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Bounds for the solutions of decomposable form equations

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Abstract. We give considerable improvements of the previous bounds on the solutions of decomposable form equations. Further, we generalize and make more efficient our earlier method for solving such equations via unit equations. Some applications are also presented to polynomials and algebraic integers of given discriminant and to power integral bases of number fields.

1. Introduction

Many diophantine problems can be reduced to decomposable form equations, i.e. to equations of the form

$$F(\mathbf{x}) = F(x_1, \dots, x_m) = b \qquad \text{in } \mathbf{x} = (x_1, \dots, x_m) \in \mathbb{Z}^m, \tag{1.1}$$

where $b \in \mathbb{Z} \setminus \{0\}$ and $F \in \mathbb{Z}[\mathbf{X}]$ is a decomposable form (homogeneous polynomial which factorizes into linear forms with algebraic coefficients); for references see e.g. [2], [19], [31], [9], [30], [34]. Important classes of such equations are Thue equations (when $m = 2$), norm form equations, discriminant form equations and index form equations. In 1968, Baker [1] gave an explicit upper bound for the solutions of Thue equations by means of his powerful effective method concerning linear forms in logarithms. The first explicit bounds on the solutions of discriminant form and index form

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equations were established by the author (cf. [12], [13]). These results were later extended by Papp and the author [26] to a large class of norm form equations and, more generally, to so-called triangularly connected decomposable form equations (cf. Section 2). In the last twenty years various generalizations have been published, among others for a shlightly larger class of equations and for the case when the ground ring is an arbitrary finitely generated integral domain over $\mathbb{Z}$ (for references see [19], [23], [9], [25]).

In [13], [26] and [18] we first reduced the equation in question to a system of homogeneous unit equations in three unknowns. Then we utilized explicit bounds form [13], [17], [16], respectively, on the solutions of unit equations. These bounds were derived by using Baker's method. Denoting by M_i the number field generated by the coefficients of a linear factor l_i of F, the final bounds obtained in [13], [26], [18] on the solutions of decomposable form equations depend on some parameters of the splitting field of F or of those number fields $M_iM_jM_k$ for which the corresponding linear factors l_i, l_j, l_k are linearly dependent.

Recently, Bugeaud and the author [7] have considerably improved the previous bounds on the solutions of Thue equations and norm form equations. This was achieved by using among others some recent results from [6] on S-units and S-regulators and generalizing an approach of Papp and the author [27], not involving unit equations.

In the present paper we give significant improvements of the earlier bounds on the solutions of decomposable form equations of general type (cf. Theorems 1 and 2). Our results are established over the ring of integers (and more generally over the ring of T-integers) of an arbitrary number field. Theorem 1 is a slight generalization and a considerable improvement of the previously known best result (cf. [22]). It provides a bound for all integral solutions lying in the splitting field of F. As a consequence, we improve upon (cf. Theorem 3 and Corollaries 4, 5) our earlier quantitative results [21], [24] concerning polynomials of given discriminant over number fields. The improvement in Theorem 1 is mainly due to the use of a recent theorem (cf. Lemma 1) of Bugeaud and the author [6] which is an improvement of earlier results of the author [16], [17] on unit equations.

We emphasize the novelty of our Theorem 2 and its proof. In contrast with the former applications of our method involving unit equations, in the present proof it suffices to work with parameters of the fields M_i in place of those of the fields $M_i M_j M_k$ or of the splitting field of F. This fact yields significant improvement in Theorem 2 and in its consequences for discriminant form equations (cf. Theorem 4), index form equations (cf. Corollary 8) as well as for algebraic integers of given discriminant (cf. Corollary 6) and power integral bases (cf. Corollary 7). Further, our Theorem 2 implies (cf. Corollaries 2, 3) the above-mentioned results of [7] on norm form equations with essentially the same bounds. In the proof of Theorem 2 a crucial tool is our Lemma 3 which can be regarded as a new, improved version of Lemma 1 on unit equations in the special case when at least two of the unknowns are conjugate to each other.

Our new, improved bounds are still large for practical use. Smart [32], [33] has recently made more efficient our method concerning discriminant form equations and triangularly connected decomposable form equations, taking into consideration the action of the Galois group of the splitting field on the unit equations involved in the proof. As he pointed out, it suffices e.g. to solve only one equation from each Galois orbit of unit equations under consideration. We hope that a combination of results of Smart with the present improvement of our method will provide a recent, efficient algorithm for solving higher degree discriminant form and index form equations, as well as more general decomposable form equations satisfying the assumptions of Theorem 2.

Most of the results and the method of our paper were presented (cf. [25]) at the Number Theory Conference held in Eger, Hungary, July 29 – August 2, 1996.

2. Decomposable form equations

Let $F(\mathbf{X}) = F(X_1, \ldots, X_m)$ be a form (homogeneous polynomial) of degree $n \geq 3$ in $m \geq 2$ variables, and suppose that F is decomposable into linear factors over an algebraic number field K. The linear factors of F are uniquely determined over K up to proportional factors from K. Fix a factorization of F into linear factors, and denote by $\mathcal{L}_F$ the system of these linear factors.

Let S be a finite set of places on K containing the set of infinite places S_∞, and denote by O_S the ring of S-intergers in K. For a given $\beta \in K \setminus \{0\}$, consider the decomposable form equation

$$F(\mathbf{x}) = \beta \tag{2.1}$$

to be solved in $\mathbf{x} = (x_1, \ldots, x_m) \in O_S^m$.

