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Title: On the divisibility of matrices associated with multiplicative functions

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Suppose that n and k are positive integers. Let $S = \{x_1, \ldots, x_n\}$ be a sequence of n distinct positive integers, and let f be an integer-valued multiplicative function. The sequence S is called a divisor chain if $x_{\sigma(1)}|\ldots|x_{\sigma(n)}$ for some permutation σ on $\{1, \ldots, n\}$. We say that the sequence S consists of k coprime divisor chains if S can be partitioned as the union of k divisor chains $S_1, \ldots, S_k$ such that each element of S_i is coprime to each element of S_j for all integers i and j with $1 \le i \ne j \le k$. In this paper, we show that for any divisor chain S, the matrix $(f(S))$ with entries $f(\gcd(x_i, x_j))$ divides the matrix $(f[S])$ with entries $f(\mathrm{lcm}(x_i, x_j))$ in the ring $M_n(\mathbf{Z})$ of $n \times n$ matrices over the integers if and only if $f(\min(S))|f(x_i)$ for all integers $i \in \{1, \ldots, n\}$. This strengthens a result of Hong [14]. For any positive integer a and any sequence S consisting of two coprime divisor chains with $1 \notin S$, we show that the matrix $(f(S^a))$ divides the matrix $(f[S^a])$ in $M_n(\mathbf{Z})$, where $S^a := \{x_1^a, \ldots, x_n^a\}$. This confirms a conjecture of Chen and Hong. We show also that such factorization is no longer true in general if S consists of at least three coprime divisor chains with $1 \notin S$. We conjecture that if $k \ge 3$, then the GCD matrix (S) does not divide the LCM matrix $[S]$ in the ring $M_k(\mathbf{Z})$ if S consists of the first k odd prime numbers.

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