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Einstein-Finsler vector bundles

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Abstract. In the present paper, we shall be concerned with Einstein-Finsler bundles, and study the semi-stability of them.

§0. Introduction

Let $\pi : E \to M$ be a holomorphic vector bundle over a complex manifold M, and $p : PE \to M$ the projective bundle of E. If a complex Finsler structure F is given on E, a natural connection on the pull-back $\tilde{E} = p^{-1}E$ is defined, and the differential geometry of (E, F) has been studied (cf. Abate–Patrizio [1], Aikou [2, 3], Faran[4], Kobayashi [5, 7], Patrizio–Wong [9], Royden [10], Rund [11]).

In this paper, we are concerned with Einstein-Finsler bundles. This notion has been introduced by Kobayashi [7] as a natural generalization of Hermitian case, and the following problem is proposed:

What are algebro-geometric consequences of the Einstein-condition?

The original definition of Einstein-Finsler condition due to Kobayashi [7], however, has no invariant meanings as he pointed out himself. Hence the first purpose of this paper is to give the definition of Einstein-Finsler condition so that it has an invariant meaning, and to discuss Einstein-Finsler bundles (cf. Chapt. IV in Kobayashi [6]). We define Einstein-Finsler condition in terms of the curvature tensor of a partial

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connection, not of a Hermitian connection. The second purpose is to discuss the semi-stability of Einstein-Finsler bundle over a compact Kähler manifold.

In §1, we recall the notion of complex Finsler structures on a holomorphic vector bundle and its *partial connection*, and in §2, we shall show some properties of its curvature which plays an important role in this paper.

In §3, we introduce the notion of *Einstein-Finsler bundles* in terms of the curvature tensor field of the partial connection, and in §4, we shall show a vanishing theorem of Bochner-type for holomorphic sections of complex Finsler bundles. In §5, we are concerned with the second fundamental form of partial connections. In §6, we shall discuss the semi-stability of Einstein-Finsler bundles over a compact Kähler manifold under a special condition.

We shall introduce some basic notations. Let M be a complex manifold of dimension n, and E a holomorphic vector bundle of rank r over M. Each fibre E_z is a complex vector space of complex dimension r. In the case of $r = 1$, any complex Finsler structure is Hermitian. Hence, throughout the remainder of the paper, we shall always assume $r > 1$. We denote by $p : PE \to M$ the projective bundle associated to E, and $\tilde{E}$ the induced bundle $p^{-1}E$ over PE:

$$\begin{array}{ccc} \tilde{E} & \xrightarrow{\tilde{p}} & E \\ {\scriptstyle \tilde{\pi}}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ PE & \xrightarrow[p]{} & M \end{array}$$

We denote by LE the tautological line subbundle of $\tilde{E}$:

$$LE := \{(v, V) \in E \times PE; v \in V\}.$$

There exists a natural homomorphism $\tau : E \to LE$ which maps $E^{\times}$ biholomorphically to $LE^{\times}$, where $E^{\times}$ (resp. $LE^{\times}$) denotes the open submanifold of E (resp. LE) consisting of the non-zero elements.

Let $\{U, (z^i)\}$ be a complex coordinate system of M, and $\{\pi^{-1}(U), (z^{\alpha}, \xi^i)\}$ the induced complex coordinate system on E with respect to a holomorphic frame field $\{s_1, \ldots, s_r\}$ on U.

Definition 0.1. A *complex Finsler structure* F on E is a real valued function satisfying the following conditions:

(1) F is C^∞-class on $E^\times$;
(2) $F(z,\xi) \geq 0$, and "0" if and only if $\xi = 0$;
(3) $F(z,\lambda\xi) = |\lambda|^2 F(z,\xi)$ for $\forall\, \lambda \in \mathbb{C}$.

If a complex Finsler structure F is given on E, the norm $\|\xi\|$ of an arbitrary section $\xi \in \Gamma(E)$ is defined by $\|\xi(z)\| = \sqrt{F(z,\xi(z))}$. Moreover, a Hermitian structure h on $L(E)$ is introduced by $h(\tau(\xi(z))) = F(z,\xi(z))$, and so there exists a one-to-one corresponding between Hermitian structures on LE and Finsler structures on E (cf. Kobayashi [5]).

We shall use the following notation throughout this paper:

a^p (resp. A^p): the space of p-forms on M (resp. PE),

$a^{p,q}$ (resp. $A^{p,q}$): the space of (p,q)-forms on M (resp. PE),

$A^p(\tilde{E})$ (resp. $A^{p,q}(\tilde{E})$): the space of $\tilde{E}$-valued p-forms (resp. (p,q)-forms) on PE,

$a^p(E)$ (resp. $a^{p,q}(E)$): the space of E-valued p-forms (resp. (p,q)-forms) on M.

§1. Finsler structures and partial connections

For any local holomorphic frame field $s = (s_1,\ldots,s_r)$ of E, we denote by $(z,\xi) = (z^1,\ldots,z^n,\xi^1,\ldots,\xi^r)$ the induced local coordinate system on E, and by $(z,[\xi])$ the point of PE represented by (z,ξ). A complex Finsler structure F is said to be *convex* if the Hermitian matrix $(F_{i\bar{j}})$ defined by

$$F_{i\bar{j}} := \frac{\partial^2 F}{\partial \xi^i \partial \bar{\xi}^j} \tag{1.1}$$

is positive-definite. In the following, we always assume the convexity of F. By the condition (3) in Definition 0.1, matrix components $F_{i\bar{j}}$ defined by (1.1) are functions on PE.

Putting $Z^i = \xi^i \circ p$, we take $(z,[\xi],Z) = (z^1,\ldots,z^n,\xi^1 : \cdots : \xi^r, Z^1,\ldots,Z^r)$ as a local coordinate system for $\tilde{E}$. Then, for $\forall\, Z, W \in A^0(\tilde{E})$, a Hermitian structure H on $\tilde{E}$ is defined by

$$H(Z,W) := \sum_{i,j} F_{i\bar{j}} Z^i \bar{W}^j.$$

The Hermitian connection $\nabla : A^0(\tilde{E}) \to A^1(\tilde{E})$ in $(\tilde{E}, H)$ is given by the form

$$\theta^i_j := \sum_l F^{\bar{l}i} \partial F_{j\bar{l}} = \sum_{l,\alpha} F^{\bar{l}i} \frac{\partial F_{j\bar{l}}}{\partial z^\alpha} dz^\alpha + \sum_{l,k} F^{\bar{l}i} \frac{\partial F_{j\bar{l}}}{\partial \xi^k} d\xi^k. \tag{1.2}$$

A Finsler structure F on E is a Hermitian structure on E if and only if the second term in (1.2) vanishes identically.

