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A note on the strong Schwarz inequality in Hilbert A-modules

By LAJOS MOLNÁR (Debrecen)

Abstract. The aim of this note is to show that one of the axioms of Hilbert A-modules due to SAWOROTNOW is redundant and, at the same time, to give a direct proof of the so-called strong Schwarz inequality which avoids the explicit use of the spectral representation of a normal element of a proper H^*-algebra.

Throughout this note A denotes a proper H^*-algebra, i.e. A is a Banach algebra whose norm is a Hilbert space norm and which has an involution $^* : x \longmapsto x^*$ such that $(x, yz^*) = (xz, y) = (z, x^*y)$ for all $x, y, z \in A$. A projection in A is a nonzero element e of A for which $e^2 = e = e^*$; e is called primitive if it cannot be represented as a sum of two mutually orthogonal projections of A. In the sequel $\{e_\alpha : \alpha \in \Lambda\}$ stands for a fixed maximal family of mutually orthogonal projections. An element $a \in A$ is called positive $(a \geq 0)$ if $(ax, x) \geq 0$ holds whenever $x \in A$. For every $a \in A$ there exists a unique positive element $|a|$ of A such that $|a|^2 = a^*a$. If $a \in A$, then let L_a be the bounded linear operator defined by $L_a x = ax \, (x \in A)$ and denote $C(A)$ the closure of $\{L_a : a \in A\}$ in the norm of the B^*-algebra of bounded linear operators on A.

By the trace-class of A we mean the ideal $\tau(A) = \{xy : x, y \in A\}$. There is a positive linear functional tr (called trace) on $\tau(A)$ for which $tr\, xy^* = tr\, y^*x = (x, y)$, $\overline{tr\, a} = tr\, a^*$ whenever $x, y \in A$, $a \in \tau(A)$. Then one can define a norm τ on $\tau(A)$ by letting $\tau(a) = tr|a| \; (a \in \tau(A))$.

As for the detailed discussion of H^*-algebras and their trace-classes we refer to [1], [3] and [4].

Definition. Let H be a (right) A-module. Suppose that $[\,.\,,.\,] : H \times H \to \tau(A)$ is a function with the following properties:

$$[f, g+h] = [f,g] + [f,h]; \tag{1}$$

$$[f, ga] = [f,g]a; \tag{2}$$

$$[f,g]^* = [g,f]; \tag{3}$$

$$[f,f] \geq 0 \tag{4}$$

for every $f, g, h \in H$ and $a \in A$. Then $[\,.\,,.\,]$ is called a ($\tau(A)$-valued) generalized semi-inner product on H.

In the proof of our theorem we need the following

Lemma. *If $x \in A$, then*

$$|([f,g]x, x)|^2 \leq ([f,f]x,x)\ ([g,g]x,x)$$

holds for every $f, g \in H$.

Proof. Let $f, g \in H$. Suppose that $e \in A$ is a projection. Then one can easily verify that

$$\begin{aligned} 0 &\leq ([f + g(\lambda e), f + g(\lambda e)]e, e) \\ &= ([f,f]e,e) + 2\,\mathrm{Re}\,\lambda([f,g]e,e) + |\lambda|^2([g,g]e,e) \end{aligned}$$

for each $\lambda \in \mathbf{C}$. It implies that

$$|([f,g]e,e)|^2 \leq ([f,f]e,e)\ ([g,g]e,e).$$

Substitute now f, g and e by fx, gx and the sum of a finite subset of $\{e_\alpha : \alpha \in \Lambda\}$, respectively. Then, by [1, Theorem 4.1], we have the desired inequality.

Remark. We note that the proof of this lemma would be much simpler if there were an adequate vector space structure on H.

Remark also that the inequality in the Lemma directly implies the validity of the so-called weak Schwarz inequality [2, Axiom 5 in Definition 1].

Now we are in a position to prove the following

Theorem. *If $f, g \in H$, then $(\tau[f,g])^2 \leq \tau[f,f]\tau[g,g]$.*

Proof. Let $a \in A$. Suppose that F is a finite subset of Λ. If $e = \sum\{e_\alpha : \alpha \in F\}$, then, by the Lemma, we have

$$\begin{aligned} \Big|\sum(a[f,g]e_\alpha, e_\alpha)\Big|^2 &= |(a[f,g]e,e)|^2 \leq \\ \leq ([f,f]a^*e, a^*e)([g,g]e,e) &\leq tr\, a[f,f]a^*\tau[g,g] = \\ &= tr\, a^*a[f,f]\tau[g,g], \end{aligned}$$

where the involved sum is taken over F. By [3, Corollary 2 and Lemma 5], the last expression is not greater than $\|L_a\|^2\tau[f,f]\tau[g,g]$ which implies that

$$|trS[f,g]|^2 \leq \|S\|^2\tau[f,f]\tau[g,g]$$

holds for every $S \in C(A)$. The inequality follows from [4, Theorem 1].

Remark. Using a similar argument it is easy to show that

$$[f,g]^*[f,g] \leq (\tau[f,f])\ [g,g]$$

$(f,g \in H)$. In fact, let $x \in A$. Then, by the Lemma, we have

$$([f[f,g],g]x,x)^2 \leq ([f,f][f,g]x,[f,g]x)\ ([g,g]x,x) \leq$$
$$\leq \|[f,f]^{\frac{1}{2}}\|^2\ \|[f,g]x\|^2([g,g]x,x) = \tau[f,f]([f[f,g],g]x,x)\ ([g,g]x,x)$$

where $[f,f]^{\frac{1}{2}}$ denotes the unique positive element of A whose square is $[f,f]$. Now the statement is obvious.

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LAJOS MOLNÁR
LAJOS KOSSUTH UNIVERSITY
INSTITUTE OF MATHEMATICS
H–4010, DEBRECEN, P.O.BOX 12.
HUNGARY

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