To obtain finiteness results on (2.1), we have to make some assumption on $\mathcal{L}_F$. For a system $\mathcal{L}$ of non-zero linear forms with algebraic coefficients, we denote by $\mathcal{G}(\mathcal{L})$ the graph with vertex set $\mathcal{L}$ in which $l, l' \in \mathcal{L}$ are connected by an edge if l, l' are linearly dependent, or if l, l' are linearly independent and $\lambda l + \lambda' l' + \lambda'' l'' = 0$ for some $l'' \in \mathcal{L}$ and some non-zero constants $\lambda, \lambda', \lambda''$. When $\mathcal{G}(\mathcal{L})$ is connected we say that $\mathcal{L}$ is *triangularly connected* (cf. [26]). For a partition $\mathcal{L}_1, \dots, \mathcal{L}_k$ of $\mathcal{L}$ into subsystems and for a finite set $\mathcal{L}'$ of non-zero linear forms with algebraic coefficients, denote by $\mathcal{H}_{\mathcal{L}'} = \mathcal{H}_{\mathcal{L}'}(\mathcal{L}_1, \dots, \mathcal{L}_k)$ the graph with vertex set $\{\mathcal{L}_1, \dots, \mathcal{L}_k\}$ in which $\mathcal{L}_i, \mathcal{L}_j$ are connected if there is an l_{ij} in $\mathcal{L}'$ which can be expressed as a linear combination both of the forms in $\mathcal{L}_i$ and of the forms in $\mathcal{L}_j$.

Suppose that $\mathcal{L}_F$, the system of linear factors of F considered above satisfies the following conditions:

(i) $\mathcal{L}_F$ has rank m;

(ii) $\mathcal{L}_F$ can be partitioned into triangularly connected subsystems $\mathcal{L}_1, \dots, \mathcal{L}_k$;

(iii) If $k > 1$, then there exists a finite set $\mathcal{L}'$ of non-zero linear forms with algebraic coefficients such that the graph $\mathcal{H}_{\mathcal{L}'}(\mathcal{L}_1, \dots, \mathcal{L}_k)$ is connected.

When $k = 1$, the decomposable form F and equation (2.1) are also called *triangularly connected*.

To formulate our Theorem 1 on equation (2.1) we need some further notation. Let d, r, R_K, h_K and O_K denote the degree, unit rank, regulator, class number and the ring of integers, respectively, of K. Let $s = \mathrm{Card}(S)$, and denote by P and Q the greatest and the product, respectively, of the rational primes lying below the finite places of S (with the convention that $P = Q = 1$ if $S = S_\infty$). Further, let R_S denote the S-regulator of K (for its definition see e.g. [6]). We note that for $S = S_\infty$, $O_S = O_K$ and $R_S = R_K$ hold. For $\alpha \in \bar{\mathbb{Q}}$, $h(\alpha)$ will denote the (absolute) height of α (cf. Section 5). Suppose, for convenience, that proportional linear factors of F are equal, and that the heights of the coefficients of the linear factors in $\mathcal{L}_F$ do not exceed $A (\geq e)$. Assume that $h(\beta) \leq B$.

Further, we write throughout the paper $\log^* a$ for $\max\{\log a, 1\}$.

Theorem 1. *With the above notation and assumptions, all solutions* $\mathbf{x} = (x_1, \dots, x_m) \in O_S^m$ *of* (2.1), *with* $l(\mathbf{x}) \neq 0$ *for* $l \in \mathcal{L}'$ *if* $k > 1$, *satisfy*

$$\max_{1 \leq i \leq m} h(x_i) < \exp\{c_1 k P^d R_S (\log^* R_S)(\log^*(P R_S)/\log^* P) \times (R_K + h_K \log Q + mn \log A + \log B)\}, \tag{2.2}$$

where $c_1 = m^2 n \cdot 5^{s+20} \cdot d^{3s+2r+5} \cdot s^{5s+10}$*. Further, if in particular* $S = S_\infty$*, all solutions* $\mathbf{x} = (x_1, \ldots, x_m) \in O_K^m$ *of* (2.1) *with* $l(\mathbf{x}) \neq 0$ *for* $l \in \mathcal{L}'$ *if* $k > 1$*, satisfy*

$$(2.3) \qquad \max_{1 \le i \le m} h(x_i) < \exp\{c_2 k R_K (\log^* R_K)(R_K + mn \log A + \log B)\}$$

where $c_2 = mn \cdot 3^{27} d^{2r+5} (r+1)^{6r+20}$.

Remark 1. In (2.2), the factor $\log^*(PR_S)/\log^* P$ can be estimated from above by $2 \log^* R_S$, and if $\log^* R_S \le \log^* P$ then by 2. Further, if $S \supsetneq S_\infty$ and $\wp_1, \ldots, \wp_{s_0}$ denote the prime ideals associated to the finite places in S, then we have (cf. [6] or [3])

$$(2.4) \qquad R_S \le R_K h_K \prod_{i=1}^{s_0} \log N(\wp_i).$$

Remark 2. Theorem 1 generalizes and considerably improves Theorem 1 of [22]. This theorem in [22] corresponds to the special case $\mathcal{L}' = \{X_q\}$ of our Theorem 1, where q is a fixed integer with $1 \le q \le m$. The improvement arises mainly from the use of our Lemma 1 which gives a significant improvement of previously known bounds on the solutions of S-unit equations. We remark that in the first version of the present paper a proof of Lemma 1 was also included. Independently, Y. Bugeaud obtained a similar result on S-unit equations, and later we published Lemma 1 and its proof in our joint paper [6].

For some applications it is important the case when in (2.1) the coefficients of F and the solutions are contained in a subfield of K. In this case Theorem 2 below provides in general much better bounds for the solutions. To state our Theorem 2, we introduce some definitions and notation.