We shall introduce a *partial connection* in $(\tilde{E}, H)$ which is the main tool in this paper. The partial connection D is defined as a covariant derivation in transversal direction to the fibres of PE. For the cotangent bundle T^*_{PE} of PE, we shall introduce a C^∞-splitting by defining the left splitting σ of the exact sequence

$$0 \longrightarrow \mathcal{H}^* \longrightarrow T^*_{PE} \underset{\sigma}{\overset{\longrightarrow}{\longleftarrow}} (\ker dp)^* \longrightarrow 0$$

by

$$\sigma(d\xi^i) := d\xi^i + \sum_j \theta^i_j \xi^j.$$

The set of local 1-forms $\{dz^1, \ldots, dz^n, \theta^1, \ldots, \theta^r\}$, $\theta^i := \sigma(d\xi^i)$, is a local co-frame field for T^*_{PE}, and it defines a C^∞-splitting

$$T^*_{PE} = \mathcal{H}^* \oplus \mathcal{V}^*, \tag{1.3}$$

where $\mathcal{H}^*$ is locally spanned by $\{dz^1, \ldots, dz^n\}$, and $\mathcal{V}^*$ by $\{\theta^1, \ldots, \theta^r\}$. We shall denote by $p'_{\mathcal{H}} : A^0(T^*_{PE}) \to A^0(\mathcal{H}^*)$ and $p''_{\mathcal{H}} : A^0(T^*_{PE}) \to A^0(\bar{\mathcal{H}}^*)$ the natural projections with respect to the splitting (1.3) respectively. Then we shall define a homomorphism $D := D' + D'' : A^0(\tilde{E}) \to A^0\Big((\mathcal{H} \oplus \bar{\mathcal{H}})^* \otimes \tilde{E}\Big)$ so that the following diagram is commutative.

$$\begin{array}{ccc} A^0(\tilde{E}) & \xrightarrow{\nabla} & A^0\Big((T_{PE} \oplus \bar{T}_{PE})^* \otimes \tilde{E}\Big) \\ \Big\| & & \Big\downarrow{\scriptstyle p_{\mathcal{H}} \otimes 1} \\ A^0(\tilde{E}) & \xrightarrow[D]{} & A^0\Big((\mathcal{H} \oplus \bar{\mathcal{H}})^* \otimes \tilde{E}\Big) \end{array}$$

For $\forall\, Z \in A^0(\tilde{E})$, we shall define

$$DZ := (p_{\mathcal{H}} \otimes 1) \circ \nabla Z.$$

In the following, we shall show the local expression of D. For this purpose, we rewrite the form θ^i_j as follows:

$$\theta^i_j = \sum_\alpha \Gamma^i_{j\alpha} dz^\alpha + \sum_k C^i_{jk}\theta^k, \tag{1.4}$$

where we put

$$\Gamma^i_{j\alpha} := \sum_l F^{\bar{l}i}\left(\frac{\partial F_{j\bar{l}}}{\partial z^\alpha} - \sum_m N^m_\alpha \frac{\partial F_{j\bar{l}}}{\partial \xi^m}\right), \quad C^i_{jk} := \sum_l F^{\bar{l}i}\frac{\partial F_{j\bar{l}}}{\partial \xi^k},$$

$$N^m_\alpha := \sum_j \xi^j \Gamma^m_{j\alpha} = \sum_{m,r} F^{\bar{r}m}\frac{\partial F_{l\bar{r}}}{\partial z^\alpha}\xi^l.$$

We define operators $\partial_{\mathcal{H}}, \partial_{\mathcal{V}} : A^0 \to A^{1,0}$ by $\partial_{\mathcal{H}} := p'_{\mathcal{H}} \circ \partial$, and by $\partial_{\mathcal{V}} := \partial - \partial_{\mathcal{H}}$ respectively. Then

$$\partial_{\mathcal{H}} f := \sum_\alpha \frac{\delta f}{\delta z^\alpha} dz^\alpha := \sum_\alpha \left(\frac{\partial f}{\partial z^\alpha} - \sum_m N^m_\alpha \frac{\partial f}{\partial \xi^m}\right) dz^\alpha,$$

$$\partial_{\mathcal{V}} f := \sum_m \frac{\partial f}{\partial \xi^m}\theta^m = \partial f - \partial_{\mathcal{H}} f$$

for $\forall\, f \in A^0$ respectively. Then, by definition of $\partial_{\mathcal{H}}$, we get the following:

Proposition 1.1. *For the given complex Finsler structure, the following identity holds:*

$$\partial_{\mathcal{H}} F \equiv 0.$$

In terms of $\partial_{\mathcal{H}}$, the first term of (1.4) is written as follows:

$$\omega^i_j := \sum_\alpha \Gamma^i_{j\alpha} dz^\alpha = \sum_r F^{\bar{r}i}\partial_{\mathcal{H}} F_{j\bar{r}}. \tag{1.5}$$

The $(1,0)$-part $D' : A^0(\tilde{E}) \to A^{1,0}(\tilde{E})$ and the $(0,1)$-part $D'' : A^0(\tilde{E}) \to A^{0,1}(\tilde{E})$ of D are given by

$$D'Z := (p'_{\mathcal{H}} \otimes 1)(\nabla Z), \quad D''Z := (p''_{\mathcal{H}} \otimes 1)(\nabla Z)$$

respectively. Their local expression are given by

$$D'Z = \sum_i \left(\partial_{\mathcal{H}} Z^i + \sum_m Z^m \omega^i_m\right) \otimes s_i := \sum_\alpha \left(D_\alpha Z^i\right) dz^\alpha \otimes s_i,$$
$$D''Z = \sum_i \bar{\partial}_{\mathcal{H}} Z^i \otimes s_i$$

for $\forall\, Z = \sum_i Z^i s_i \in A^0(\tilde{E})$. $D = D' + D''$ is a *partial connection* in the following sense.

Proposition 1.2. *Let (E, F) be a complex Finsler bundle. Then the homomorphism $D : A^0(\tilde{E}) \to A^1(\tilde{E})$ is uniquely determined in (E, F) and satisfies*

$$D(fZ) = d_{\mathcal{H}} f \otimes Z + f DZ$$

for $\forall\, f \in A^0$ and $\forall\, Z \in A^0(\tilde{E})$, where we put $d_{\mathcal{H}} := \partial_{\mathcal{H}} + \bar{\partial}_{\mathcal{H}}$.

