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On Finsler spaces of Douglas type
A generalization of the notion of Berwald space

By S. BÁCSÓ (Debrecen) and M. MATSUMOTO (Kyoto)

Dedicated to Professor Lajos Tamássy on his 75th birthday

Abstract. A new generalization of the notion of Berwald space is proposed from the viewpoint of the equation of geodesics. A Douglas space is characterized by the vanishing Douglas tensor. Various examples of Douglas spaces are given in relation to other special Finsler spaces.

1. Introduction

We consider a geodesic curve $C : x^i = x^i(t)$, $t_0 \leq t \leq t_1$, of an n-dimensional Finsler space $F^n = (M^n, L(x,y))$ on a smooth n-manifold M^n, equipped with the fundamental function $L(x,y)$, $x = (x^i)$, $y = (y^i)$. C is the extremal of the length integral $s = \int_{t_0}^{t_1} L(x,\dot{x})dt$, $\dot{x}^i = dx^i/dt$, given by the Euler equation

$$E_i(C) = \frac{d}{dt}L_{(i)} - L_i = 0, \tag{1.1}$$

where $L_{(i)} = \dot{\partial}_i L$ and $L_i = \partial_i L$. Putting $F = L^2/2$, we get the fundamental tensor $g_{ij} = \dot{\partial}_j\dot{\partial}_i F$ and the well-known functions

$$2G_j = (\dot{\partial}_j\partial_r F)y^r - \partial_j F.$$

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Then, $(g^{ij}) = (g_{ij})^{-1}$ and $G^i = g^{ij}G_j$, we get

$$Lg^{ij}E_j(C) = \ddot{x}^i + 2G^i(x,\dot{x}) - \frac{\ddot{s}}{\dot{s}}\dot{x}^i = 0.$$

Consequently C is given by the system of differential equations

$$\ddot{x}^i\dot{x}^j - \ddot{x}^j\dot{x}^i + 2D^{ij}(x,\dot{x}) = 0, \tag{1.2}$$

where we put

$$D^{ij}(x,y) = G^i(x,y)y^j - G^j(x,y)y^i. \tag{1.3}$$

We are, in particular, concerned with a two-dimensional Finsler space F^2 with a local coordinate system $(x^1,x^2) = (x,y)$ and we put $(y^1,y^2) = (p,q)$. Let us take x as a parameter of curves and denote $y' = dy/dx = q/p$ and $y'' = d^2y/dx^2 = (p\dot{q} - \dot{p}q)/p^3$. Since $D^{ij}(x,y;p,q)$ are positively homogeneous in (p,q) of degree three, we have $D^{ij}(x,y;p,q) = p^3D^{ij}(x,y;1,q/p)$, provided $p > 0$. Consequently (1.2) can be written in the form

$$y'' = 2\{G^1(x,y;1,y')y' - G^2(x,y;1,y')\}.$$

We consider the Berwald connection $B\Gamma = (G_i{}^j{}_k, G^i_j)$ ([1], [13]), which is given by $G^i_j = \dot{\partial}_jG^i$ and $G_j{}^i{}_k = \dot{\partial}_kG^i_j$. Then we get $2G^i = G_j{}^i{}_ky^jy^k$, and the equation above can be written in the form

$$y'' = X_3(y')^3 + X_2(y')^2 + X_1y' + X_0, \tag{1.4}$$

where we put

$$\begin{aligned} &X_3 = G_2{}^1{}_2, \quad X_2 = 2G_1{}^1{}_2 - G_2{}^2{}_2,\\ &X_1 = G_1{}^1{}_1 - 2G_1{}^2{}_2, \quad X_0 = -G_1{}^2{}_1. \end{aligned} \tag{1.4a}$$

Suppose that the F^2 under consideration is a Berwald space ([1], [13]), that is, $G_j{}^i{}_k$ are functions of position (x,y) alone. Then the X's of (1.4a) are functions of (x,y) and, in consequence, (1.4) shows that the right-hand side of the equation $y'' = f(x,y,y')$ of a geodesic is a polynomial in y' of degree at most three. If our discussion is restricted to Riemannian space of dimension two, then $G_j{}^i{}_k$ are Christoffel symbols, and hence $f(x,y,y')$ is, of course, a polynomial in y' of degree at most three for all geodesics of any two-dimensional Riemannian space.

The remarkable property of $y'' = f(x,y,y')$ as above given does not depend on the choice of coordinates (x,y). In fact, we have the following

Lemma. *We consider an ordinary differential equation of second order having the following form*

$$y'' = Y_3(y')^3 + Y_2(y')^2 + Y_1 y' + Y_0, \tag{1.5}$$

where the Y*'s are functions of* (x, y). *This special form is preserved under any transformation of variables.*

Proof. Suppose that we have a differential equation

$$\bar{y}'' = \bar{Y}_3(\bar{y}')^3 + \bar{Y}_2(\bar{y}')^2 + \bar{Y}_1 \bar{y}' + \bar{Y}_0,$$

where $\bar{y}' = d\bar{y}/d\bar{x}$ and the $\bar{Y}$'s are functions of $(\bar{x}, \bar{y})$. Let us consider a transformation $(\bar{x}, \bar{y}) \to (x, y)$, given by $\bar{x} = f(x, y)$ and $\bar{y} = g(x, y)$. Then it is easy to find the transformed equation from (1.5) with the Y's as coefficients as follows: Putting $J = f_x g_y - f_y g_x$, we obtain

$$\begin{aligned}
JY_3 &= g_y^3 \bar{Y}_3 + g_y^2 f_y \bar{Y}_2 + g_y f_y^2 \bar{Y}_1 + f_y^3 \bar{Y}_0 - f_y g_{yy} + f_{yy} g_y,\\
JY_2 &= 3 g_x g_y^2 \bar{Y}_3 + (f_x g_y + 2 f_y g_x) g_y \bar{Y}_2 + (f_y g_x + 2 f_x g_y) f_y \bar{Y}_1\\
&\quad + 3 f_x f_y^2 \bar{Y}_0 - f_x g_{yy} + g_x f_{yy} + 2(f_{xy} g_y - g_{xy} f_y),\\
JY_1 &= 3 g_x^2 g_y \bar{Y}_3 + (f_y g_x + 2 f_x g_y) g_x \bar{Y}_2 + (f_x g_y + 2 f_y g_x) f_x \bar{Y}_1\\
&\quad + 3 f_x^2 f_y \bar{Y}_0 - f_y g_{xx} + g_y f_{xx} + 2(f_{xy} g_x - g_{xy} f_x),\\
JY_0 &= g_x^3 \bar{Y}_3 + g_x^2 f_x \bar{Y}_2 + g_x f_x^2 \bar{Y}_1 + f_x^3 \bar{Y}_0 - f_x g_{xx} + f_{xx} g_x.
\end{aligned}$$

It is observed from the above that "$Y_3 = 0$", for instance, is not preserved by such a transformation of variables.

