Year: 2022 | Vol.: 100 | Fasc.: 1-2

Title: Sharp inequalities for sine polynomials

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Let $F_n(x) = \sum_{k=1}^{n} \frac{\sin(kx)}{k}$ and $C_n(x) = \sum_{k=1}^{n} \frac{\sin((2k-1)x)}{2k-1}$.

The classical inequalities

$$0 < F_n(x) < \int_0^{\pi} \frac{\sin(t)}{t} dt = 1.85193\ldots \quad \text{and} \quad 0 < C_n(x) \leq 1$$

are valid for all $n \geq 1$ and $x \in (0, \pi)$. All constant bounds are sharp. We present the following refinements of the lower bound for $F_n(x)$ and the upper bound for $C_n(x)$.

(i) Let $\mu \geq 1$. The inequality $\frac{\sin(x)}{\mu - \cos(x)} < F_n(x)$ holds for all odd $n \geq 1$ and $x \in (0, \pi)$ if and only if $\mu \geq 2$.

(ii) For all $n \geq 2$ and $x \in [0, \pi]$, we have $C_n(x) \leq 1 - \lambda \sin(x)$ with the best possible constant factor $\lambda = \sqrt[3]{9} - 2$.

Moreover, we offer a companion to the inequality $C_n(x) > 0$.

(iii) Let $n \geq 1$. The inequality $0 \leq \sum_{k=1}^{n} (\delta(n) - (k-1)k) \sin((2k-1)x)$ holds for all $x \in [0, \pi]$ if and only if $\delta(n) \geq (n^2 - 1)/2$.

This extends a result of Dimitrov and Merlo, who proved the inequality for the special case $\delta(n) = n(n+1)$. The following inequality for the Chebyshev polynomials of the second kind plays a key role in our proof of (iii).

(iv) Let $m \geq 0$. For all $t \in \mathbb{R}$, we have $\left(m^2(1-t^2) - 1\right) U_m^2(t) + (m+1) U_{2m}(t) \leq m(m+1)$. The upper bound is sharp.

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