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Hypercentre-by-finite groups

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1. Introduction

It was observed by R. BAER (see [9]) that a consequence of B. H. NEUMANN's result on groups covered by finitely many cosets [8] is the characterization of centre-by-finite groups as those groups which are the union of finitely many abelian subgroups. In [13], we gave a bound for the index of the centre in terms of the number of abelian subgroups required to cover G. If G is the union of n abelian subgroups then any set of $n+1$ elements contains a pair which commute; we shall say that "G satisfies the condition $(\mathbf{A}, n)$". This notation indicates that the largest set of noncommuting elements in G (or the largest set of elements in G in which no two generate an abelian subgroup) has size n. The apparently weaker condition that any infinite set of elements contains a pair which commute is denoted by $(\mathbf{A}, <\infty)$. B. H. NEUMANN [10] and V. FABER, R. LAVER and R. MCKENZIE [2] showed that G satisfying $(\mathbf{A}, <\infty)$ is equivalent to G being centre-by-finite and so G satisfying $(\mathbf{A}, n)$ for some integer n. L. PYBER [11] gave a bound for the index of the centre in a group G satisfying $(\mathbf{A}, n)$.

The results of this note concern the corresponding conditions for nilpotent subgroups rather than abelian subgroups. R. BAER showed that a group G is the union of finitely many hypercentral subgroups if and only if the hypercentre $Z^*(G)$ of G has finite index in G (see [12], Theorem 4.18). It is a fairly simple matter to show that G is the union of finitely many nilpotent subgroups if and only if some finite term $Z_n(G)$ of the upper central series of G has finite index in G. This result is a consequence of the following.

Theorem A. *If G is the union of n subgroups each of which is nilpotent of class c, then $|G/Z_k(G)| \le (n!)!$, where $k = c(r + c^{\log_2 r})$ and $r = (n!)!$.*

If G is the union of finitely many normal nilpotent subgroups then G is itself nilpotent. The usual proof of FITTING's Theorem gives a bound for the nilpotency class of G and a similar argument shows that the intersection of the normal subgroups is contained in a particular term of the upper central series so that we obtain

Theorem B. *If G is the union of n normal subgroups each of which is nilpotent of class c, then $|G/Z_m(G)| \leq n!$, where $m = n(c-1)+1$. G is nilpotent of class at most nc.*

The conditions $(\mathbf{N}, n)$ and $(\mathbf{N}, <\infty)$ were considered by J. C. LENNOX and J. WIEGOLD [6] for finitely generated soluble groups. Here we say that G satisfies the condition $(\mathbf{N}, n)$ (the condition $(\mathbf{N}, <\infty)$) if any set of $n+1$ elements of G (any infinite set of elements of G) contains a pair which generate a nilpotent subgroup. LENNOX and WIEGOLD showed that G satisfies $(\mathbf{N}, <\infty)$ if and only if it is finite-by-nilpotent and hence satisfies $(\mathbf{N}, n)$ for some integer n. We concentrate on the bounded condition $(\mathbf{N}, n)$ and prove

Theorem C. *If G is a finitely generated soluble group satisfying $(\mathbf{N}, n)$ then $|G/Z^*(G)| \leq n^{n^4}$.*

It should be noted that a group G is finite-by-nilpotent if and only if $|G/Z_j(G)|$ is finite for some integer j [4]. This implies that the hypercentre $Z^*(G)$ has finite index in G. Conversely, if $Z^*(G)$ has finite index in the finitely generated group G then $Z^*(G) = Z_j(G)$ for some integer j.

We should stress, however, that our Theorem C says nothing about the condition $(\mathbf{N}, <\infty)$ and, in particular, the result of LENNOX and WIEGOLD does not follow from this theorem. The bounds given in all three theorems are all rather large and unlikely to be even close to best possible.

In connection with Theorems A and B we mention related results of L.–C. KAPPE [5] who considered groups which are the union of finitely many 2-Engel groups. In this case the subgroup of right 2-Engel elements of G

$$L(G) = \{a \in G : [a, x, x] = 1, \text{ for all } x \in G\}$$

has finite index in G and if the covering subgroups are normal then G is a 3-Engel group. As the 2-Engel condition is rather special it is unlikely that these results will extend to a covering by n-Engel subgroups. On the other hand they are much more satisfactory than the bounds given by Theorems A and B above when-interpreted for subgroups which have nilpotency class two.