The following proposition is obvious from the definition (1.3) and the homogeneity of D^{ij}:

Proposition 1. *The right-hand side of the equation* (1.4) *is a polynomial in* y' *of degree at most three, if and only if* $D^{12}(x, y; p, q)$ *is a homogeneous polynomial in* (p, q) *of degree three.*

The Lemma suggests that the differential equation of the type (1.5) will also be of great value and interest from a geometrical point of view. In fact, E. Cartan ([7], p. 242) defines a projective connection and constructs two-dimensional differential-geometric entities.

2. Douglas space

Definition. A Finsler space is said to be *of Douglas type* or a *Douglas space*, if $D^{ij} = G^i y^j - G^j y^i$ are homogeneous polynomials in (y^i) of degree three.

Theorem 1. *A two-dimensional Finsler space is a Douglas space if and only if, in a local coordinate system (x, y), the right-hand side $f(x, y, y')$ of the equation of geodesics $y'' = f(x, y, y')$ is a polynomial in y' of degree at most three.*

We treat of a Finlser space F^n with the Berwald connection $B\Gamma = (G_j{}^i{}_k, G^i_j)$. F^n is by definition a Douglas space if and only if

$$\dot\partial_k \dot\partial_j \dot\partial_i \dot\partial_h (G^l y^m - G^m y^l) = 0.$$

We have first for $D^{lm} = G^l y^m - G^m y^l$

$$\dot\partial_h D^{lm} = G^l_h y^m + G^l \delta^m_h - [l, m],$$

where $[l, m]$ denotes the interchange of indices (l, m) of the preceding terms. Next

$$\dot\partial_i \dot\partial_h D^{lm} = G_h{}^l{}_i y^m + G^l_h \delta^m_i + G^l_i \delta^m_h - [l, m],$$
$$\dot\partial_j \dot\partial_i \dot\partial_h D^{lm} = G_h{}^l{}_{ij} y^m + \{G_h{}^l{}_i \delta^m_j + (h, i, j)\} - [l, m],$$

where (h, i, j) denotes the cyclic permutation of the indices (h, i, j) of the preceding terms in the parentheses and $G_h{}^l{}_{ij} = \dot\partial_j \; G_h{}^l{}_i = \dot\partial_j \dot\partial_i \dot\partial_h G^l$ are the components of the *hv-curvature tensor* of $B\Gamma$ ([1], p. 86; [13], p. 118). Further, introducing the tensor $G_h{}^l{}_{ijk} = \dot\partial_k G_h{}^l{}_{ij}$, we obtain

$$\dot\partial_k \dot\partial_j \dot\partial_i \dot\partial_h D^{lm} (= D^{lm}_{hijk}) = G_h{}^l{}_{ijk} y^m + \{G_h{}^l{}_{ij} \delta^m_k + (h, i, j, k)\} - [l, m], \tag{2.1}$$

where (h, i, j, k) is the symbol analogous to (h, i, j). D^{lm}_{hijk} are components of a tensor and $D^{lm}_{hijk} = 0$ is necessary and sufficient for F^n to be a Douglas space. By $G_h{}^l{}_{ijr} y^r = -G_h{}^l{}_{ij}$, (2.1) yields

$$D^{lr}_{hijr} = (n + 1) D_h{}^l{}_{ij}, \tag{2.2}$$

where ${D_h}^l{}_{ij}$ are components of the well-known *Douglas tensor* ([1], (3.3.2.7); [4]; [11]):

$$D_h{}^l{}_{ij} = G_h{}^l{}_{ij} - \frac{1}{n+1} G_{hij} y^l - \frac{1}{n+1} \{G_{hi} \delta^l_j + (h, i, j)\}, \tag{2.3}$$

where $G_{hi} = G_h{}^r{}_{ir}$ is the *hv-Ricci tensor* of $B\Gamma$ and $G_{hij} = \dot{\partial}_j G_{hi} = G_h{}^r{}_{irj}$. Therefore the Douglas tensor must vanish for F^n.

Conversely, if the Douglas tensor of an F^n vanishes identically, then F^n is a Douglas space, because it is easy to show the following equality:

$$D^{lm}_{hijk} = (\dot{\partial}_k D_h{}^l{}_{ij}) y^m + \{D_i{}^l{}_{jk} \delta^m_h + (h, i, j, k)\} - [l, m]. \tag{2.4}$$

Therefore we have the following

Theorem 2. *A Finsler space is of Douglas type, if and only if the Douglas tensor vanishes identically.*

Historical remark. It is well-known that the Douglas tensor (2.3) has been introduced first by J. Douglas in 1928 in his paper on the general geometry of paths [8]. He used the German letter $\mathfrak{H}$ to show this tensor ([8], (5.10), (5.11)), because the Roman letter H was already used to show the hv-curvature tensor G. In Berwald's paper [5], published in 1941, he proved that a two-dimensional Landsberg space with vanishing Douglas tensor is a Berwald space ([5], p. 110). In one of his posthumous papers [6], published in 1947, he used the letter D to denote the Douglas tensor which is preserved invariant under projective change, and showed that $D = 0$ is one half of the necessary and sufficient conditions for a generalized affine space to be projectively flat. The Douglas tensor also appeared in H. Rund's monograph ([20], p. 143), denoted by the letter B.

In 1980 Z. I. Szabó's paper [21] on the global foundations of Finsler projective geometry was published; he denoted the Douglas tensor by the letter D ([21], (3.3), (3.11)) and proposed first the name "Douglas tensor". Almost simultaneously the second author of the present paper was concerned with this tensor ([11], (2.10)) and called it the projective hv-curvature tensor or the Douglas tensor.

Recently the first author of the present paper considered $n(>2)$-dimensional Lansdberg spaces with vanishing Douglas tensor, and proved an extension of Berwald's theorem as above. This theorem was supplemented and completed by the present authors [4].