Proof. Since the operator $\partial_{\mathcal{H}}$ is uniquely determined from F, and by (1.5), the operator D is also determined uniquely from F. The second assertion is obvious from its definition. □

The Hermitian connection ∇ in $(\tilde{E}, H)$ is metrical:

$$dH(Z, W) = H(\nabla Z, W) + H(Z, \nabla W).$$

The partial connection D, however, is partially metrical, that is,

Proposition 1.3. *For $\forall\, Z, W \in A^0(\tilde{E})$, the following identities hold:*

$$d_{\mathcal{H}} H(Z, W) = H(DZ, W) + H(Z, DW), \tag{1.6}$$

$$D' F_{i\bar{j}} := \partial_{\mathcal{H}} F_{i\bar{j}} - \sum_m F_{m\bar{j}} \omega^m_i = 0. \tag{1.7}$$

§2. Curvature of partial connection D

In this section, we shall state some important properties of the partial connection D. For this purpose, we shall prepare some lemmas.

First, extending the operator $\partial_{\mathcal{H}}$ to the space A^p, we have

Lemma 2.1. *The* $(1,0)$*-form* ω^i_j *defined by* (1.5) *satisfies*

$$\partial_{\mathcal{H}}\omega^i_j + \sum_r \omega^i_r \wedge \omega^r_j = 0.$$

Proof. This equality is obtained by direct calculations. In fact, if we write

$$\partial_{\mathcal{H}}\omega^i_j + \sum_r \omega^i_r \wedge \omega^r_j := \sum_{\alpha,\beta} R^i_{j\alpha\beta} dz^\alpha \wedge dz^\beta,$$

we get

$$R^i_{j\alpha\beta} = \frac{\delta}{\delta z^\alpha}\mathit{\Gamma}^i_{j\beta} - \frac{\delta}{\delta z^\beta}\mathit{\Gamma}^i_{j\alpha} + \sum_l \mathit{\Gamma}^i_{l\alpha}\mathit{\Gamma}^l_{j\beta} - \sum_l \mathit{\Gamma}^i_{l\beta}\mathit{\Gamma}^l_{j\alpha}.$$

From $\mathit{\Gamma}^i_{j_\alpha} = \sum\limits_l F^{\bar{m}i}\delta F_{j\bar{m}}/\delta z^\alpha$, direct calculations give

$$R^i_{j\alpha\beta} = \sum_{l,m} C^i_{jl} R^l_{\alpha\beta},$$

where we put

$$R^i_{\alpha\beta} := \frac{\delta N^i_\alpha}{\delta z^\beta} - \frac{\delta N^i_\beta}{\delta z^a}.$$

On the other hand, by definition, we get also the following relation:

$$\sum_j \xi^j R^i_{j\alpha\beta} = R^i_{\alpha\beta}.$$

These relations and the identity $\sum\limits_j \xi^j C^i_{jl} \equiv 0$ imply $R^i_{j\alpha\beta} \equiv 0$. □

By the proof above, we also have the following

Lemma 2.2. $\partial^2_{\mathcal{H}} f \equiv 0$ *for* $\forall\, f \in A^0$.

Proof. By direct calculations, we have

$$\begin{aligned}\partial^2_{\mathcal{H}} f &= \frac{1}{2}\sum_{m,\alpha,\beta}\left(\frac{\delta N^m_\beta}{\delta z^\alpha} - \frac{\delta N^m_\alpha}{\delta z^\beta}\right)\frac{\partial f}{\partial \xi^m} dz^\alpha \wedge dz^\beta \\ &= \frac{1}{2}\sum_{m,\alpha,\beta} R^m_{\alpha\beta}\frac{\partial f}{\partial \xi^m} dz^\alpha \wedge dz^\beta.\end{aligned}$$

Since $R^m_{\alpha\beta} \equiv 0$, we get $\partial^2_{\mathcal{H}} f \equiv 0$. □

Secondly we shall extend D to the space $A^p(\tilde{E})$ in the usual way, that is, for $\forall \sum_i \phi(z,[\xi])^i \otimes s_i \in A^p(\tilde{E})$,

$$D'\Big(\sum_i \phi^i \otimes s_i\Big) = \sum_i \partial_{\mathcal{H}}\phi^i \otimes s_i + (-1)^p \sum_i \phi^i \wedge D's_i,$$

$$D''\Big(\sum_i \phi^i \otimes s_i\Big) = \sum_i \bar{\partial}_{\mathcal{H}}\phi^i \otimes s_i.$$

Then we introduce an $\mathrm{End}(\tilde{E})$-valued $(1,1)$-form $R = (R^i_j)$ by

$$\Omega^i_j := \bar{\partial}_{\mathcal{H}}\omega^i_j = \sum_{\alpha,\beta} R^i_{j\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta, \tag{2.1}$$

where we put

$$R^i_{j\alpha\bar{\beta}} := -\frac{\delta \Gamma^i_{j\alpha}}{\delta \bar{z}^\beta} = -\sum_m F^{\bar{m}i}\left(\frac{\delta^2 F_{j\bar{m}}}{\delta z^\alpha \delta \bar{z}^\beta} - \sum_{k,l} F^{\bar{l}k}\frac{\delta F_{j\bar{l}}}{\delta z^\alpha}\,\frac{\delta F_{k\bar{m}}}{\delta \bar{z}^\beta}\right).$$

Then we have the following fundamental theorem.

Theorem 2.1. *The partial connection $D = D' + D''$ satisfies*

$$D'D' \equiv 0,\ D''D'' \equiv 0,\ DD = D'D'' + D''D',$$

and for $\forall\, Z \in A^0(\tilde{E})$

$$DDZ = R(Z), \tag{2.2}$$

where $R \in A^{1,1}(\mathrm{End}(\tilde{E}))$ is defined by

$$R(Z) = \sum_{i,j} Z^j \Omega^i_j \otimes s_i.$$

Proof. The first assertion is directly derived from Lemma 2.1 and 2.2. We shall prove the equation (2.2) only. Since $DD(fZ) = fDDZ$ for $\forall\, f \in A^0$, it is sufficient to prove $DDs_i = \sum_m \Omega^m_i s_m$. By definition of D' and D'', we get

$$\begin{aligned}(D'D'' + D''D')s_i = D''D's_i &= D''\Big(\sum_m \omega^m_i s_m\Big)\\ &= \bar{\partial}_{\mathcal{H}}\Big(\sum_m \omega^m_i s_m\Big) = \sum_m \Omega^m_i s_m.\end{aligned}$$

So we have completed the proof. □

Proposition 2.1. *The partial connection D satisfies*

$$\partial_{\mathcal{H}}\bar{\partial}_{\mathcal{H}}H(Z,Z) = H(DZ,DZ) - H(R(Z),Z)$$

for any holomorphic section Z of $\tilde{E}$,

In this paper, we call R the *curvature* of D.