Let again $\mathcal{L}$ denote a system of non-zero linear forms with algebraic coefficients. For a given number field L, consider the subgraph $\mathcal{G}_L(\mathcal{L})$ of $\mathcal{G}(\mathcal{L})$ with vertex set $\mathcal{L}$ in which $l, l' \in \mathcal{L}$ are connected if l, l' are linearly dependent, or if they are linearly independent and $\lambda l + \lambda' l' + \lambda'' l'' = 0$ for some non-zero constants $\lambda, \lambda', \lambda''$ and some $l'' \in \mathcal{L}$ such that at least two of l, l' and l'' are conjugate over L. We say that $\mathcal{L}$ is *triangularly connected over* L when $\mathcal{G}_L(\mathcal{L})$ is connected.

Suppose that the decomposable form $F(\mathbf{X}) = F(X_1, \ldots, X_m)$ considered in (2.1) has its coefficients in a subfield, say L, of K. We may assume without loss of generality that for every $l \in \mathcal{L}_F$, the conjugates of l over L also belong to $\mathcal{L}_F$. Further, we assume that $\mathcal{L}_F$ satisfies the assumptions (i) to (iii) above with (ii) replaced by

(ii') $\mathcal{L}_F$ can be partitioned into triangularly connected subsystems $\mathcal{L}_1, \dots, \mathcal{L}_k$ over L.

When $k = 1$, we say that F and (2.1) are *triangularly connected over* L.

Let T be a finite set of places on L, containing the set of infinite places T_∞. Denote by O_L, O_T the ring of integers and the ring of T-integers in L. Let P, Q denote the greatest and the product, respectively, of the rational primes lying below the finite places of T (with $P = Q = 1$ if $T = T_\infty$). For $l_i \in \mathcal{L}_F$, denote by M_i the number field generated by the coefficients of l_i over L, and by V_i the set of all extensions to M_i of the places in T. Denote by v an upper bound for the cardinalities of the V_i, and by r, R, h and R_V upper bounds for the unit ranks, regulators, class numbers and V_i-regulators, respectively, of the number fields M_i. Further, denote by d_3 the maximum of the degrees of those number fields $M_iM_jM_p$ for which l_i, l_j, l_p are pairwise non-proportional and form a triangle in $\mathcal{G}_L(\mathcal{L})$.

Theorem 2. *Under the above assumptions, all solutions* $\mathbf{x} = (x_1, \dots, x_m) \in O_T^m$ *of* (2.1), *with* $l(\mathbf{x}) \neq 0$ *for* $l \in \mathcal{L}'$ *if* $k > 1$, *satisfy*

$$(2.5)\qquad \max_{1 \le i \le m} h(x_i) < \exp\{c_3 k P^{d_3} R_V (\log^* R_V)(\log^*(P R_V)/\log^* P) \times (R + h \log Q + mn \log A + \log B)\},$$

where $c_3 = m^2 n \cdot 3^{2v+29} \cdot d_3^{4v+5} \cdot v^{5v+11}$. *Further, if* $T = T_\infty$, *all solutions* $\mathbf{x} = (x_1, \dots, x_m) \in O_L^m$ *of* (2.1), *with* $l(\mathbf{x}) \neq 0$ *for* $l \in \mathcal{L}'$ *if* $k > 1$, *satisfy*

$$(2.6)\qquad \max_{1 \le i \le m} h(x_i) < \exp\{c_4 k R (\log^* R)(R + mn \log A + \log B)\},$$

where $c_4 = mn \cdot 3^{r+29} \cdot d_3^{3r+5} (r+1)^{6r+18}$.

Theorem 2 gives a significant sharpening and generalization of Theorem 1 of [26] which was established in the case when $T = T_\infty$ and $\mathcal{G}(\mathcal{L}_F)$ is connected. The upper bound in [26] depends on parameters of the number fields $M_iM_jM_p$ involved which are in general larger than those of the fields M_i considered in Theorem 2. Similarly, the parameters of the field K which occur in (2.2) and (2.3) are, in general, much larger than those of the M_i, appearing in (2.5) and (2.6). Hence, in most cases, Theorem 2 gives much better bounds for the solutions than Theorem 1 or Theorem 1 of [26]. This improvement is a consequence of the use of our Lemma 3 which is a new, improved version of Lemma 1 on unit equations.

Our Theorems 1 and 2 can be compared with effective, but non-explicit results of Evertse and Győry [9] on equation (2.1).

We now present some consequences of Theorems 1 and 2. Let K, S and β be as in Theorem 1, with the parameters specified there. Let $F(X_1, X_2)$ denote a binary form of degree n which factorizes into linear factors over K. Suppose that among the linear factors of F at least three are pairwise non-proportional, and that the heights of the coefficients of F do not exceed $A(\geq e)$.

The next corollary is a significant improvement of Corollary 1.1 of [22]. It follows from Theorem 1 above in the same way as Corollary 1.1 from Theorem 1 in [22].

Corollary 1. *All solutions* $(x_1, x_2) \in O_S^2$ *of the Thue equation*

$$F(x_1, x_2) = \beta \tag{2.7}$$

satisfy (2.2), *and for* $S = S_\infty$ (2.3), *with* m, k, c_1 *and* c_2 *replaced by* 2, 1, $c_1(8d^2n)^3$ *and* $c_2(8d^2n)^3$, *respectively.*

For $m = 2$, $S = S_\infty$, Theorem 1 with $k = 1$ gives also an improvement of Corollary 1.1 of [26].

Of particular importance is the case when the coefficients of F and the solutions x_1, x_2 are contained in a subfield L of K. When F is irreducible over L, the best known bounds for the solutions have recently been established by Bugeaud and Győry [7] without using unit equations. We remark that similar bounds can be derived from our Theorem 2 with the choice $m = 2$, $k = 1$, provided that F has at least three pairwise non-proportional linear factors over K. These estimates can be compared with a recent bound of Bombieri [4] derived by a different method.