Using the name "Douglas space", this theorem can be stated as follows:

Theorem 3. *If a Finsler space $F^n (n \geq 2)$ is a Landsberg space and a Douglas space, then it is a Berwald space. Conversely a Berwald space is a Landsberg space and a Douglas space.*

As is well-known, the Douglas tensor is projectively invariant. Hence we have the

Theorem 4. *If a Finsler space is projectively related to a Douglas space, then it is also a Douglas space.*

Example 1. In a previous paper of the second author [16] it is shown that the family of solutions of a second order linear differential equation

$$y'' + P(x)y' + Q(x)y = R(x)$$

coincides with the family of geodesics of the two-dimensional Finsler space F^2 with the metric

$$L(x, y; p, q) = \frac{1}{p} e^{\int P dx} [(2R - Qy)yp^2 + q^2] + E_x p + E_y q,$$

where $E = E(x, y)$ is an arbitrary function. Consequently this F^2 is a Douglas space. It is observed that this metric is of Kropina type (cf. Theorem 8).

3. Douglas spaces with projective connection

It seems that the Douglas tensor has made a contribution only to the theory of projective changes.

A Finsler space F^n is said to be projectively related or projective to another Finsler space $\bar{F}^n$, if any geodesic of F^n is a geodesic of $\bar{F}^n$ and vice versa. The condition for it is written as

$$\bar{G}^i = G^i + P y^i, \tag{3.1}$$

where $P = P(x, y)$ is a scalar function. The change $F^n \to \bar{F}^n$ is called projective.

From (3.1) we have

$$\bar{G}^h_i = G^h_i + P_i y^h + P \delta^h_i, \tag{3.2}$$

$$\bar{G}_i{}^h{}_j = G_i{}^h{}_j + P_{ij} y^h + P_i \delta^h_j + P_j \delta^h_i, \tag{3.3}$$

where $P_i = \dot\partial_i P$ and $P_{ij} = \dot\partial_j P_i$. If we put $G = G^r_r$, then (3.1) and (3.2) give

$$\bar G = G + (n+1)P, \qquad \bar G^h - \frac{\bar G}{n+1} y^h = G^h - \frac{G}{n+1} y^h,$$

which gives rise to a projective invariant

$$Q^h = G^h - \frac{G}{n+1} y^h, \qquad G = G^r_r. \tag{3.4}$$

Putting $Q^h_i = \dot\partial_i Q^h$ and $Q_i{}^h{}_j = \dot\partial_j Q^h_i$, we have

$$Q_i{}^h{}_j = G_i{}^h{}_j - \frac{1}{n+1}(G_{ij} y^h + G_r{}^r{}_i \delta^h_j + G_r{}^r{}_j \delta^h_i). \tag{3.5}$$

Finally we get a remarkable expression of the Douglas tensor as follows:

$$\dot\partial_k \dot\partial_j \dot\partial_i Q^h = D_i{}^h{}_{jk}. \tag{3.6}$$

Remark. Though K. Yano's notation in his monograph [23] is quite different from our notation, his (9.28), p. 197 is just the same with our (3.6). See [11], (5.1).

We are led from (3.6) and Theorem 2 to the

Theorem 5. *A Finsler space is of Douglas type, if and only if Q^h of* (3.4) *are homogeneous polynomials in (y^i) of degree two.*

Let F^n be a Douglas space. Then $Q_i{}^h{}_j$ of (3.5) are functions of position (x) alone and we may put

$$Q_i{}^h{}_j(x) y^i y^j = 2G^h - \frac{2}{n+1} G^r_r y^h,$$

which implies

$$2D^{hk} = (Q_i{}^h{}_j(x) y^i y^j) y^k - [h, k]. \tag{3.7}$$

Proposition 2. *For a Douglas space, $D^{hk} = G^h y^k - G^k y^h$ are homogeneous polynomials in (y^i) of degree three as written in the form* (3.7).

In particular, for a two-dimensional Douglas space, the equation of geodesics $y'' = f(x, y, y')$ is written as (1.5). As it is seen from (1.4), (1.5) is written in more convenient form as

(3.8) $y'' = Q_2{}^1{}_2(y')^3 + (2Q_1{}^1{}_2 - Q_2{}^2{}_2)(y')^2 + (Q_1{}^1{}_1 - 2Q_1{}^2{}_2)y' - Q_1{}^2{}_1,$

where $Q_j{}^i{}_k = Q_j{}^i{}_k(x)$.

We consider a Finsler space F^n with the *projective connection* $P\Gamma = (P_j{}^i{}_k, G^i_j)$ [14], where the h-connection coefficients $P_j{}^i{}_k$ are given by

$$P_j{}^i{}_k = G_j{}^i{}_k - \frac{1}{n+1} G_{jk} y^i. \tag{3.9}$$

Since $G_{jk} = G_j{}^r{}_{kr}$ is a tensor, called the hv-Ricci tensor, $(P_j{}^i{}_k)$ define certainly a horizontal connection ([13], §9). It is noteworthy that $P_j{}^i{}_k$ appear in (3.5):

$$Q_j{}^i{}_k = P_j{}^i{}_k - \frac{1}{n+1}(G_r{}^r{}_j \delta^i_k + G_r{}^r{}_k \delta^i_j). \tag{3.5'}$$

Remark. In Yano's monograph [23] the $P_j{}^i{}_k$ are denoted by the Greek letter Π ((9.11), p. 196), and called the connection coefficients of the normal projective connection. On the other hand, the name "projective connection coefficients" appears in Rund's monograph [20], p. 142, and coincides with $Q_i{}^h{}_j$ of (3.5). Rund writes:

They define a projective covariant derivative in the same manner as the $G_j{}^i{}_k$ give rise to a covariant derivative. However the projective covariant derivatives of a tensor are not, in general, tensors.

Thus it is sure that Rund did not recognize the entity $(Q_j{}^i{}_k)$ as a connection in the modern sense; nevertheless, how many strange papers have been devoted to studying Rund's theory of projective connection!