§3. Einstein-Finsler condition

Let (E,F) be a complex Finsler bundle over a Hermitian manifold (M,g), $g = \sum_{\alpha,\beta} g_{\alpha\bar{\beta}}(z)dz^{\alpha} \otimes d\bar{z}^{\beta}$. We shall define the *mean curvature* $K_{i\bar{j}}$ of (E,F) by the partial mean curvature of $(\tilde{E},H)$, that is, the g-trace

$$K^i_j := \sum_{\alpha,\bar{\beta}} g^{\bar{\beta}\alpha} R^i_{j\alpha\bar{\beta}},$$

where $R^i_{j\alpha\bar{\beta}}$ is the *curvature tensor* defined by (2.1). Putting $K_{i\bar{j}} := \sum_m F_{m\bar{j}} K^m_i$, we shall define a Hermitian form K by

$$K(Z,W) = \sum_{i,j} K_{i\bar{j}} Z^i \bar{W}^j$$

for $\forall\, Z,W \in A^0(\tilde{E})$.

Definition 3.1. A complex Finsler bundle (E,F) is said to be *weakly Einstein-Finsler* if the partial mean curvature K of $(\tilde{E},H)$ satisfies

$$K_{i\bar{j}} = \varphi(z)F_{i\bar{j}} \tag{3.1}$$

for a function φ on M. If the factor φ is constant, (E,F) is said to be *Einstein-Finsler.*

Remark 3.1. (1) We note that the original definition of K^i_j of Kobayashi [7] has no invariant meaning, since the quantities $R^i_{j\alpha\bar{\beta}}$ in [7] is not a tensor field (See (5.6) in [7]). Our definition of K^i_j, however, has an invariant meaning because of the tensority of $R^i_{j\alpha\bar{\beta}}$.

(2) If the given F is a Hermitian structure, that is, $F(z,\xi) = \sum_{i,j} h_{i\bar{j}}(z)\xi^i\bar{\xi}^j$, the curvature $\Omega^i_j = \sum_{\alpha,\beta} R^i_{j\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$ of D is just the one of $h = (h_{i\bar{j}})$. Hence our definition is a natural generalization of Hermitian case.

In the rest of this section, we are concerned with conformal changes of complex Finsler structures. A conformal change of F is defined by $F \to \tilde{F} := e^{u(z)}F$ for a smooth function $u(z)$ on M.

Lemma 3.1. *The curvature $\tilde{R}$ of $\tilde{D}$ in $(E, \tilde{F})$ is given by*

$$\tilde{\Omega}^i_j = \Omega^i_j + \bar{\partial}\partial u \delta^i_j.$$

Proof. Since the connection form $\tilde{\omega}^i_j$ of $\tilde{D}$ is given by $\tilde{\omega}^i_j = \omega^i_j + \partial u \delta^i_j$, the functions $\tilde{N}^i_\alpha$ derived from $\tilde{F}$ is given by

$$\tilde{N}^i_\alpha = N^i_\alpha + \frac{\partial u}{\partial z^\alpha}\xi^i.$$

Hence the operator $\partial_{\tilde{\mathcal{H}}}$ satisfies

$$\partial_{\tilde{\mathcal{H}}} f = \partial_{\mathcal{H}} f - \sum_m \xi^m \frac{\partial f}{\partial \xi^m} \partial u$$

for $\forall\, f \in A^0$. By this relation and the formula

$$\sum_m \frac{\partial \Gamma^i_{j\alpha}}{\partial \bar{\xi}^m} \bar{\xi}^m \equiv 0,$$

we get easily our assertion. □

By this lemma, the mean curvature $\tilde{K}^i_j$ of $(E, \tilde{F})$ is given by

$$\tilde{K}^i_j = K^i_j + \Box u \delta^i_j, \tag{3.2}$$

where we put

$$\Box u := -\sum_{\alpha,\beta} g^{\bar{\beta}\alpha} \frac{\partial^2 u}{\partial z^\alpha \partial \bar{z}^\beta}.$$

Hence we have

Proposition 3.1. *Let (E, F) be a weakly Einstein-Finsler bundle with factor φ. Then $(E, e^{u(z)}F)$ satisfies the Einstein-Finsler condition* (3.1) *with factor $\varphi + \Box u$.*

Moreover we have

Proposition 3.2. *Let (E, F) be a weakly Einstein-Finsler bundle over a compact Kähler manifold (M, Φ). Then there exists a conformal change of $F \to \tilde{F}$ such that $(E, \tilde{F})$ is an Einstein-Finsler bundle.*

Proof. The proof is similar to Proposition 2.4 of Kobayashi [6]. For the constant c defined by

$$c\int_M \Phi^n = \int_M \varphi\Phi^n,$$

the harmonic theory on M implies that there exists a function $u(z)$ on M satisfying $c - \varphi(z) = \Box u$. If we consider the conformal change $F \to \tilde{F} := e^{u(z)}F$ for this function $u(z)$, Proposition 3.1 implies $\tilde{K}^i_j = c\delta^i_j$, and so $(E, \tilde{F})$ is an Einstein-Finsler bundle with constant factor c. □

Remark 3.2. In the case of $F = \sum_{i,j} h_{i\bar{j}}(z)\xi^i\overline{\xi^j}$, the constant c is given by

$$c = \frac{\int_M \varphi\Phi^n}{\int_M \Phi^n} = \frac{2n\pi \deg(E)}{r \cdot \mathrm{vol}(M)}, \tag{3.3}$$

where the degree $\deg(E)$ of E is defined by

$$\deg(E) := \int_M c_1(E) \wedge \Phi^{n-1}.$$

Hence the constant c depends only on the cohomology class of Φ and the first Chern class $c_1(E)$, not on the Hermitian structure h (cf. Kobayashi [6], p. 104).

The mean curvature K^i_j in (3.1) may be considered as an endomorphism of the bundle E. We do not know whether there exists an Hermitian structure on E whose mean curvature is K^i_j in (3.1), except the case where (E, F) is modeled on a complex Minkowski space (cf. §6). Hence, in general, the constant c in Proposition 3.4 depends on the given Finsler structure F.