Let L, T and β be as in Theorem 2, with the parameters specified there. Further, let M denote an extension of degree $n \geq 3$ of L, and let $\alpha_1 = 1, \alpha_2, \ldots, \alpha_m$ be linearly indepent elements of M over L with $m \geq 2$ and with heights $\leq A$. Let V denote the set of extensions to M of the places in T, and let v be the cardinality of V and R_V the V-regulator of M. Denote by d, r, R, h the degree, unit rank, regulator and class number of

M, respectively. Further, let d_3 denote the maximum of the degrees of the fields $M^{(i)}M^{(j)}M^{(p)}$, where $M^{(i)}, M^{(j)}, M^{(p)}$ denote arbitrary conjugates of M over L. If in particular M/L is a normal extension then clearly $d_3 = d$. Consider the norm form equation

$$N_{M/L}(\alpha_1 x_1 + \cdots + \alpha_m x_m) = \beta \tag{2.8}$$

in $(x_1, \ldots, x_m) \in O_T^m$. Under the assumptions of Corollaries 2 or 3 below, equation (2.8) has only finitely many solutions. In these cases the best known bounds on the solutions have recently been obtained by Bugeaud and Győry [7] without using unit equations. From Theorem 2 we deduce similar bounds for the solutions. We shall show in Section 5 that the norm form involved on the left-hand side of (2.8) satisfies the conditions concerning F of Theorem 2.

Corollary 2. *Suppose that α_m is of degree ≥ 3 over $L(\alpha_1, \ldots, \alpha_{m-1})$. Then all solutions $(x_1, \ldots, x_m) \in O_T^m$ of (2.8) with $x_m \neq 0$ satisfy (2.5), and for $T = T_\infty$ (2.6).*

This implies the following.

Corollary 3. *Suppose that α_{i+1} is of degrre ≥ 3 over $L(\alpha_1, \ldots, \alpha_i)$ for $i = 1, \ldots, m-1$. Then all solutions of (2.8) satisfy (2.5), and for $T = T_\infty$ (2.6).*

Similar bounds can be deduced from Theorem 1 for those solutions which are contained in the normal closure of M over L and are integral over O_T.

Remark 3. We note that our results above can be applied to equations of Mahler-type (cf. Remark 2 in [22]). Further, they can be easily applied to equations of the form (2.1) with F replaced by $F \cdot G$, where G is a given polynomial in $\mathbf{X} = (X_1, \ldots, X_m)$ with coefficients in L (or K). Indeed, if $\mathbf{x}$ is a solution, by Lemma 2 there is a unit ε such that $F(\varepsilon \mathbf{x}) = \beta_1$ with some β_1 of bounded height. If F satisfies e.g. the assumptions of Theorem 2 (or 1), we get a bound for $h(x_i/x_j)$ and so, from the equation, for $\max_i h(x_i)$, too.

3. Applications to polynomials of given discriminant

We keep the notation of Section 2. Let L denote an algebraic number field with degree l and discriminant D_L, and T a finite set of places on L containing the set of infinite places T_∞. Suppose that K is a finite normal extension of L with the parameters d, r, R_K, h_K specified above, and that S is the set of extensions to K of the places in T. Let s, P, Q and R_S have the same meaning as in Theorem 1.

If f is a polynomial with coefficients in O_T, the ring of T-integers of L, and $f^*(X) = f(X + a)$ with some $a \in O_T$, then their discriminants $D(f), D(f^*)$ coincide. Such polynomials $f, f^* \in O_T[X]$ are called *O_T-equivalent* (and, for $T = T_\infty$, *O_L-equivalent*). For a monic polynomial $f \in O_T[X]$, f_0 will denote its monic polynomial divisor of maximal degree over L with non-zero discriminant. Clearly $f_0 \in O_T[X]$. Further, if $D(f) \neq 0$ then $f_0 = f$. For linear f_0 let $D(f_0) = 1$.

For $f \in L[X]$, we denote by $h(f)$ the maximum of the heights of the coefficients of f. Let $\beta \in O_T \setminus \{0\}$ with height not exceeding $B(\geq e)$.

From Theorem 1 we deduce the following.

Theorem 3. *If $f \in O_T[X]$ is a monic polynomial with $\deg(f) = n$, $\deg(f_0) = m$, $D(f_0) = \beta$ and with roots in K, then f is O_T-equivalent to a polynomial f^* for which*

$$(3.1) \qquad h(f^*) < |D_L|^{n/2} \exp\{c_5 P^d R_S (\log^* R_S)^2 \times (R_K + h_K \log Q + \log B + m^3)\}$$

where $c_5 = mc_1$ with the c_1 specified in Theorem 1. Further, if $T = T_\infty$ (i.e. if $O_T = O_L$), then this bound can be replaced by

$$(3.2) \qquad |D_L|^{n/2} \exp\{c_6 R_K (\log^* R_K)(R_K + \log B + m^3)\}$$

where $c_6 = m^2 c_2$ with the c_2 occuring in Theorem 1.

Theorem 3 and its Corollaries 4 and 5 below give significant improvements of the main results of our papers [14], [15], [21] and, in the number field case, [24]. The other results in these articles can also be considerably improved by applying Theorem 3 and Corollaries 4, 5 instead of their earlier versions.

Under the assumptions of Theorem 3, $m = \deg(f_0)$ can be estimated from above (cf. [8]), Theorem 2) by an explicit bound which depends only on d, s and the number of distinct prime ideal divisors of β.