According to the theory of the paper [14], our $P\Gamma$ is symmetric and L-metrical: $L_{;i} = 0$. Its deflection tensor $y^r P_r{}^i{}_k - G^i_k$ vanishes and the $(v)hv$-torsion tensor $U_j{}^i{}_k = \dot\partial_k G^i_j - P_k{}^i{}_j$ is given by

$$U_j{}^i{}_k = \frac{1}{n+1} y^i G_{jk}. \tag{3.10}$$

The hv-curvature tensor of $P\Gamma$, denoted by $U_i{}^h{}_{jk}$, is equal to $\dot\partial_k P_i{}^h{}_j$, which is written as

$$U_i{}^h{}_{jk} = G_i{}^h{}_{jk} - \frac{1}{n+1}(G_{ijk} y^h + G_{ij} \delta^h_k). \tag{3.11}$$

Since the hv-Ricci tensor is $U_{ij} = U_i{}^r{}_{jr} = 2G_{ij}/(n+1)$, (2.3) and (3.11) lead to the expression of the Douglas tensor in terms of $P\Gamma$ as follows:

$$D_i{}^h{}_{jk} = U_i{}^h{}_{jk} - \frac{1}{2}(\delta_i^h U_{jk} + \delta_j^h U_{ik}). \tag{3.12}$$

Consequently we have the

Theorem 6. *In terms of the projective connection $P\Gamma$ a Douglas space is characterized by the equation*

$$U_i{}^h{}_{jk} = \frac{1}{2}(\delta_i^h U_{jk} + \delta_j^h U_{ik}),$$

where $U_i{}^h{}_{jk}$ and U_{ij} are the hv-curvature tensor and the hv-Ricci tensor.

Remark. On p. 244 of Cartan's monograph [7] we find the differential equation

$$\frac{d^2v}{du^2} = \Pi_2{}^1{}_2\left(\frac{dv}{du}\right)^3 + (2\Pi_1{}^1{}_2 + \Pi_2{}^1{}_1)\left(\frac{dv}{du}\right)^2 + (2\Pi_1{}^1{}_1 - \Pi_1{}^2{}_2)\frac{dv}{du} - \Pi_1{}^2{}_1,$$

where $\Pi_j{}^i{}_k = \Pi_j{}^i{}_k(u,v)$ are coefficients of Cartan's projective connection. It is seemingly analogous to (3.8), but $Q_j{}^i{}_k$ of (3.8) are not the projective connection $P\Gamma$; however $P_j{}^i{}_k$ may depend on (y^i) of (x^i, y^i).

4. Wagner spaces of Douglas type

The notion of Wagner space was originally defined by V.V. Wagner in 1943 [22] and established strictly from the modern standpoint by M. Hashiguchi in 1975 [9] (cf. [2]; [13], Definition 25.4).

Let $s_i(x)$ be components of a covariant vector field on an n-manifold M. Then the *Wagner connection* $W\Gamma(s) = (F_j{}^i{}_k, N_j^i, C_j{}^i{}_k)$ of a Finsler space $F^n = (M^n, L(x,y))$ is by definition a Finsler connection which is uniquely determined by the following five axioms:

(1) h-metrical: $g_{ij|k} = 0$,
(2) $(h)h$-torsion tensor $T_j{}^i{}_k = F_j{}^i{}_k - F_k{}^i{}_j$ is given by $T_j{}^i{}_k = \delta_j^i s_k - \delta_k^i s_j$.
(3) deflection tensor: $D_j^i = y^r F_r{}^i{}_j - N_j^i = 0$,
(4) v-metrical: $g_{ij}|_k = 0$,
(5) $(v)v$-torsion tensor: $S_j{}^i{}_k = C_j{}^i{}_k - C_k{}^i{}_j = 0$.

Thus $W\Gamma(s)$ is not intrinsically defined in F^n, but is a geometrical structure on M^n given by $L(x,y)$ together with $s_i(x)$.

A Finsler space F^n is called a *Wagner space*, if its $W\Gamma(s)$ is linear, that is, $F_j{}^i{}_k$ are functions of position (x^i) alone. Consequently the notion of Wagner space is regarded as a generalization of Berwald space.

Let $C\Gamma = (\Gamma^{*i}_{jk}, G^i_j, C_j{}^i{}_k)$ be the Cartan connection of a Finsler space with a Wagner connection $W\Gamma(s)$. The difference $D_j{}^i{}_k = F_j{}^i{}_k - \Gamma^{*i}_{jk}$ of $W\Gamma(s)$ from $C\Gamma$ is given [2] by

$$D_j{}^i{}_k = V_j{}^i{}_{kr} s^r, \qquad s^r = g^{ri} s_i(x), \tag{4.1}$$

$$\begin{aligned} V_j{}^i{}_{kh} = {} & g_{jk}\delta^i_h - g_{jh}\delta^i_k - C_j{}^i{}_h y_k - C_k{}^i{}_h y_j + C_j{}^i{}_k y_h \\ & + C_{jkh} y^i + L^2(S_j{}^i{}_{kh} + C_j{}^i{}_r C_k{}^r{}_h), \end{aligned} \tag{4.1a}$$

where $y_i = g_{ir}y^r = LL_{(i)}$ and $S_j{}^i{}_{kh}$ is the v-curvature tensor of $C\Gamma$. Consequently we have

$$V_0{}^i{}_{0h} = L^2 h^i_h, \qquad D_0{}^i{}_0 = L^2 s^i - s_0 y^i,$$

where $h^i_h = \delta^i_h - l^i l_h$ is the angular metric tensor. Thus we get

$$F_0{}^i{}_0 = 2G^i + L^2 s^i - s_0 y^i, \tag{4.2}$$

which implies

$$F_0{}^i{}_0 y^j - F_0{}^j{}_0 y^i = 2(G^i y^j - G^j y^i) + L^2(g^{ir}y^j - g^{jr}y^i)s_r.$$

Therefore from the definition of Wagner space we obtain

Proposition 3. *For a Wagner space with $W\Gamma(s)$,*

$$2(G^i y^j - G^j y^i) + L^2(g^{ir}y^j - g^{jr}y^i)s_r$$

are homogeneous polynomials in (y^i) of degree three.

From the definition of Douglas type and Proposition 3 it follows

Theorem 7. *Let F^n be a Wagner space with a Wagner connection $W\Gamma(s)$. F^n is of Douglas type, if and only if*

$$W^{ij} = L^2(g^{ir}y^j - g^{jr}y^i)s_r$$

are homogeneous polynomials in (y^i) of degree three.