In general, for the given two Finsler bundles (E, F) and (E', F'), we do not know the natural way to define a Finsler structure on the tensor product $E \otimes E'$ in a computable form. In the case where E' is a line bundle L, however, we can define a Finsler structure $F^{E\otimes L}$ on $E \otimes L$ as follows. Since any Finsler structure F^L on a line bundle L is a Hermitian structure, for any section λ of L we may put

$$F^L(z,\lambda) = a(z)\,|\lambda|^2\,, \tag{3.4}$$

where $a(z)$ is a positive-valued C^∞-function. Then, for $\forall\,\xi = \sum \xi^i s_i \otimes t \in a^0(E \otimes L)$, we shall define $F^{E\otimes L}$ by

$$F^{E\otimes L}(z,\xi) := a(z)F(z,\xi).$$

Then, by Lemma 3.1, the curvature $R^{E\otimes L}$ of $F^{E\otimes L}$ is given by

$$R^{E\otimes L} = R^E \otimes 1 + I_E \otimes \bar{\partial}\partial \log a(z),$$

and the mean curvature $K^{E\otimes L}$ is given by

$$K^{E\otimes L} = K^E \otimes 1 + I_E \otimes \Box \log a(z).$$

Hence we have

Proposition 3.3. *Let (E,F) be a weak Einstein-Finsler bundle with factor φ, and (L,F^L) an arbitrary line bundle with a Hermitian metric* (3.4). *Then the tensor product $E \otimes L$ admits a weak Einstein-Finsler structure with factor $\varphi + \Box \log a(z)$.*

§4. A vanishing theorem for holomorphic sections

In this section, we shall show a Bochner-type vanishing theorem for holomorphic sections of complex Finsler bundle (E,F).

Let $\zeta = \sum_i \zeta^i(z)s_i$ be a non-vanishing holomorphic section over an open set U. We denote by $PE_{\zeta(U)} \subset PE$ the image of $\zeta(U)$ by the natural projection $E^\times \to PE$, that is,

$$PE_{\zeta(U)} := \{(z,[\zeta(z)]) \in PE;\ z \in U\}\,.$$

We also denote by ζ_P the corresponding holomorphic section of LE over $PE_{\zeta(U)}$. For the holomorphic mapping $f_\zeta : z \in U \to (z,[\zeta(z)]) \in PE$, we get the following commutative diagram:

$$\begin{array}{ccc} LE^{\times} & \xleftarrow{\ \tau\ } & E^{\times} \\ {\scriptstyle \zeta_P}\uparrow & & \uparrow{\scriptstyle \zeta} \\ PE_{\zeta(U)} & \xleftarrow[f_\zeta]{} & U \end{array}$$

We say that a holomorphic section $\zeta = \sum_i \zeta^i(z)s_i$ is *parallel with respect to* D if it satisfies $D\zeta_P = 0$ on $PE_{\zeta(U)}$, that is,

$$D_\alpha \zeta^i := \frac{\partial \zeta^i}{\partial z^\alpha} + \sum_{m=1}^{r} \zeta^m(z) \varGamma^i_{m\alpha}(z, [\zeta(z)]) = 0. \tag{4.1}$$

For any holomorphic section ζ of E, we show the following Weitezenböck-type formula.

Proposition 4.1. *Let (E, F) be a complex Finsler bundle over a Hermitian manifold (M, g). For any holomorphic section ζ of E, the following identity holds:*

$$\Box F(z, \zeta(z)) = \|D'\zeta_P\|^2 - K(\zeta_P, \zeta_P),$$

where

$$\|D'\zeta_P\|^2 := \sum_{\alpha,\beta,i,j} g^{\bar\beta\alpha}(z) F_{i\bar j}(z, [\zeta(z)]) D_\alpha \zeta^i \overline{D_\beta \zeta^j}.$$

Proof. For any function f, we have $D''D'f = \bar\partial_{\mathcal H}\partial_{\mathcal H} f$. We shall apply this to the function $f(z) = F(z, \zeta(z)) = H(\zeta_P, \zeta_P)$. Proposition 2.1 implies

$$\partial_{\mathcal H}\bar\partial_{\mathcal H} H(\zeta_P, \zeta_P) = -H(R(\zeta_P), \zeta_P) + H(D'\zeta_P, D'\zeta_P).$$

Hence we get

$$\partial\bar\partial f = -H(R(\zeta_P), \zeta_P) + H(D'\zeta_P, D'\zeta_P).$$

By taking the g-trace of the equation above, we complete the proof. □

By this formula and the maximum principle of E. Hopf (cf. Theorem 1.10 in p. 52 of Kobayashi [6]), we can show the following Bochner-type vanishing theorem for holomorphic sections:

Theorem 4.1. *Let (E, F) be a complex Finsler bundle over a compact Hermitian manifold (M, g).*

(1) *If the mean curvature K is negative semi-definite on PE, then every holomorphic section ζ of E is parallel with respect to D, that is,*

$$D\zeta_P = 0,$$

and satisfies

$$K(\zeta_P, \zeta_P) = 0.$$

(2) *If K is negative semi-definite on PE and negative definite at some point of PE, then E admits no nonzero holomorphic sections.*

By this theorem, we have

Proposition 4.2. *Let (E, F) be an Einstein-Finsler bundle over a compact Hermitian manifold with constant factor φ.*

(1) *If $\varphi = 0$, then every non-vanishing holomorphic section of E is parallel with respect to D.*

(2) *If $\varphi < 0$, then E admits no nonzero holomorphic sections.*

§5. Partial second fundamental form

In this section, we shall define a $(1,0)$-form which plays a role of the so-called second fundamental form. Let (E, F) be a Finsler vector bundle over a compact Kähler manifold (M, Φ) with a convex Finsler structure F. The Hermitian structure and the partial connection on the induced bundle $\tilde{E}$ is also denoted by H and by D respectively.

Let S be a holomorphic subbundle of E with rank s, and $\tilde{S}$ the induced bundle $p^{-1}S$ over PE. We denote by H_S the restriction of H to $\tilde{S}$, and by D_S the partial connection on $(\tilde{S}, H_S)$. Then we define

$$A(Z) := (D - D_S)Z$$

for a section $Z \in A^0(\tilde{S})$. For the quotient bundle $\tilde{Q} := \tilde{E}\big/\tilde{S}$, it is proved easily that A is an $\mathrm{End}(\tilde{S}, \tilde{Q})$-valued $(1,0)$-form. We shall call A the *partial second fundamental form* of (E, F_S). We say that a section Z of $\tilde{E}$ is *partial-holomorphic* if it satisfies

$$D''Z = \bar{\partial}_{\mathcal{H}} Z = 0.$$

Then we have

Proposition 5.1. *The partial second fundamental form A vanishes identically if and only if the exact sequence*

$$0 \to \tilde{S} \to \tilde{E} \to \tilde{Q} \to 0 \tag{5.1}$$

splits H-orthogonally and partial-holomorphically.