Let $\wp_1, \dots, \wp_t$ $(t \geq 0)$ be distinct prime ideals in L, and P, Q the largest and the product, respectively, of the rational primes lying below $\wp_1, \dots, \wp_t$. Let W denote the product of the logarithms of the primes under consideration. Further, denote by $\mathcal{T}$ the set of integers in L which are not divisible by prime ideals different from $\wp_1, \dots, \wp_t$. Finally, let β denote a non-zero integer in L with $|N_{L/\mathbb{Q}}(\beta)| \leq b$, and let $[K : L] = d_0$.

Corollary 4. *If $f \in O_L[X]$ is a monic polynomial with $\deg(f) = n$, $\deg(f_0) = m$, $D(f_0) \in \beta\mathcal{T}$ and with roots in K, then f is O_L-equivalent to a polynomial of the form $\eta^n f^*(\eta^{-1}X)$, where $\eta \in \mathcal{T}$, $f^* \in O_L[X]$ and*

$$(3.3) \quad h(f^*) < |D_L|^{n/2} \exp\{m^6 n (c_7(t+1))^{5d_0(t+1)} P^d W^{d_0+1} R_K h_K \times (\log^* R_K h_K)^2 (R_K + h_K \log Q + \log^* b)\}$$

with an effectively computable positive constant $c_7 = c_7(d)$ which depends only on d.

Denote by $w_L(P)$ the number of distinct irreducible factors over L of a polynomial $P \in L[X]$.

Corollary 5. *If $f \in O_L[X]$ is a monic polynomial with $\deg(f) = n$, $\deg(f_0) = m \geq 2$, $w_L(f_0) = w$ and with $D(f_0) \in \beta\mathcal{T}$, then f is O_L-equivalent to a polynomial of the form $\eta^n f^*(\eta^{-1}X)$, where $\eta \in \mathcal{T}$, $f^* \in O_L[X]$ and*

$$(3.4) \qquad h(f^*) < \exp\{n[(c_8(t+1))^{5(t+1)} (P(b|D_L|^m Q^{l(m+1)})^w)^l]^{m!}\}$$

with an effectively computable positive constant $c_8 = c_8(l, m)$ which depends only on l and m.

Corollaries 4 and 5 have immediate consequence (cf. [21]) for algebraic numbers with discriminants contained in $\beta\mathcal{T}$.

Remark 4. Clearly, we have $P \leq Q$ and $W < Q$. Further, using well-known estimates from prime number theory (cf. [29]) one can easily show that if c_9, c_{10} are effectively computable constants depending only on d (respectively, on l, m) then $(c_9(t+1))^{c_{10}(t+1)} \leq (c_{11}Q)^{c_{12}}$ with effective constants c_{11}, c_{12} depending on d (respectively, on l, m). Hence the bounds in (3.3) and (3.4) can be replaced by bounds which depend on Q, but not on P, W and t.

4. Bounds for the solutions of discriminant form and index form equations

We keep the notation of Section 2, used in the statements of Theorem 2 and Corollaries 2, 3. Let L be an algebraic number field of degree l, T a finite set of places on L containing T_∞, t the cardinality of T, P, Q as in Theorem 2, and W the product of the logarithms of the primes considered in Q. Let M be an extension of degree $n \geq 3$ over L with discriminant D_M over $\mathbb{Q}$, and $1, \alpha_1, \ldots, \alpha_m \in M$ linearly independent numbers over L with heights at most $A(\geq e)$, such that $M = L(\alpha_1, \ldots, \alpha_m)$. For $k = 2$ and 3, denote by n_k the maximum of the degrees of the fields $M^{(i_1)} \ldots M^{(i_k)}$ over L, where $M^{(i_1)}, \ldots, M^{(i_k)}$ denote arbitrary conjugates of M over L. Let K denote the normal closure of M over L, and let $\beta \in L \setminus \{0\}$ with height $\leq B$.

Consider the discriminant form equation

$$D_{M/L}(\alpha_1 x_1 + \cdots + \alpha_m x_m) = \beta \tag{4.1}$$

in $(x_1, \ldots, x_m) \in O_T^m$. It is known (see [26]) that the discriminant form involved on the left-hand side of (4.1) is triangularly connected. Hence Theorem 1 gives explicit upper bounds for those solutions in K which are integral over O_T. These bounds provide a considerable improvement of Corollary 4.1 of [22]. In Section 7 we show that much better bounds can be derived if K is "large" with respect to M. Namely, if $n_3 > n_2$ then the above discriminant form satisfies the conditions of Theorem 2 as well. In this case Theorem 2 can be applied, while if $n_3 = n_2$, Theorem 1 applies to (4.1) to prove the following.

Theorem 4. *All solutions $(x_1, \ldots, x_m) \in O_T^m$ of (4.1) satisfy*

$$\max_{1 \leq i \leq m} h(x_i) < \exp\{c_{13} P^{l n_3} W^{n_1+1} |D_m|^{n_2/n} \times (\log |D_M|)^{2 l n_2} (|D_M|^{n_2/n} + \log(AB))\} \tag{4.2}$$

where $c_{13} = 3^{2tn_2+29} (l n_3)^{7tn_2+5} (tn_2)^{5tn_2+11}$. Further, if $T = T_\infty$, then all solutions $(x_1, \ldots, x_m) \in O_L^m$ of (4.1) satisfy

$$\max_{1 \leq i \leq m} h(x_i) < \exp\{c_{14} |D_M|^{n_2/n} (\log |D_M|)^{2 l n_2 - 1} \times (|D_M|^{n_2/n} + \log(AB))\} \tag{4.3}$$

where $c_{14} = 3^{l n_2 + 28} (l n_3)^{9 l n_2 + 14}$.

If in particular $M^{(i_1)}M^{(i_2)}/L$ is normal for some i_1, i_2, then we may take $n_3 = n_2$. Further, if M/L is normal then $n_3 = n_2 = n$ and, as will be clear from the proof, $|D_M|^{n_2/n}$ may be replaced by $|D_M|^{1/2}$.