Example 2. We consider a Kropina space $F^n = (M^n,\ L = \alpha^2/\beta)$, $\alpha^2 = a_{ij}(x)y^i y^j$, $\beta = b_i(x)y^i$ ([13], p. 107). The fundamental tensor g_{ij} and g^{ij} of F^n are given on account of the formulae (30.4) and (30.7) of [13] as follows:

$$g_{ij} = \frac{2\alpha^2}{\beta^2}a_{ij} + \frac{3\alpha^4}{\beta^4}b_i b_j - \frac{4\alpha^2}{\beta^3}(b_i Y_j + b_j Y_i) + \frac{4}{\beta^2}Y_i Y_j,$$
$$g^{ij} = \frac{\beta^2}{2\alpha^2}a^{ij} - \frac{\beta^2}{2b^2\alpha^2}B^i B^j + \frac{\beta^3}{b^2\alpha^2}(B^i y^j + B^j y^i) + \frac{\beta^2}{\alpha^4}\left(1 - \frac{2\beta^2}{b^2\alpha^2}\right)y^i y^j,$$

where $Y_i = a_{ir}y^r$, $B^i = a^{ir}b_r$ and $b^2 = b_r B^r$. Consequently we get

$$W^{ij} = \frac{1}{2b^2}(b^2\alpha^2 a^{ir} - \alpha^2 B^i B^r + 2\beta B^i y^r)s_r y^j - [i,j].$$

It is obvious that W^{ij} above are homogeneous polynomials in (y^i) of degree three. Therefore F^n is of Douglas type, provided that it is a Wagner space. In particular, a Kropina space F^2 is a Wagner space, as shown by the second author of the present paper [12] (cf. [2], [19]). Therefore we have the

Theorem 8. *Let F^n be a Kropina space.* (1) *If F^n $(n > 2)$ is a Wagner space, then it is a Douglas space.* (2) *F^2 is a Douglas space.*

Remark. The second author is so sorry to correct the equation (30.7') of his monograph [13] as follows:

$$s_{-1} = \frac{1}{\tau p}\{pp_{-1} - (p_0 p_{-2} - p_{-1}^2)\beta\}.$$

We shall continue to consider Wagner spaces of Douglas type, according to Theorem 5. For a Wagner space F^n with $W\Gamma(s)$ we have from (4.2)

$$F_j{}^i{}_0 + F_0{}^i{}_j = 2G^i_j + 2Ls^i l_j - 2L^2 C_j{}^i{}_r s^r - y^i s_j - s_0\delta^i_j.$$

The second axiom $F_j{}^i{}_k - F_h{}^i{}_j = \delta^i_j s_k - \delta^i_k s_j$ yields

$$F_0{}^i{}_j = G^i_j - L^2 C_j{}^i{}_r s^r - s_0\delta^i_j + s^i y_j,$$

which implies

$$F_0{}^r{}_r = G^r_r - L^2 C^r s_r - (n-1)s_0, \qquad C^r = g^{ij}C_i{}^r{}_j.$$

Consequently Q^i of (3.4) is written as

$$(4.3)\qquad Q^i = \left\{\frac{1}{2}F_0{}^i{}_0 - \frac{1}{n+1}F_0{}^r{}_r y^i + \left(\frac{1}{2} - \frac{n-1}{n+1}\right) s_0 y^i\right\} + L^2\left(\frac{1}{2}s^i + \frac{1}{n+1}C^r s_r y^i\right).$$

The terms in the first parentheses of (4.3) are homogeneous polynomials in (y^i) of degree two, and hence we have

Theorem 9. *Let F^n be a Wagner space with a Wagner connection $W\Gamma(s)$. F^n is a Douglas space, if and only if*

$$V^i = L^2\{(n+1)g^{ir} + 2y^i C^r\}s_r$$

are homogeneous polynomials in (y^i) of degree two.

Similarly as in the case of (3.7), W^{ij} of Theorem 7 are in relation to V^i as

$$(4.4)\qquad (n+1)W^{ij} = V^i y^j - V^j y^i.$$

5. Two-dimensional Douglas spaces

We consider two-dimensional Douglas spaces based on the Berwald frame and the main scalar $I(x,y)$ ([13], §28; [1], 3.5; [4]).

In a two-dimensional Finsler space F^2 we have an orthonormal frame field, called the Berwald frame (l,m); the vector fields are defined by

$$(5.1)\qquad \begin{cases} l^i = \frac{1}{L}y^i, \quad l_i = L_{(i)},\ h_{ij} = \varepsilon m_i m_j,\ \varepsilon = \pm 1, \\ l_i m^i = 0, \quad m_i m^i = \varepsilon, \end{cases}$$

where h_{ij} is the angular metric tensor $h_{ij} = LL_{(i)(j)}$ and the sign ε is the signature of F^2. Then we get

$$(5.2)\qquad \begin{cases} g_{ij} = l_i l_j + \varepsilon m_i m_j, \\ (m_i) = h(-l^2, l^1), \quad (m^i) = k(-l_2, l_1),\ hk = \varepsilon, \\ g(= \det(g_{ij})) = \varepsilon h^2. \end{cases}$$

As is shown in [17], we have

$$2G^i = L_0 l^i + \frac{L^2 M}{h} m^i, \tag{5.3}$$

$$L_0 = L_r y^r, \qquad M = L_{1(2)} - L_{2(1)}, \tag{5.3a}$$

where $L_r = \partial_r L$ and $L_{(i)} = \dot{\partial}_i L$.

For an F^2 we can introduce the Weierstrass invariant

$$W = \frac{L_{(1)(1)}}{(y^2)^2} = \frac{-L_{(1)(2)}}{y^1 y^2} = \frac{L_{(2)(2)}}{(y^1)^2}. \tag{5.4}$$

Hence we have

$$h_{11} = L L_{(1)(1)} = L W (y^2)^2, \qquad h_{11} = \varepsilon (m_1)^2 = \varepsilon h^2 (l^2)^2,$$

which implies

$$L^3 W = \varepsilon h^2 = g. \tag{5.5}$$

Consequently (5.3) is rewritten in a simpler form as

$$2G^1 = \frac{1}{L}\left(L_0 y^1 - \frac{M}{W} L_{(2)}\right), \qquad 2G^2 = \frac{1}{L}\left(L_0 y^2 + \frac{M}{W} L_{(1)}\right). \tag{5.3'}$$

Therefore D^{12} of (1.3) is written in the remarkable form

$$2D^{12} = -\frac{1}{W}(L_{1(2)} - L_{2(1)}). \tag{5.6}$$

Theorem 10. *A two-dimensional Finsler space is a Douglas space, if and only if* $(L_{1(2)} - L_{2(1)})/W$ *is a homogeneous polynomial in* (y^1, y^2) *of degree three, where* W *is the Weierstrass invariant.*

We have, particularly in an F^2, a simple form of the equation of geodesics, called the *Weierstrass form* ([1], (1.1.3.2); [16], p. 296; [17], (1.4)):

$$p\dot{q} - \dot{p}q + \frac{1}{W}(L_{xq} - L_{yp}) = 0, \tag{5.7}$$

where $(x^i) = (x, y)$ and $(y^i) = (p, q)$. Thus Theorem 10 is shown directly from the definition of Douglas space.