Proof. Suppose $A \equiv 0$. This assumption implies $D\,|_{\tilde{S}} = D_S$, and so $D(A^0(\tilde{S})) \subset A^1(\tilde{S})$. We decompose $\tilde{E}$ as $\tilde{E} = \tilde{S} \oplus \tilde{S}^\perp$, where $\tilde{S}^\perp \cong \tilde{Q}$ is the H-orthogonal complement. For $\forall\, \xi \in A^0(\tilde{S}),\ \xi^\perp \in A^0(\tilde{S}^\perp)$, we have

$$H(D\xi, \xi^\perp) + H(\xi, D\xi^\perp) = d_{\mathcal{H}} H(\xi, \xi^\perp) = 0.$$

Thus we get $D(A^0(\tilde{S}^\perp)) \subset A^1(\tilde{S}^\perp)$. For an arbitrary holomorphic section σ of $\tilde{E}$, we write

$$\sigma = \xi + \xi^\perp$$

where $\xi \in A^0(\tilde{S}),\ \xi^\perp \in A^0(\tilde{S}^\perp)$. Since σ is holomorphic, $\bar{\partial}_{\mathcal{H}}\sigma = \bar{\partial}_{\mathcal{H}}\xi + \bar{\partial}_{\mathcal{H}}\xi^\perp = 0$. Moreover $\bar{\partial}_{\mathcal{H}} = D''$ implies $\bar{\partial}_{\mathcal{H}}\xi \in A^{0,1}(\tilde{S}),\ \bar{\partial}_{\mathcal{H}}\xi^\perp \in A^{0,1}(\tilde{S}^\perp)$. Consequently we get

$$\bar{\partial}_{\mathcal{H}}\xi = 0,\ \bar{\partial}_{\mathcal{H}}\xi^\perp = 0,$$

which means that the splitting $\tilde{E} = \tilde{S} \oplus \tilde{S}^\perp$ is partial-holomorphically.

Conversely, if we denote by H_Q the restriction of H to the holomorphic bundle $\tilde{Q}$, the partial connections $D_S \oplus D_Q$ of $(\tilde{S} \oplus \tilde{Q}, H_S \oplus H_Q)$ defines the one of $(\tilde{E}, H)$. Thus we get $A \equiv 0$. □

By direct calculations, the curvature form Ω of D can be written in the following form:

$$\Omega := \begin{pmatrix} \Omega_S + A \wedge {}^t\bar{A} & * \\ * & \Omega_Q + {}^t\bar{A} \wedge A \end{pmatrix}, \tag{5.2}$$

where Ω_S and Ω_Q is the curvature form of the partial connection induced on the bundle $\tilde{S}$ and $\tilde{Q}$ respectively. Then we have

$$H(\Omega(X, \bar{X})Z, Z) = H_S(\Omega_S(X, \bar{X})Z, Z) - \|A(X)Z\|^2$$

for $\forall\, X \in TPE$ and $\forall\, Z \in A^0(\tilde{S})$.

Now we shall introduce an analogy of first Chern form of vector bundle. We shall set as

$$c(\tilde{E}) := \frac{\sqrt{-1}}{2\pi}\sum_{j=1}^{r}\Omega^j_j, \quad c(\tilde{S}) := \frac{\sqrt{-1}}{2\pi}\sum_{j=1}^{s}{\Omega_S}^j_j\,.$$

Moreover, for a non-vanishing holomorphic section ζ of E on an open set $U \subset M$, we shall consider the pull-back of $c(\tilde{E})$ and $c(\tilde{S})$ by f_ζ:

$$c_\zeta(E) := f^*_\zeta c(\tilde{E}), \quad c_\zeta(S) := f^*_\zeta c(\tilde{S}).$$

$c_\zeta(E)$ and $c_\zeta(S)$ are $(1,1)$-forms on U. For the Kähler form $\Phi = \sqrt{-1}\sum g_{\alpha\bar{\beta}}(z)dz^\alpha \wedge d\bar{z}^\beta$ on M, we know the following (cf. Kobayashi [6], p. 55):

$$d_\zeta(E) := c_\zeta(E)\wedge\Phi^{n-1} = \frac{1}{2n\pi}\tilde{g}\left(f^*_\zeta\sum_{j=1}^{r}\Omega^j_j\right)\Phi^n,$$

$$d_\zeta(S) : = c_\zeta(S)\wedge\Phi^{n-1} = \frac{1}{2n\pi}\tilde{g}\left(f^*_\zeta\sum_{j=1}^{s}{\Omega_S}^j_j\right)\Phi^n,$$

where the notion $\tilde{g}$ means that $\tilde{g}(\sigma) := \sum g^{\alpha\bar{\beta}}\sigma_{\alpha\bar{\beta}}$ for an arbitrary $(1,1)$-form $\sigma = \sum \sigma_{\alpha\bar{\beta}}dz^\alpha\wedge d\bar{z}^\beta$ on M.

Proposition 5.2. *Let (E,F) be an Einstein-Finsler bundle with constant factor φ over a compact Kähler manifold (M,Φ). For any holomorphic subbundle S of E, we have*

$$\frac{d_\zeta(S)}{\operatorname{rank} S} \le \frac{d_\zeta(E)}{\operatorname{rank} E} \tag{5.3}$$

for any non-vanishing holomorphic section ζ. If the equality holds, the exact sequence (5.1) *splits partial-holomorphically, and S, Q are Einstein-Finsler vector bundle with the same factor φ.*