2. Groups covered by finitely many nilpotent subgroups

PROOF of Theorem A. Let $G = H_1 \cup \ldots \cup H_n$, where each H_i is nilpotent of class c. By omitting some of the H_i if necessary we may assume that this is an irredundant cover of G; that is $H_i \not\subseteq \bigcup_{j \neq i} H_j$ for each $i = 1, \ldots, n$. If $H = H_1 \cap \ldots \cap H_n$ then, by Theorem 2.2 of [13], $|G : H| \leq n!$ (In fact, Theorem 3.4 of [13] gives a slightly better bound for $|G : H|$ but as this is still unlikely to be best possible we use the simpler form $n!$). If $N = \mathrm{core}_G(H)$ then $|G/N| \leq (n!)! = r$. We show that $N \leq Z_k(G)$.

Certainly N is nilpotent of class at most c and so has a central series of length c. It is sufficient to show that $A = Z(N)$ is contained in the $(r + c^{\log_2 r})$-th term of the upper central series of G and then, by factoring out $Z_{i-1}(N)$, the same result can be obtained for $Z_i(N)/Z_{i-1}(N)$.

Let $T = \mathrm{Dr}\, A_p$ be the torsion part of A, where A_p is the maximal p-subgroup of A. If $a \in A_p$ then $\langle a^G \rangle$ is a finite normal p-subgroup of G. Let xN be a p'-element of G/N; then $x \in H_i$, for some $i = 1, \ldots, n$, and so $\langle a^G, x \rangle$ is nilpotent. It follows that x centralizes $\langle a^G \rangle$ and so $G/C_G(\langle a^G \rangle)$ is a p-group of order at most r. Now $\overline{G} = G/C_G(\langle a^G \rangle)$ has a finite central series of length $s \leq \log_2 r$ in which each factor is cyclic

$$\overline{1} < \langle \overline{x}_1 \rangle < \langle \overline{x}_1, \overline{x}_2 \rangle < \ldots < \langle \overline{x}_1, \ldots, \overline{x}_s \rangle = \overline{G}.$$

Each $\overline{x}_j$ can be written as $x_j C_G(\langle a^G \rangle)$. Since $x_1 \in H_i$, for some $i = 1, \ldots, n$, $\langle a^G, x_1 \rangle$ is nilpotent of class at most c. Therefore there is a chain of subgroups.

$$\langle a^G \rangle \geq [\langle a^G \rangle, x_1] \geq [\langle a^G \rangle, x_1, x_1] \geq \ldots \geq [\langle a^G \rangle, {}_c x_1] = 1$$

and, since $\overline{x}_1 \in Z(\overline{G})$, each of these subgroups is normal in G. Each factor in this series is centralized by $\langle C_G(\langle a^G \rangle), x_1 \rangle$ and so we can consider the action of x_2 on each factor. This gives a series of normal subgroups of G of length c^2 each factor of which is centralized by $\langle C_G(\langle a^G \rangle), x_1, x_2 \rangle$. Continuing in this way we obtain a series of normal subgroups of G of length c^s each factor of which is central in G. This means that $\langle a^G \rangle \leq Z_{c^s}(G)$ and hence $T \leq Z_{c^s}(G)$.

Factoring out T we now assume that A is torsion-free. If $a \in A$, then $\langle a^G \rangle$ is free abelian of rank at most r. Consider $\langle a^G \rangle / \langle a^G \rangle^p$; as above this is contained in the hypercentre of $G/\langle a^G \rangle^p$ and, since it has order at most p^r, we have $[\langle a^G \rangle, {}_r G] \leq \langle a^G \rangle^p$, for each prime p. Thus $[\langle a^G \rangle, {}_r G] = 1$ and $a \in Z_r(G)$.

It follows that in the general case $A = Z(N) \leq Z_{r+c^s}(G)$ and so $N \leq Z_k(G)$, where $k = c(r + c^s)$.

PROOF of Theorem B. Let $G = H_1 \cup \ldots \cup H_n$, where each H_i is a normal nilpotent subgroup of class c. As in the proof of Theorem A we may assume that this is an irredundant covering and $|G : H_1 \cap \ldots \cap H_n| \leq n!$. Let $g \in H_1 \cap \ldots \cap H_n$ and $g_1, \ldots, g_m \in G$, where $m = n(c-1)+1$. Then c of the $g_1, \ldots, g_m$ are in one of the H_i and so

$$[g, g_1, \ldots, g_m] \in \gamma_{c+1}(H_i) = 1.$$

Hence $g \in Z_m(G)$.