Theorem 4 is a significant improvement of our earlier bounds (cf. [13], [26], [19], [22]) on the solutions of (4.1).

We present some consequences of Theorem 4. For simplicity, we restrict ourselves here to some special cases. General versions can be deduced from Theorem 4 in a similar way.

First consider the equation

$$D_{M/L}(\alpha) = \beta \qquad \text{in } \alpha \in O_M. \tag{4.4}$$

The elements $\alpha, \alpha^* \in O_M$ are called O_L-*equivalent* if $\alpha - \alpha^* \in O_L$. In this case their discriminants $D_{M/L}(\alpha), D_{M/L}(\alpha^*)$ coincide. From Theorem 4 we deduce the following.

Corollary 6. *Every solution α of (4.4) is O_L-equivalent to a solution α^* for which*

$$h(\alpha^*) < \exp\{c_{15}|D_M|^{n_2/n}(\log|D_M|)^{2ln_2-1}(|D_M|^{n_2/n} + \log B)\} \tag{4.5}$$

where $c_{15} = n^4 c_{14}$ with the c_{14} occurring in Theorem 4.

If $O_M = O_L[\alpha]$ for some $\alpha \in O_M$, then $O_M = O_L[\varepsilon\alpha + a]$ for all $\varepsilon \in O_L^*$ and all $a \in O_L$. We deduce from Corollary 6 the following.

Corollary 7. *Suppose that $O_M = O_L[\alpha]$ for some $\alpha \in O_M$. Then α is O_L-equivalent to an algebraic integer of the form $\varepsilon\alpha^*$, where ε is a unit in L and*

$$h(\alpha^*) < \exp\{c_{16}|D_M|^{2n_2/n}(\log|D_M|)^{2ln_2-1}\}$$

where $c_{16} = 3l^3n^2c_{15}$ with the c_{15} specified in Corollary 6.

Corollaries 6 and 7 considerably improve the corresponding earlier results of the author [13], [14].

If in particular M/L is normal, then in Corollaries 6 and 7 $|D_M|^{n_2/n}$ may be replaced by $|D_M|^{1/2}$. From Theorem 4 one can easily deduce in a similar way more general results, for the solutions $\alpha \in O_S$ of (4.4) and for the elements $\alpha \in O_S$ with $O_S = O_T[\alpha]$, where O_S denotes the integral closure of O_T in M.

Finally, we present a further consequence of Theorem 4 for index form equations. For simplicity, we restrict ourselves to the important special case $L = \mathbb{Q}$. In the general case, a similar consequence can be easily deduced from Theorem 4.

Let M be an algebraic number field of degree $n \geq 3$ with discriminant D_M over $\mathbb{Q}$, and let n_2, n_3 have the same meaning as before. Let $\{1, w_2, \ldots, w_n\}$ be an integral basis for M over $\mathbb{Q}$ with $\max_{2\leq k\leq n} h(w_k) \leq H$ $(H \geq e)$, and let $F(X_2, \ldots, X_n)$ denote the index form of this basis. As is known, F has its coefficients in $\mathbb{Z}$ and satisfies

$$D_{M/\mathbb{Q}}(w_2X_2 + \cdots + w_nX_n) = F^2(X_2, \ldots, X_n)D_M. \tag{4.6}$$

Let a denote a non-zero rational integer, and consider the index form equation

$$F(x_2, \ldots, x_n) = \pm a \qquad \text{in } x_2, \ldots, x_n \in \mathbb{Z}. \tag{4.7}$$

Corollary 8. *All solutions of* (4.7) *satisfy*

$$\max_{2\leq k\leq n} |x_k| < \exp\{c_{17}|D_M|^{n_2/n}(\log|D_M|)^{2n_2-1}(|D_M|^{n_2/n} + \log(H \cdot |a|))\} \tag{4.8}$$

where $c_{17} = 3^{n_2+29} \cdot n_3^{9n_2+15}$.

We note that if $M/\mathbb{Q}$ is normal then, in (4.8), $n_3 = n_2 = n$ and $|D_M|^{n_2/n}$ may be replaced by $|D_M|^{1/2}$.

Corollary 8 is a significant improvement of earlier bounds of the author [13], [14].

5. Proofs of Theorems 1, 2 and Corollaries 2, 3

We adopt the notation of Sections 2 and 3. We recall that K denotes an algebraic number field of degree d with unit rank r, regulator R_K, class number h_K and ring of integers O_K. Denote by M_K the set of places on K. Assume that for every $v \in M_K$, the valuation $|\cdot|_v$ is normalized as in [6]. Then the (absolute) *height* of an algebraic number α contained in K is defined by

$$h(\alpha) = \left(\prod_{v\in M_K} \max(1, |\alpha|_v)\right)^{1/d}.$$

This height is independent of the choice of K. For $\alpha \in K$, the denominator of α is at most $h(\alpha)^d$. If in particular α is an integer in K, then $h(\alpha) \leq \lceil\overline{\alpha}\rceil \leq (h(\alpha))^d$ where $\lceil\overline{\alpha}\rceil$ denotes the maximum absolute value of the conjugates of α. Further, $|N_{K/\mathbb{Q}}(\alpha)| \leq (h(\alpha))^d$. Finally, it will be frequently used that $h(\alpha^{-1}) = h(\alpha)$ for $\alpha \in K \setminus \{0\}$, and

$$h(\alpha_1 + \cdots + \alpha_m) \leq mh(\alpha_1)\dots h(\alpha_m), h(\alpha_1 \dots \alpha_m) \leq h(\alpha_1)\dots h(\alpha_m)$$

for $\alpha_1, \dots, \alpha_m \in K$.