Example 3. We consider a two-dimensional Finsler space with the metric

$$L(x,y;p,q) = q\tan^{-1}\frac{q}{p} - p\log\sqrt{1+\left(\frac{q}{p}\right)^2} - xq.$$

The differential equation of the geodesics of F^2 is given by

$$y'' = (y')^2 + 1,$$

which shows that F^2 is a Douglas space ([16], Example 4). The finite equation of geodesics is

$$y = c_1 - \log|\cos(x+c_2)|,$$

where the c's are arbitrary constants. We have

$$L_{xq} - L_{yp} = -1, \quad L_0 = -pq, \quad L_{pp} = \frac{q^2}{p(p^2+q^2)}, \quad W = \frac{1}{p(p^2+q^2)}.$$

Thus (5.3') leads to

$$2LG^1 = -p^2q + p(p^2+q^2)\left(\tan^{-1}\frac{q}{p} - x\right),$$
$$2LG^2 = -pq^2 + p(p^2+q^2)\log\sqrt{1+\left(\frac{q}{p}\right)^2}.$$

Consequently F^2 is certainly not a Berwald space, but we have $2(G^1q - G^2p) = p(p^2+q^2)$, which implies again that F^2 is a Douglas space.

The *main scalar* $I(x,y;p,q)$ of F^2 is a scalar, positively homogeneous in (p,q) of degree zero, defined from the C-tensor as

$$LC_{ijk} = Im_im_jm_k.$$

Then the hv-curvature tensor $G_i{}^h{}_{jk}$ of $B\Gamma$ is written ([13], §28; [1], 3.5; [4]) as

$$LG_i{}^h{}_{jk} = (-2I_{,1}l^h + I_2m^h)m_im_jm_k, \qquad I_2 = I_{,1;2} + I_{,2}.$$

The scalar derivatives $(S_{,1}, S_{,2})$ and $(S_{;1}, S_{;2})$ of a scalar field S are defined by

$$S_{|i} = S_{,1}l_i + S_{,2}m_i, \qquad LS|_i = S_{;1}l_i + S_{;2}m_i,$$

where $S_{|i} = \partial_i S - (\dot{\partial}_r S)G_i^r$ and $S|_i = \dot{\partial}_i S$. We have $S_{;1} = 0$ if S is of degree zero in (p, q). Then (2.3) leads to the expression of the Douglas tensor as follows:

$$3LD_i{}^h{}_{jk} = -(6I_{,1} + \varepsilon I_{2;2} + 2II_2)m_i l^h m_j m_k. \tag{5.8}$$

Theorem 11. *A two-dimensional Finsler space is a Douglas space, if and only if the main scalar I satisfies the equation*

$$D^2 = 6I_{,1} + \varepsilon I_{2;2} + 2II_2 = 0, \qquad I_2 = I_{,1;2} + I_{,2}.$$

We first deal with a Douglas space F^2 with vanishing T-tensor. This tensor is defined by

$$T_{hijk} = LC_{hij}|_k + l_h C_{ijk} + l_i C_{jkh} + l_j C_{khi} + l_k C_{hij},$$

where $|k$ denotes the v-covariant differentiation in $C\Gamma$. In terms of the Berwald frame we have ([13], §28; [1], 3.5.3.)

$$LT_{hijk} = I_{;2} m_h m_i m_j m_k.$$

Consequently the conditions for F^2 under consideration are as follows:

$$(1) \quad D^2 = 6I_{,1} + \varepsilon I_{2;2} + 2II_2 = 0, \qquad (2) \quad I_{;2} = 0. \tag{5.9}$$

We have to pay attention to two of the Ricci identities for the scalar derivatives;

$$\begin{aligned} &(1) \qquad S_{,1;2} - S_{;2,1} = S_{,2}, \\ &(2) \qquad S_{,2;2} - S_{;2,2} = -\varepsilon(S_{,1} + IS_{,2} + I_{,2}S_{;2}). \end{aligned} \tag{5.10}$$

Consequently we observe for F^2 that

$$I_{,1;2} = I_{,2}, \qquad I_{,2;2} = -\varepsilon(I_{,1} + II_{,2}).$$

Hence we get $I_2 = 2I_{,2}$ and (1) of (5.9) is reduced to

$$2I_{,1} + II_{,2} = 0. \tag{5.11}$$

Differentiating (;) we get from (5.11) an equation, which is rewritten as

$$-\varepsilon II_{,1} + (2 - \varepsilon I^2)I_{,2} = 0.$$

This together with (5.11) yields $4-\varepsilon I^2=0$, if $I_{,1}=I_{,2}=0$ do not hold. Even if they hold, we get $I=\text{const}$ from (2) of (5.9). Consequently I is reduced to a constant, and hence F^2 is a Berwald space ([1], Theorem 3.5.3.1; [13]).

Theorem 12. *If a two-dimensional Douglas space has vanishing T-tensor, then it is a Berwald space with constant main scalar.*

Next we consider Wagner spaces F^2 of dimension two. In terms of the Berwald frame, W^{ij} of Theorem 7 is written as

$$W^{ij}=L^3\varepsilon(m^il^j-m^jl^i)m^rs_r.$$

Putting $s_i=s_1l_i+s_2m_i$, we get $m^rs_r=\varepsilon s_2$, and hence $W^{12}=L^3(m^1l^2-m^2l^1)s_2=-L^3ks_2$, $k^2=\varepsilon/g$. Therefore we have from Theorem 7 the following

Theorem 13. *Let F^2 be a two-dimensional Wagner space with $W\Gamma(s)$. F^2 is a Douglas space, if and only if $(L^3/\sqrt{|g|}\,)s_2$ is a homogeneous polynomial in (y^1,y^2) of degree three, where $s_i=s_1l_i+s_2m_i$.*

Remark. This is an interesting result, compared with the following fact from the theory of Berwald spaces of dimension two ([5]; [13], p. 189; [1], p. 139); $I_{;2}=0$, that is, I is a function of postition alone, if and only if $L^2/\sqrt{|g|}$ is a homogeneous polynomial in (y^1,y^2) of degree two.