It is possible to make some improvements to the bound given for the nilpotency class of G but it is not clear what the correct bound should be. For example, if G is the union of three normal subgroups each of class two, then it can be shown that G has class at most 5.

3. Proof of Theorem C

One of the weaknesses of our main theorem is the requirement that G be finitely generated and soluble. We will see in the proof that we make considerable use of the solubility condition in particular, using many properties of finite soluble groups. Our first task is to show that $G/Z^*(G)$ is finite so that we can consider a finite soluble group with trivial centre. It is sufficient to show that $G/Z^*(G)$ is periodic and in order to do this we require the characterization of $Z^*(G)$ as the set of right Engel elements given by BROOKES [1] for finitely generated soluble groups.

We begin with an elementary number-theoretic result which may be well known.

Lemma 3.1. *For any integer $k \geq 2$, there is a set of k distinct positive integers $\{a_1, \dots, a_k\}$ such that $\gcd(a_i, a_j) = |a_i - a_j|$, for all $i \neq j$.*

PROOF. We construct the sets $\{a_1, \dots, a_k\}$ by induction on k and satisfying the additional condition

$$\mathrm{lcm}\{a_1, \dots, a_k\} > \max\{a_1, \dots, a_k\}.$$

For $k = 2$, we can take the set $\{2, 3\}$.

So suppose that $k > 2$ and that we have a set $\{b_1, \dots, b_{k-1}\}$ of positive integers such that

$$b_1 < b_2 < \dots < b_{k-1} < m = \mathrm{lcm}\{b_1, \dots, b_{k-1}\}$$

and $\gcd(b_i, b_j) = b_i - b_j$, whenever $i > j$. Let $a_k = m$ and, for $i = 1, \dots, k-1$, let $a_i = m - b_{k-i}$. Then $a_1 < a_2 < \dots < a_k$. If $i < j < k$ then $\gcd(a_i, a_j) = \gcd(m - b_{k-i}, m - b_{k-j}) = \gcd(b_{k-i}, b_{k-j}) = b_{k-i} - b_{k-j} = a_j - a_i$. If $i < k$, then $\gcd(a_i, a_k) = \gcd(m - b_{k-i}, m) = b_{k-i} = a_k - a_i$.

If $\mathrm{lcm}\{a_1, \dots, a_k\} = a_k = m$, then $a_i | m$ and so $m - b_i | m$. This implies that $b_i \geq m/2$. But also $b_i | m$ and so $b_i = m/2$. This clearly can not be true for different values of b_i and since $k - 1 \geq 2$ we have a contradiction. Therefore $\mathrm{lcm}\{a_1, \dots, a_k\} > \max\{a_1, \dots, a_k\}$.

Lemma 3.2. *Let G be a finitely generated soluble group satisfying the condition $(\mathbf{N}, n)$ for some n. Then $G/Z^*(G)$ is finite.*

PROOF. We need only show that $G/Z^*(G)$ is periodic. Suppose not; then there is an element $x \in G$ such that no power of x is in $Z^*(G)$. Let $a_1, \dots, a_{n+1}$ be distinct positive integers such that $\gcd(a_i, a_j) = |a_i - a_j|$, for all $i \neq j$, and let $m = \text{lcm}\{a_1, \dots, a_{n+1}\}$. Then $x^m \notin Z^*(G)$ and since $Z^*(G) = R(G)$, the set of right Engel elements of G, there is an element $y \in G$ such that $[x^m, {}_k y] \neq 1$, for any integer k, and hence $\langle x^m, y\rangle$ is not nilpotent.

But consider the set $\{x^{a_1}y, \dots, x^{a_{n+1}}y\}$. The $(\mathbf{N}, n)$-condition implies that this set of $n+1$ elements must contain a pair which generate a nilpotent subgroup. But

$$\langle x^{a_i}y, x^{a_j}y\rangle = \langle x^{a_i - a_j}, x^{a_i}y\rangle = \langle x^{a_i-a_j}, y\rangle,$$

since $a_i - a_j$ divides a_i. Also $a_i - a_j$ divides m and so $\langle x^{a_i-a_j}, y\rangle \geq \langle x^m, y\rangle$, which is not nilpotent. We have therefore obtained a contradiction and so $G/Z^*(G)$ must be periodic and a periodic finitely generated soluble group is finite.