There exists a positive constant δ_K, depending only on K, such that for every $\alpha \in K\setminus\{0\}$ which is not a root of unity, we have $\log h(\alpha) \geq \delta_K/d$. For $d = 1$ we may take $\delta_K = \log 2$. Recently Voutier (cf. [37] and the references given there) has shown that one can take

$$\delta_K = 2(\log(3d))^{-3} \quad \text{for} \quad d \geq 2. \tag{5.1}$$

Let S denote a finite subset of M_K with $S \supseteq S_\infty$ and with cardinality s. Further, denote by O_S the ring of S-integers, by O_S^* the group of S-units and by R_S the S-regulator of K (see e.g. [6]). $N_S(\alpha)$ denotes the S-norm of $\alpha \in K$ (see also e.g. [6]). Let P, Q have the same meaning as in Theorem 1. Consider the equation

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0 \quad \text{in} \quad x_1, x_2, x_2 \in O_S^* \tag{5.2}$$

where $\alpha_1, \alpha_2, \alpha_3 \in K \setminus \{0\}$ with $\max_{1\leq i\leq 3} h(\alpha_i) \leq H$ $(H \geq e)$.

Lemma 1. *For every solution x_1, x_2, x_3 of (5.2), we have*

$$\max_{i,j} h\left(\frac{x_i}{x_j}\right) < \exp\{c_{18} P^d R_S (\log^* R_S)(\log^*(PR_S)/\log^* P)\log H\} \tag{5.3}$$

where $c_{18} = 3^{26}(9d^2/\delta_K)^{s+1}s^{5s+10}$. Further, if $S = S_\infty$, then the bound in (5.3) can be replaced by

$$\exp\{c_{19} R_K (\log^* R_K)\log H\}$$

where $c_{19} = 3^{r+28}(r+1)^{5r+17}d^3\delta_K^{-(r+1)}$.

Proof. This is an immediate consequence of the Theorem in [6]. As was remarked before, the proof of this lemma (with slightly different values of c_{18} and c_{19}) had been also included in the first version of this paper. □

Lemma 2. *For every* $\alpha \in O_S \setminus \{0\}$ *and every integer* $n \geq 1$ *there exists an* $\varepsilon \in O_S^*$ *such that*

$$(5.4) \qquad h(\varepsilon^n \alpha) < (N_S(\alpha))^{1/d} \exp\{n(c_{20} R_K + h_K \log Q)\}$$

where $c_{20} = r^{r+1}(\log(3d))^{3(r-1)}/2^r$.

Proof. This is Lemma 2 of [6] with δ_K replaced by (5.1) and $s_0 \log P$ replaced by $\log Q$, where s_0 denotes the number of finite places in S. The proof of Lemma 2 in [6] is based on the proofs of Lemmas 9 and 10 of [10]. In the proofs of these lemmas or [6] and [10] Q can be taken everywhere in place or P^{s_0}, and (5.4) immediately follows. □

Proof of Theorem 1. We prove Theorem 1 by means of Lemma 1 and some arguments used in [19], [20], [22]. Hence we shall only sketch the proof.

After multiplying (2.1) by the product of the denominators of the coefficients of the linear factors of F, (2.1) can be written in the form

$$(2.1') \qquad \prod_{i=1}^{n} l_i(\mathbf{x}) = \beta$$

where the solutions $\mathbf{x}$ are taken from O_S, the linear forms $l_i(\mathbf{X})$ have already integral coefficients in K with heights at most $A_1 = A^{md}$ and β is of height at most $B_1 = A^{mnd}B$. Then we may assume that $\beta \in O_S$, since otherwise (2.1') is not solvable.

Let now $\mathbf{x} \in O_S^m$ be a solution of (2.1'), with $l(\mathbf{x}) \neq 0$ for $l \in \mathcal{L}'$ if $k > 1$. Put $l_i(\mathbf{x}) = \beta_i$ for $i = 1, \dots, n$. Using $\beta_i \in O_S$ and properties of the S-form (see e.g. [9]), we get $N_S(\beta_i) \leq N_S(\beta) \leq B_1^d$. Further, by Lemma 2 we can write $\beta_i = \gamma_i \varepsilon_i$ with $\varepsilon_i \in O_S^*$, $\gamma_i \in O_S$, $i = 1, \dots, n$, such that

$$(5.5) \qquad h(\gamma_i) < B_1 \exp\{c_{20} R_K + h_K \log Q\} := E_1.$$

If l_{i_1}, l_{i_2} in $\mathcal{L}_F$ is an edge of $\mathcal{G}(\mathcal{L}_F)$ and l_{i_1}, l_{i_2} are linearly independent, then there are $l_{i_{1,2}} \in \mathcal{L}_F$ and non-zero integers $\lambda_{i_1}, \lambda_{i_2}, \lambda_{i_{1,2}}$ in K with heights at most $A_2 = 2A_1^4$ such that $\lambda_{i_1} l_{i_1} + \lambda_{i_2} l_{i_2} + \lambda_{i_{1,2}} l_{i_{1,2}} = 0$. If l_{i_1}, l_{i_2} are linearly dependent then, by our assumption made in Section 2, $l_{i_1} = l_{i_2}$. In the first case put $\alpha_i = \lambda_i \gamma_i$; then $h(\alpha_i) \leq E_1 A_2 = H$ for each i in question. Further, we have

$$(5.6) \qquad \alpha_{i_1} \varepsilon_{i_1} + \alpha_{i_2} \varepsilon_{i_2} + \alpha_{i_{1,2}} \varepsilon_{i_{1,2}} = 0.$$

By applying Lemma 1 to this equation in $\varepsilon_{i_1}, \varepsilon_{i_2}, \varepsilon_{i_{1,2}}$ we infer that

$$\max_{q=1,2} h(\varepsilon_{i_q}/\varepsilon_{i_{1,2}}) \leq E_2$$

where E_2 denotes the upper bound occurring in the estimate (5.3) of Lemma 1. With the choice $\varepsilon = \varepsilon_{i_{1,2}}^{-1}$ we deduce that

$$\max_{q=1,2} h(\varepsilon\beta_{i_q}) \leq E_1 E_2 = E_3.$$

By (5.5) it is clear that this holds in the second case, too.