Proof. From (5.2) we get

$$\frac{2\pi}{\sqrt{-1}}c(\tilde{S}) = \sum_{j=1}^{s}\Omega^j_j - \sum_{\lambda=1}^{s}\sum_{\mu=1}^{r-s}A^\mu_\lambda\wedge\bar{A}^\mu_\lambda,$$

where we put $A = (A^\mu_\lambda)$, $A^\mu_\lambda = \sum_{\alpha=1}^{n} A^\mu_{\lambda\alpha} dz^\alpha$. The Einstein condition (3.1) implies $\tilde{g}\big(f^*_\zeta \sum_{j=1}^{r} \Omega^j_j\big) = r\varphi$. Hence we get

$$\frac{d_\zeta(E)}{r} = \frac{\varphi}{2n\pi}\Phi^n,$$

$$\frac{d_\zeta(S)}{s} = \frac{\varphi}{2n\pi}\Phi^n - \frac{1}{s}\tilde{g}\left(f^*_\zeta \sum_{\lambda=1}^{s}\sum_{\mu=1}^{r-s} A^\mu_\lambda \wedge \bar{A}^\mu_\lambda\right)\Phi^n.$$

Because of $\tilde{g}\big(f^*_\zeta A \wedge {}^t\bar{A}\big) \geq 0$, we get (5.3). The equality holds if and only if $A \equiv 0$. The second assertion is obtained from Proposition 5.1. □

If ζ is defined on M, the constant factor φ is given by

$$\varphi = \frac{2n\pi}{r \cdot \mathrm{vol}(M)} \int_M d_\zeta(E). \tag{5.4}$$

§6. Semi-stability: Special Cases

In this section, we shall consider the semi-stability of Einstein-Finsler bundles. We recall the definition of semi-stability in the sense of Mumford-Takemoto (cf. Kobayashi [6], p. 134).

Let $\mathcal{O}(E) := \mathcal{E}$ be the sheaf of germs of holomorphic sections of E. E is said to be *Φ-stable* (resp. *Φ-semi-stable*) if for any coherent subsheaf $\mathcal{F} \subset \mathcal{E}$ with $0 < \operatorname{rank}\mathcal{F} < \operatorname{rank} E$, the following inequality holds:

$$\mu(\mathcal{F}) := \frac{\deg(\mathcal{F})}{\operatorname{rank}\mathcal{F}} < \frac{\deg(E)}{\operatorname{rank} E} = \mu(E)$$

(resp. $\leq$). Lübke [8] gave a proof of the following

Theorem 6.1 (Kobayashi's theorem). *Let (E,h) be an Einstein-Hermitian vector bundle over a compact Kähler manifold (M,Φ). Then E is Φ-semi-stable and (E,h) is a direct sum*

$$(E,h) = (E_1,h_1) \oplus \cdots \oplus (E_k,h_k)$$

of Φ-stable Einstein-Hermitian vector bundles $(E_1,h_1),\dots,(E_k,h_k)$ with the same factor c as (E,h).

As shown in Proposition 3.3, for any (weak) Einstein-Finsler bundle (E,F) and Hermitian line bundle (L,h), the tensor product $E\otimes L$ admits a natural (weak) Einstein-Finsler structure. Then, by the idea of Lübke [8], we get the following

Proposition 6.1. *Let (E, F) be an Einstein-Finsler bundle over a compact Kähler manifold (M, Φ) with constant factor φ. Let $\mathcal{F}$ be any reflexive subsheaf of $\mathcal{E}$ of rank $\mathcal{F} = 1$, i.e. line bundle $\mathcal{F}$. Then the following inequality holds:*

$$\varphi \geq \frac{2n\pi}{vol(M)}\mu(\mathcal{F}).$$

Proof. Since $\mathcal{F}$ is of rank $\mathcal{F} = 1$, it may be considered as (the sheaf of germs of holomorphic sections of) a holomorphic line bundle L. For any holomorphic line bundle over a compact Kähler manifold (M, Φ), we may consider it as Einstein-Hermitian with constant factor $\psi = \frac{2n\pi}{\mathrm{vol}(M)}\mu(L)$. The monomorphism $\mathcal{F} \to \mathcal{E}$ induce a non-trivial holomorphic section $f : \mathcal{O}_M \to \mathcal{E} \otimes \mathcal{F}^*$, which is considered as a global non-trivial holomorphic section of $E \otimes L^*$. By Proposition 3.3, $E \otimes L^*$ is an Einstein-Finsler bundle with constant factor $\varphi - \psi$. Since f is a non-trivial holomorphic section of $E \otimes L^*$, Proposition 4.2 completes the proof.

If the equality hold, since the bundle $E \otimes L^*$ is Einstein-Finsler with its factor $\varphi - \psi = 0$, Propsition 4.2 implies that the holomorphic section f is parallel with respect to the partial connection of $E \otimes L^*$, that is, L is parallel with respect to D. Then the pull pack $\tilde{E}$ splits partial-holomorphically as $\tilde{E} = \tilde{E}' \oplus \tilde{L}$. □

At this time, we have no information about the semi-stability of Einstein-Finsler bundles. The difficulty lies in the facts that, for the case of rank $\mathcal{F} > 1$, there exists no computable way to define a Finsler structure on the tensor product $\otimes E$ from the given Finsler structure F on E, and that the constant φ depends on the given Finsler structure F.

On the other hand, if it is always possible to find a Hermitian structure on E such that its mean curvature is given by K^i_j in (3.1), then any Einstein-Finsler bundle is Φ-semi-stable. In general, we do not know the existence of such a Hermitian structure. So we shall treat special cases which are reducible to the case of Hermitian-Einstein.

§**6.1. Special case I.** We recall the following definition (cf. AIKOU [2, 3]).

Definition 6.1. A complex Finsler bundle (E, F) is said to be *modeled on a complex Minkowski space* if its partial connection D is induced from a connection in E, that is, $\Gamma^i_{j\alpha} = \Gamma^i_{j\alpha}(z)$.

In a previous papers [2], we have proved

Theorem 6.2. *Let (E, F) be a complex Finsler bundle which is modeled on a complex Minkowski space. Then there exists a Hermitian structure h_F in E such that D is induced from the Hermitian connection of h_F.*

Example 6.1. Let (E, h) be a *reducible* Hermitian vector bundle in the following sense, that is, we suppose that (E, h) is splits holomorphically as an h-orthogonal sum

$$(E, h) = (E', h') \oplus (L, h^L),$$

where (L, h^L) is a trivial Hermitian line bundle. (e.g., E is the holomorphic tangent bundle of the product manifold of a compact Hermitian manifold and a complex torus.) Then, the structure group of E is reducible to $U(r-1) \times 1$. The Hermitian connection ∇ of (E, h) is also splits as

$$\nabla = \nabla' \oplus d,$$

where d is the exterior differentiation on L, that is, the connection form ω of (E, h) with respect to a suitable holomorphic frame field of E is written as $\omega := \begin{pmatrix} \omega' & 0 \\ 0 & 0 \end{pmatrix}$, where ω' is the connection form of (E', h').