In fact, the proof shows that $G/Z^*(G)$ must have exponent dividing m but the value of m is large and seems to make no reduction in the estimates obtained below.

By considering the group $G/Z^*(G)$, which also satisfies the condition $(\mathbf{N}, n)$, we may now restrict our attention to a finite soluble group with trivial centre. Some of the lemmas leading to our final theorem do not require all of these conditions.

Lemma 3.3. *Let G be a finite group satisfying $(\mathbf{N}, n)$ and with $Z(G)=1$. If p is a prime dividing $|G|$, then $p < n$.*

PROOF. Let P be a Sylow p-subgroup of G and let $Q = O^P(G)$, the subgroup generated by all p'-elements of G. Then $G = PQ$ and if $[P, Q] = 1$ then $Z(P) \leq Z(G) = 1$. But $Z(P) \neq 1$ and so we must have $[P, Q] \neq 1$. Therefore there is a p-element a and a p'-element b such that $[a, b] \neq 1$. Consider the set

$$\{a, b, ab, a^2b, \dots, a^{p-1}b\}.$$

This is a set of $p+1$ elements and any two of the elements generate the subgroup $\langle a, b\rangle$ which is not nilpotent. Therefore $p+1 < n+1$ and so $p < n$.

Lemma 3.4. *Let G be a finite group satisfying $(\mathbf{N}, n)$, let P be a p-subgroup of G and x a p'-element of $N_G(P)$. Then $|P : C_P(x)| \leq n$.*

PROOF. Let $|P : C_P(x)| = k$ and let $y_1, \dots, y_k$ be coset representatives of $C_P(x)$ in P so that $y_i y_j^{-1} \notin C_P(x)$, whenever $i \neq j$. Consider the set of k elements $\{x^{y_1}, \dots, x^{y_k}\}$. If $k > n$, then there are i and j such that $\langle x^{y_i}, x^{y_j}\rangle$ is nilpotent. Thus $\langle x^{y_i y_j^{-1}}, x\rangle$ is nilpotent. But x and $x^{y_i y_j^{-1}}$ are distinct p'-elements of $P\langle x\rangle$. Thus $\langle x\rangle$ is a maximal p'-subgroup of $P\langle x\rangle$ and so is the unique maximal p'-subgroup of $\langle x^{y_i y_j^{-1}}, x\rangle$. Therefore $x^{y_i y_j^{-1}} \in \langle x\rangle$ and so $y_i y_j^{-1} \in N_P(\langle x\rangle) = C_P(x)$ contrary to the choice of $y_1, \dots, y_k$. Hence $k \leq n$.

Lemma 3.5. *Let G be a finite group satisfying (N, n). Let $R = R_1 \times \ldots \times R_k$ be an elementary abelian p-subgroup of G and let $x_1, \ldots, x_k$ be elements of an abelian p'-subgroup Q such that $[R_i, x_i]{=}R_i$ and $[R_i, x_j] = 1$, if $i \neq j$. Then $|R| \leq n$.*

PROOF. Let $x = x_1 x_2 \ldots x_k$; then x is a p'-element normalizing the p-subgroup R. By Lemma 3.4, we have $|R : C_R(x)| \leq n$. Let $y \in C_R(x)$; then we can write $y = y_1 y_2 \ldots y_k$, with $y_i \in R_i$. Then

$$1 = [y, x] = [y_1 y_2 \ldots y_k,\ x_1 x_2 \ldots x_k] = [y_1, x_1] \ldots [y_k,\ x_k].$$

Since $R = R_1 \times \ldots \times R_k$ and $[y_i, x_i] \in R_i$, we have $[y_i, x_i] = 1$ and so $y_i \in C_{R_i}(x_i) = 1$. Hence $C_R(x) = 1$ and so $|R| \leq n$.

Lemma 3.5 will be used to consider various subgroups acting on the elementary abelian p-subgroup R of order at most n. Bounds for the orders of such groups are easily derived from known results on the orders of soluble linear groups.

Lemma 3.6. *Let p be a prime such that $p^k \leq n$. Then*

(i) a p-subgroup of $\mathrm{GL}(k, p)$ has order at most $n^{\frac{1}{2}(\log_2 n - 1)}$,

(ii) an abelian p'-subgroup of $\mathrm{GL}(k, p)$ has order at most $n - 1$,

(iii) a soluble p'-subgroup of $\mathrm{GL}(k, p)$ has order less than $\frac{1}{2} n^{9/4}$,

(iv) a soluble subgroup of $\mathrm{GL}(k, p)$ has order at most $n^{\frac{1}{2}(\log_2 n + 1)}$.

