(*)$ is also sufficient for every multiplier to be inner.

Theorem 4. *If*

I. $\operatorname{ess} M$ *is dense in* M*;*

II. $(F_w \mid ma) := (w \mid ma)$ *defines a continuous linear functional also in* $\| \, \|_\tau$ *for every* $w \in W$*;*

III. there exists $c > 0$ *so that* $c\|w\| \le \|T_w\|$*; then*

$$(\operatorname{ess} M;\ \tau)^* = M^* = W.$$

Proof. $\{F_w;\ w \in W\}$ is weak*-dense in $(\operatorname{ess} M; \tau)^*$ since if $m_0 \in \operatorname{ess} M$ and $(w \mid m_0) = 0$ for every $w \in W$ then $m_0 = 0$, obviously.

Let S^* be the unit sphere in $(\operatorname{ess} M; \tau)^*$. We claim that

$$\{F_w;\ w \in W\} \cap \lambda S^*$$

is weak*-closed for every $\lambda > 0$. If it is so, then it follows from the Krein-Shmulian theorem ([3] V.5.7) that $\{F_w;\ w \in W\}$ is weak*-closed in $(\operatorname{ess} M; \tau)^*$ and hence $(\operatorname{ess} M; \tau)^* = W$.

In fact, if $F_{w_a} \in \lambda S^*$ and $\{F_{w_a}\}$ tends to F in the weak*-topology, then

$$(w_a \mid ma) \Rightarrow (F \mid ma) \quad m \in M, \quad a \in A$$

and $F \in \lambda S^*$ by the Alaoglu theorem ([3], V.4.2).

Since $\|F_{w_a}\| \le \lambda$ and

$$c\|w\| \le \|T_w\| := \| \diamond F_w\| \le \|F_w\|$$

it follows

$$\|w_a\| < \frac{\lambda}{c}$$

and hence by the Banach-Steinhaus theorem, there is $w \in W$ so that $w_a \Rightarrow w$ in the weak*-topology in W. In particular

$$(w_a \mid ma) \Rightarrow (w \mid ma) \quad m \in M, \quad a \in A$$

and we conclude

$$F = F_w.$$

Corollary. If $I \in (\operatorname{ess} M;\, \tau)^*$ besides I-III, then $\mathcal{M}(A, W) = W$ i.e. every multiplier is inner.

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(Received January 24, 1990)