We have a remarkable theorem on two-dimensional Wagner spaces, shown by Wagner ([22], [12]): F^2 is a Wagner space, if and only if $I_{;2}$ can be written as a function $f(I)$ of I, provided that $I_{;2}\neq 0$, that is, the T-tensor $\neq 0$. Since Theorem 12 has been shown, we may consider only two-dimensional Wagner space with $I_{;2}\neq 0$, and hence $I_{;2}=f(I)$ may be supposed in the following.

Now we deal with a Wagner space F^2 of Douglas type with $I_{;2}=f(I)\neq 0$. Then (5.10) shows that

$$I_{,1;2}=f'I_{,1}+I_{,2},\quad I_{,2;2}=-\varepsilon(1+f)I_{,1}+(f'-\varepsilon I)I_{,2},$$

which implies

$$\begin{aligned}I_2&=f'I_{,1}+2I_{,2},\\ I_{2;2}&=\{ff''+(f')^2-2\varepsilon(1+f)\}I_{,1}+(3f'-2\varepsilon I)I_{,2}.\end{aligned}$$

Consequently D^2 of Theorem 11 is written in the form

$$D_1 I_{,1} + D_2 I_{,2} = 0, \tag{5.12}$$

$$\begin{cases} D_1 = \varepsilon\{ff'' + (f')^2\} + 2(f'I - f + 2)), \\ D_2 = 3\varepsilon f' + 2I. \end{cases} \tag{5.12a}$$

Proposition 4. *Let F^2 be a two-dimensional Wagner space with non-zero $I_{;2}$. F^2 is a Douglas space, if and only if $I_{;2} = f(I)$ satisfies* (5.12).

We treat only of the sufficient conditions $D_1 = D_2 = 0$ in Proposition 4. $D_2 = 0$ yields $f = c - \varepsilon I^2/3$ with a constant c and then $D_1 = 0$ gives $c = 3/2$. Consequently F^2 has $I_{;2} = 3/2 - \varepsilon I^2/3$, which implies that F^2 is nothing but a Kropina space ([12], [2]).

Up to now we have only two kinds of two-dimensional Wagner spaces whose metrics are known concretely. That is, Kropina metric and cubic metric. A *cubic metric* $L(x, y)$ [18] is defined as

$$L^3(x, y) = a_{ijk}(x) y^i y^j y^k,$$

where a_{ijk} are assumed to be symmetric. A two-dimensional Finsler space F^2 with cubic metric has $I_{;2} = f(I) = -3/2 - 3\varepsilon I^2$ [2], provided that $I_{;2} \neq 0$, which yields $D_1 = 16(3\varepsilon I^2 + 1)$ and $D_2 = -16I$. Thus $D^2 = 0$ is written in the form

$$(3\varepsilon I^2 + 1) I_{,1} - I I_{,2} = 0. \tag{5.13}$$

Differentiating (;) this, a procedure similar to (5.11) yields

$$\left(-\frac{31}{2}\varepsilon I - 39 I^3\right) I_{,1} + \left(\frac{5}{2} + 13\varepsilon I^2\right) I_{,2} = 0.$$

This together with (5.13) shows that $1 + 2\varepsilon I^2 = 0$, if $I_{,1} = I_{,2} = 0$ do not hold. But $1 + 2\varepsilon I^2 = 0$ leads to $I_{;2} = 0$, a contradiction, and hence we obtain $I_{,1} = I_{,2} = 0$. Then one of the Ricci identities [4]: $S_{,1,2} - S_{,2,1} = -RS_{;2}$, gives rise to $R = 0$, that is, F^2 is a locally Minkowski space, because R is the curvature.

On the other hand, if $I_{;2} = 0$, then Theorem 12 shows that F^2 is a Berwald space with constant main scalar I [18]. Therefore we have the

Theorem 14. *Let F^2 be a two-dimensional Douglas space with cubic metric. Then F^2 is* (1) *a locally Minkowski space, or* (2) *a Berwald space with $\varepsilon = -1$, $I^2 = 1/2$ and $L^3 = \beta\gamma^2$, where β and γ are 1-forms.*

6. Douglas spaces with special metric

First we consider the Randers spaces. They, with the Kropina spaces which appeared in Example 1 etc., have played a central role in the theory of (α,β)-metrics, where $\alpha^2 = a_{ij}(x)y^i y^j$ and $\beta = b_i(x)y^i$ (cf. [1], 1.3 and 1.4; [13]; [19]).

For a Finsler space $F^n = (M^n, L(\alpha,\beta))$ with (α,β)-metric the Riemannian space $R^n = (M^n,\alpha)$ is said to associate with F^n. In R^n we have the Levi-Civita connection $(\gamma_j{}^i{}_k(x))$, in which we have the symbols as follows:

$$r_{ij} = \frac{1}{2}(b_{i;j} + b_{j;i}), \quad s_{ij} = \frac{1}{2}(b_{i;j} - b_{j;i}) = \frac{1}{2}(\partial_j b_i - \partial_i b_j),$$
$$s^i_j = a^{ir}s_{rj}, \qquad s_i = b_r s^r_i.$$

Now we consider a Randers space $F^n = (M^n, L = \alpha+\beta)$. On account of the simplicity of its metric, the functions G^i of F^n are easily written [15] as

$$2G^i = \gamma_0{}^i{}_0 + 2(Ay^i + \alpha s^i_0),$$

where $A = (r_{00} - 2\alpha s_0)/2(\alpha+\beta)$. Then we get

$$2D^{ij} = (\gamma_0{}^i{}_0 y^j - \gamma_0{}^j{}_0 y^i) + 2\alpha(s^i_0 y^j - s^j_0 y^i).$$

It is obvious that the terms in the first (..) of the right-hand side are homogeneous polynomials in (y^i) of degree three, while the Riemannian α is surely irrational in (y^i). Therefore F^n is a Douglas space, if and only if $s^i_0 y^j - s^j_0 y^i = 0$; transvection by $Y_j = a_{jr}y^r$ gives $s^i_0 = 0$, that is, $s_{ij} = 0$. Therefore

Theorem 15. *A Randers space is of Douglas type, if and only if $\partial_j b_i - \partial_i b_j = 0$, that is, β is a closed form. Then*

$$2G^i = \gamma_0{}^i{}_0 + \frac{r_{00}}{\alpha+\beta}y^i,$$

where r_{ij} is equal to $b_{i;j}$.