PROOF. (i) $|\mathrm{GL}(k, p)| = (p^k - 1)(p^k - p) \ldots (p^k - p^{k-1})$ and so the Sylow p-subgroups of $\mathrm{GL}(k, p)$ have order $p^{\frac{1}{2}k(k-1)} = n^{\frac{1}{2}(k-1)}$ and $k - 1 = \log_p n - 1$.

(ii) A p'-group acting on a $\mathrm{GF}(p)$-vector space is completely reducible. If G is an abelian group acting faithfully and irreducibly on a vector space V then, for each $x \in G$, $C_V(x) = 0$. It follows that if v is a nonzero element of V then the elements vg, $g \in G$, are distinct. Hence $|G| \leq |V| - 1$. A simple induction argument extends this to completely reducible groups.

(iii) A soluble p'-subgroup of $\mathrm{GL}(k, p)$ is completely reducible and so we can quote the result of T. R. WOLF [14, Theorem 3.1].

(iv) To prove (iv) we use the characterization of the soluble subgroups of maximal order in $\mathrm{GL}(k, p)$ given by A. MANN [7]. For $p > 5$ these are the triangular subgroups which have order $(p-1)^k\ p^{\frac{1}{2}k(k-1)} < p^{\frac{1}{2}k(k+1)} = n^{\frac{1}{2}(\log_p n + 1)}$. For $p = 5, 3$ or 2 the groups are block triangular, the blocks on the diagonal being 2×2 matrix groups of order 96, 48 or 6 respectively. Writing $96 < 5^3$ and $6 < 2^3$ there is no difficulty in obtaining the given result for $p = 5$ or 2. For $p = 3$, the maximal size of a soluble subgroup of $\mathrm{GL}(k, 3)$ is $48^{[\frac{1}{2}k]}\ 3^{\frac{1}{2}k(k-1) - [\frac{1}{2}k]} = 16^{[\frac{1}{2}k]}\ 3^{\frac{1}{2}k(k-1)} < 3^{3k/2 + \frac{1}{2}k(k-1)} = 3^{\frac{1}{2}k(k+2)} = n^{\frac{1}{2}(\log_3 n + 2)}$. But provided $n \geq 3^2$ we have $\log_3 n < \log_2 n - 1$

and so $\log_3 n + 2 < \log_2 n + 1$. If $n = 3$ then $\mathrm{GL}(1,3)$ has order 2 which is less than $3^{\frac{1}{2}(1+\log_2 3)}$.

Lemma 3.7. *Let $G = RX$ be an extension of an elementary abelian p-group R by an abelian p'-subgroup X such that X acts faithfully on R and $R = [R, X]$. If G satisfies (N, n) then $|X| \leq n - 1$ and $|R| \leq n^{n-2}$.*

PROOF. Choose a non-trivial element $x_1 \in X$; then $R = R_1 \times S_1$ where $R_1 = [R, x_1] = [R_1, x_1]$ and $S_1 = C_R(x_1)$. By Lemma 3.5, $|R_1| \leq n$.

Let $T_1 = C_X(R_1)$. If $T_1 = 1$, then $|X| \leq n-1$, by Lemma 3.6 (ii), and since $1 = \bigcap_{x\in X-1} C_R(x)$ and $|R : C_R(x)| \leq n$, by Lemma 3.4, we have $|R| \leq n^{n-2}$.

So suppose that $T_1 > 1$ and choose a non-trivial element $x_2 \in T_1$. Now let $S_1 = R_2 \times S_2$, where $R_2 = [S_1, x_2] = [R_2, x_2] = [R, x_2]$ and $S_2 = C_{S_1}(x_2)$. Note that $R = R_2 \times S_2 \times R_1$ and $S_2 \times R_1 = C_R(x_2)$.

Now we have $R_1 \times R_2,\ x_1, x_2$ as in Lemma 3.5 and so $|R_1 \times R_2| \leq n$. As above, we may suppose that $T_2 = C_X(R_1 \times R_2) > 1$ and choose a non-trivial element $x_3 \in T_2$.