Let $\xi = \xi' + \xi^L$ be the corresponding decomposition of $\xi \in a^0(E)$. Then, we shall define a complex Finsler structure F on E by

$$F(z, \xi) = \frac{1}{2}\left\{ \|\xi\|^2_E + \sqrt{\|\xi\|^4_E + 4\,\|\xi^L\|^4_{E^L}} \right\},$$

where $\|\xi\|^2_E = h(\xi, \xi)$ and $\left\|\xi^L\right\|^2_{E^L} = h^L(\xi^L, \xi^L)$. The convexity of F is derived by direct calculations.

Let $\xi(t) = \xi'(t) + \xi^L(t) \in a^0(E)$ be a parallel field with respect to ω along a curve $c(t) = (z(t))$. Since $\frac{d}{dt}\|\xi(t)\|_E = \frac{d}{dt}\left\|\xi^L(t)\right\|_{E^L} = 0$, we get

$$\frac{d}{dt}F(z(t), \xi(t)) = 0.$$

This means that ω is the partial connection of (E, F). Thus (E, F) is modeled on a complex Minkowski space, and its associated h_F is the given h. Moreover, (E, F) is Einstein-Finsler if and only if (E, h_F) is Einstein-Hermitian.

We suppose that an Einstein-Finsler bundle (E, F) with a constant factor φ is modeled on a complex Minkowski space. Then, by Theorem 6.2, the partial connection D is given by the Hermitian connection of the associated (E, h_F). Hence, in this case, all the results of Lübke [8] hold, and the constant φ in (5.4) is given by the c in (3.3). Moreover we shall show the following

Theorem 6.3. *Let (E, F) be an Einstein-Finsler bundle over a compact Kähler manifold (M, Φ). If (E, F) is modeled on a complex Minkowski space, E is Φ-semi-stable and (E, F) is a direct sum*

$$(E, F) = (E_1, F_1) \oplus \cdots \oplus (E_k, F_k),$$

where $F_j := F|_{E_j}$, and each (E_j, F_j) is modeled on a complex Minkowski space whose associated Hermitian vector bundles is a Φ-stable Einstein-Hermitian vector bundle with the same factor c as (E, F).

Proof. By Theorem 6.2, if (E, F) is modeled on a complex Minkowski space, then there exists a Hermitian metric h_F of E such that the mean curvature K^i_j in (3.1) is the one of h_F. Therefore (E, h_F) is an Einstein-Hermitian vector bundle over (M, Φ). So, by Theorem 6.1, E is Φ-semi-stable, and (E, h_F) is a direct sum

$$(E, h_F) = (E_1, h_{F_1}) \oplus \cdots \oplus (E_k, h_{F_k}).$$

Each Finsler bundle (E_j, F_j), $F_j := F|_{E_j}$ is obviously modeled on a complex Minkowski space, and its associated Hermitian vector bundle (E_j, h_{F_j}) is Φ-stable. □

§**6.2. Special case II.** Let (E, F) be a convex Finsler bundle. We suppose that *there exists a non-trivial holomorphic ζ section of E such that $D\zeta_P=0$.* For the function $f(z) := F(z, \zeta(z)) = \|\zeta(z)\|^2$, by Proposition 1.1 we have

$$\partial f = (\partial_{\mathcal{H}} F)_{(z,\zeta(z))} = 0,$$

and hence the norm of ζ is constant, and so is a non-vanishing section. This section ζ spans a trivial holomorphic line bundle $L = \langle\zeta\rangle$, and its pull back $\tilde{L}$ is parallel with respect to D. Hence the pull-back $\tilde{E}$ is splits partial-holomorphically as

$$(\tilde{E}, H) = (\tilde{E}', H') \oplus (\tilde{L}, H^L),$$

and the partial connection D also splits as

$$D = D' \oplus d_{\mathcal{H}}\,.$$

Then we say the triplet (E, F, ζ) is *partially reducible*. If (E, F) is Hermitian bundle, this notion is just the reducibility defined in Example 6.1.

Let (E, F, ζ) be a partially reducible convex Finsler bundle. Then we shall define a Hermitian structure $h^\zeta = \left(h^\zeta_{i\bar{j}}\right)$ on E by $h^\zeta := f_\zeta^* H$:

$$h^\zeta_{i\bar{j}}(z) := F_{i\bar{j}}(z, [\zeta(z)]).$$

By the discussions below, we can identify the bundle (E, h_ζ) with $(\tilde{E}, H)\big|_{f_\zeta(M)}$. In fact, the Hermitian connection ∇^ζ of (E, h^ζ) is given by the form $\theta^\zeta = {h^\zeta}^{-1}\partial h^\zeta$. Then, from (4.1) and $D_\alpha \zeta^i = 0$, we get

$$\partial h^\zeta = \partial(f_\zeta^* H) = f_\zeta^* \partial_{\mathcal{H}} H,$$

and so

$$\theta^{\zeta}{}^i_j = \sum_m {h^\zeta}^{\bar{m}i} f_\zeta^*(\partial_{\mathcal{H}} F_{j\bar{m}}) = f_\zeta^* \omega^i_j\,.$$

The curvature Θ^ζ of (E, h^ζ) is also given by

$$\Theta^{\zeta}{}^i_j(z) = f_\zeta^* \Omega^i_j = \sum_{\alpha,\beta} R^i_{j\alpha\bar{\beta}}(z, [\zeta(z)]) dz^\alpha \wedge d\bar{z}^\beta,$$

and its mean curvature K^ζ by $K^{\zeta}{}^i_j(z) = K^i_j(z, [\zeta(z)])$.

Then, if (E, F) is an Einstein-Finsler bundle with constant factor φ, the bundle (E, h_ζ) is Einstein-Hermitian with the factor φ. Since the degree of E is independent on the choice of Hermitian structure, the constant φ in (5.4) is given by c in (3.3). Thus, from Theorem 6.1 we get

Theorem 6.4. *Let (E, F, ζ) be a partially reducible Finsler bundle over a compact Kähler manifold (M, Φ). If (E, F) satisfies the Einstein-Finsler condition, then E is Φ-semi-stable, and (E, h_ζ) is a direct sum*

$$(E, h_\zeta) = (E_1, h_{\zeta_1}) \oplus \cdots \oplus (E_k, h_{\zeta_k}),$$

where (E_j, h_{ζ_j}) is a Φ-stable Einstein-Hermitian vector bundle with the same factor c as (E, F).

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