Remark. It is interesting for the authors to compare Theorem 15 with Kikuchi's theorem ([10], [15]). A Randers space is a Berwald space if and only if $b_{i;j}=0$, and then $2G^i=\gamma_0{}^i{}_0$.

We shall pay attention to another special metric, called the *1-form metric* ([1], 1.5). A Finsler metric L is called a 1-form metric, if we have a typical Minkowski metric $L(v^\alpha)$ on an n-dimensional vector space and $L=L(a^\alpha)$, $a^\alpha=a_i^\alpha(x)y^i$ being n 1-forms. Of course, a^α, $\alpha=1,\dots,n$, should be independent: $d=\det(a_i^\alpha)\neq 0$. Putting $L_\alpha=\partial L/\partial a^\alpha$, we have

$$L_{(i)}=L_\alpha a_i^\alpha,\qquad L_{(i)(j)}=L_{\alpha\beta}a_i^\alpha a_j^\beta.$$

We shall restrict our consideration to two-dimensional spaces F^2 with 1-form metric $L(a^1,a^2)$. Analogously to the Weierstrass invariant W of (5.4), because of the homogeneity of $L(a^1,a^2)$ we can also define

$$w=\frac{L_{11}}{(a^2)^2}=\frac{-L_{12}}{a^1a^2}=\frac{L_{22}}{(a^1)^2}\,, \tag{6.1}$$

called the *intrinsic Weierstrass invariant*. It is easy to show

$$W=wd^2. \tag{6.2}$$

In a general F^n with 1-form metric we have a linear non-symmetric connection $(\Gamma_j{}^i{}_k(x))$, which is called the 1-form connection ([1], 1.5.2) and is given by

$$\Gamma_j{}^i{}_k=b_\alpha^i\partial_k a_j^\alpha,\qquad (b_\alpha^i)=(a_i^\alpha)^{-1}.$$

The definition is rewritten as $a^\alpha_{i;j}=\partial_j a_i^\alpha-a_r^\alpha\Gamma_i{}^r{}_j=0$, and hence, in the induced Finsler connection $(\Gamma_j{}^i{}_k,\Gamma_0{}^i{}_j)$, we have

$$L_{(j);i}=(L_\alpha a_j^\alpha)_{;i}=0=L_{(j)i}-L_{(j)(r)}\Gamma_0{}^r{}_i-L_{(r)}\Gamma_j{}^r{}_i.$$

Now, let us return to F^2 with 1-form metric, and we have

$$\begin{aligned}M=L_{1(2)}-L_{2(1)}&=L_{(2)(r)}\Gamma_0{}^r{}_1-L_{(1)(r)}\Gamma_0{}^r{}_2-L_{(r)}T_1{}^{(r)}{}_2\\&=W(y^1\Gamma_0{}^2{}_0-y^2\Gamma_0{}^1{}_0)-L_\alpha T^\alpha,\qquad \alpha=1,2,\end{aligned}$$

where $T_i{}^r{}_j=b_\alpha^r\,(\partial_j a_i^\alpha-\partial_i a_j^\alpha)$ is the torsion tensor of the 1-form connection and $T^\alpha=a_r^\alpha T_1{}^r{}_2=\partial_2\ a_1^\alpha-\partial_1 a_2^\alpha$. It is noted that the terms in the parentheses are homogeneous polynomials in (y^1,y^2) of degree three.

Therefore Theorem 10 together with (6.2) shows the

Theorem 16. *Let F^2 be a two-dimensional Finsler space with 1-form metric $L(a^1, a^2)$. F^2 is a Douglas space, if and only if $L_\alpha T^\alpha / w$ is a homogeneous polynomial in (y^1, y^2) of degree three, where*

$$L_\alpha = \partial L / \partial a^\alpha, \quad T^\alpha = \partial_2 a_1^\alpha - \partial_1 a_2^\alpha, \quad \alpha = 1, 2,$$

and w is the intrinsic Weierstrass invariant.

It is noted from (5.7) that for an F^2 with 1-form metric the Weierstrass form of geodesics equation is written in the form

$$p\dot{q} - \dot{p}q + p\Gamma_0{}^2{}_0 - q\Gamma_0{}^1{}_0 - \frac{1}{wd^2} L_\alpha T^\alpha = 0, \quad \alpha = 1, 2. \tag{6.3}$$

Example 4. As is well-known ([5]; [13], §28; [1], 3.5), any two-dimensional Berwald spaces with constant main scalar are with 1-form metric. Except those trivial spaces, let us here treat of an F^2 with the metric

$$L = a^2 \log \left| \frac{a^2}{a^1} \right|^r,$$

where r is a non-zero real number. We have

$$L_1 = -r \frac{a^2}{a^1}, \quad L_2 = r \left(1 + \log \left| \frac{a^2}{a^1} \right| \right), \quad L_{12} = -\frac{r}{a^1},$$
$$w = \frac{r}{(a^1)^2 a^2}, \quad \frac{1}{w} L_\alpha T^\alpha = -a^1 (a^2)^2 T^1 + (a^1)^2 a^2 \left(1 + \log \left| \frac{a^2}{a^1} \right| \right) T^2.$$

Since T^α do not contain (p, q), $L_\alpha T^\alpha / w$ is homogeneous polynomial in (p, q) of degree three, if and only if $T^2 = 0$; $(a_1^2)_y - (a_2^2)_x = 0$. Therefore this F^2 *is a Douglas space, if and only if the form a^2 is closed*, and then the equation of the geodesics is written from (6.3) as

$$p\dot{q} - \dot{p}q + p\Gamma_0{}^2{}_0 - q\Gamma_0{}^1{}_0 + a^1 (a^2)^2 \frac{T^1}{(d)^2} = 0,$$

where $T^1 = (a_1^1)_y - (a_2^1)_x$.

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SÁNDOR BÁCSÓ
INSTITUTE OF MATHEMATICS AND INFORMATICS
LAJOS KOSSUTH UNIVERSITY
H–4010 DEBRECEN
P.O.BOX 12
HUNGARY

MAKOTO MATSUMOTO
15 ZENBU–CHO, SHIMOGAMO
SAKYO–KU, KYOTO
606 JAPAN

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