Continuing in this way, we construct $R_1 \times \ldots \times R_k$ and non-trivial elements $x_1, x_2, \ldots, x_k \in X$ such that $[R_i, x_i] = R_i$ and $[R_i, x_j] = 1$, whenever $i \neq j$. If $T_i = C_X(R_1 \times \ldots \times R_i)$, then $X > T_1 > \ldots > T_i$ so that there is a k such that $T_k = 1$. By Lemma 3.5, $|R_1 \times \ldots \times R_k| \leq n$ so that $|X| \leq n - 1$ and hence $|R| \leq n^{n-2}$.

PROOF of Theorem C. Lemma 3.2 shows that, by factoring out $Z^*(G)$, we may assume that G is a finite soluble group with trivial centre satisfying the condition (N, n).

Let $H_p/O_{p'}(G)$ be the hypercentre of $G/O_{p'}(G)$. Then, since G is finite, there is a positive integer m such that

$$[H_p,\, {}_mG] \leq O_{p'}(G),$$

for all $p \in \pi(G)$. Hence

$$\left[\bigcap_{p\in\pi(G)} H_p,\, {}_mG\right] \leq \bigcap_{p\in\pi(G)} O_{p'}(G) = 1$$

and so $\bigcap_{p\in\pi(G)} H_p = Z^*(G) = 1$. Thus, if we can show that there is a function g such that $|G/H_p| \leq g(n)$ for all $p \in \pi(G)$, then, by Lemma 3.3 we shall be able to conclude that $|G| \leq g(n)^{\pi(n)}$, where $\pi(n)$ is the number of primes less than n.

To find such a bound for $|G/H_p|$, we consider the factor group $H = G/O_{p'}(G)$, which is a finite soluble group with Fitt $H = O_p(H) = P$, say. Then H/P acts faithfully on the $\mathrm{GF}(p)$-vector space $P/\Phi(P)$ (e.g. [3] Theorem 6.3.4).

By factoring out $\Phi(P)$, we may assume for the moment that P is an elementary abelian normal p-subgroup of $H, P = O_p(H)$ and $C_H(P) = P$.

Let Q/P be the socle of H/P so that Q/P is an abelian p'-subgroup. We may write $Q = PX$, where X is an abelian p'-subgroup of Q. Let $R = [P, Q]$ so that $P = R \times C_P(Q)$. If $C = C_H(R)$, then $C \cap Q$ centralizes $R \times C_P(Q) = P$ and so $C \cap Q = P$. It follows that $C_H(R) = P$ and so H/P acts faithfully on R. Now R and X satsify the conditions of Lemma 3.7 and so $|R| \leq n^{n-2}$.

Since H/P acts faithfully on R, Lemma 3.6 (iv) shows that

$$|H/P| \leq n^{\frac{1}{2}(n-2)(1+(n-2)\log n)} = a(n),$$

say.

We now return to the general situation for H where Fitt $H = O_p(H) = P$ and no longer assume that P is elementary abelian. We still have $|H/P| \leq a(n)$. Let T be a Sylow p'-subgroup of H so that T is a soluble p'-group acting faithfully on R (which is now a subgroup of $P/\Phi(P)$). By Lemma 3.6 (iii),

$$|T| \leq \frac{1}{2} n^{9(n-2)/4} = b(n),$$

say, and T can be generated by $\log b(n)$ elements.

By Lemma 3.4, $|P : C_P(t)| \leq n$ for each $t \in T$ and so

$$|P : C_P(T)| \leq n^{\log b(n)} = c(n),$$

say. If S is a Sylow p-subgroup of H, then $H = ST$ and $[C_P(T), {}_mH] = [C_P(T), {}_mS] = 1$, for some m. Thus $C_P(T) \leq Z^*(H)$ and so

$$|G/H_p| = |H/Z^*(H)| \leq a(n)c(n) = g(n).$$

It now follows that $|G| \leq g(n)^{\pi(n)}$.

A tidier bound for $|G|$ can be obtained by noting that

$$\begin{aligned}\log_n(a(n)c(n)) &= \tfrac{1}{2}(n-2)(1+(n-2)\log n) + 9\{(n-2)\log n\}/4 - 1 \\ &\leq \tfrac{1}{4}(n-2)\{2 + 2(n-2)n + 9n\} - 1 \\ &\leq \tfrac{1}{4}\{2n^3 + n^2 - 8n - 8\} \\ &\leq n^3.\end{aligned}$$

So $|G| \leq (n^{n^3})^{\pi(n)} \leq n^{n^4}$.

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