If now l_{i_2}, l_{i_3} is an edge in $\mathcal{G}(\mathcal{L}_F)$ then we obtain in the same way that for some $\varepsilon' \in O_S^*$, $\max_{q=2,3} h(\varepsilon'\beta_{i_q}) \leq E_3$. But $\varepsilon/\varepsilon' = (\varepsilon\beta_{i_2})/(\varepsilon'\beta_{i_2})$, whence we deduce that

$$\max_{1 \leq q \leq 3} h(\varepsilon\beta_{i_q}) \leq E_3^2.$$

For $j = 1, \ldots, k$, let $\mathcal{I}_j$ denote the set of i with $l_i \in \mathcal{L}_j$. Using assumption (ii) made on the linear factors of F and repeating the above procedure by induction, we infer that for each j with $1 \leq j \leq k$ there is an $\eta_j \in O_S^*$ such that

$$h(\eta_j\beta_i) \leq E_3^{2n} \qquad \text{if} \quad i \in \mathcal{I}_j. \tag{5.7}$$

Suppose that $k > 1$. Then, by assumption (iii) concerning F, the graph $\mathcal{H}_{\mathcal{L}'}(\mathcal{L}_1, \ldots, \mathcal{L}_k)$ is connected for some fixed finite set $\mathcal{L}'$ of non-zero linear forms with algebraic coefficients. Assume, for convenience, that $\mathcal{L}_1$ and $\mathcal{L}_2$ are connected by an edge in this graph. Then there is an $l_{1,2} \in \mathcal{L}'$ such that

$$l_{1,2} = \sum_{i \in \mathcal{I}_1}{}' \lambda_i l_i = \sum_{i \in \mathcal{I}_2}{}' \lambda_i l_i \tag{5.8}$$

where the $\sum'$ means that we consider only linearly independent l_i both from $\mathcal{L}_1$ and from $\mathcal{L}_2$ with non-zero λ_i from $\overline{\mathbb{Q}}$. Then, up to a proportional factor, these λ_i provide a uniquely determined solution of the system of linear equation

$$\sum_{i \in \mathcal{I}_1}{}' \lambda_i l_i(\mathbf{X}) - \sum_{i \in \mathcal{I}_2}{}' \lambda_i l_i(\mathbf{X}) = 0.$$

in λ_i with $i \in \mathcal{I}_1 \cup \mathcal{I}_2$. By using the size it is easy to see that there is a non-zero $\lambda_{1,2}$ in K such that $\lambda_{1,2} l_{1,2}$ can be expressed in the form (5.8) with non-zero $\lambda_i \in K$ for which $h(\lambda_i) \leq m! A_1^m$. For the solution $\mathbf{x}$ considered above, we deduce by (5.7) that for $q = 1, 2$

$$h(\eta_q \lambda_{1,2} l_{1,2}(\mathbf{x})) \leq m(m! A_1^m E_3^{2n})^m = E_4$$

whence, in view of $l_{1,2}(\mathbf{x}) \neq 0$, we infer that $h(\eta_1/\eta_2) \leq E_4^2$ and so

$$h(\eta_1 \beta_i) \leq E_3^{2n} E_4^2 \qquad \text{for} \quad i \in \mathcal{I}_1 \cup \mathcal{I}_2.$$

Using the fact that $\mathcal{H}_{\mathcal{L}}, (\mathcal{L}_1, \ldots, \mathcal{L}_k)$ is connected and repeating this argument by induction, we infer that

$$h(\eta_1 \beta_i) \leq E_3^{2n} \cdot E_4^{2(k-1)} = E_5 \qquad \text{for} \quad i = 1, \ldots, n. \tag{5.9}$$

It follows from (2.1') that $\eta_1^n = (\eta_1 \beta_1) \ldots (\eta_1 \beta_n)/\beta$ and so $h(\eta_1) \leq B_1^{1/n} E_5$. Together with (5.9) this yields

$$h(\beta_i) < B_1^{1/n} E_5^2 = E_6, \qquad i = 1, \ldots, n. \tag{5.10}$$

Finally, taking into consideration assumption (i) on the linear factors of F, we can derive from $l_i(\mathbf{x}) = \beta_i$, $i = 1, \ldots, n$, a bound for the heights of the $x_1, \ldots, x_m$. To do so we assume that $l_1, \ldots, l_m$ are linearly independent. Then using both $h(\ )$ and $\lceil\,\rceil$, we infer that there are algebraic integers ρ_i, μ_j such that $\rho_i X_i = \sum_{j=1}^m \mu_j l_j(\mathbf{X})$ and $h(\rho_i)$, $h(\mu_j) \leq m! A_1^m$. Together with (5.10) this gives

$$\max_{1 \leq i \leq m} h(x_i) < m(m! A_1^m)^{2m} E_6^m$$

whence, using also (5.1), (2.2) follows.

If $S = S_\infty$, the second part of Lemma 1 can be applied, and the second assertion of Theorem 1 follows. □

Recently Voutier [36] (for $S = S_\infty$) and Bugeaud [5] (for arbitrary S) established an improved version of Lemma 1 in the special case when the solutions x_1, x_2 in (5.2) are contained in some subfields K_1 and K_2 of K, respectively. To prove our Theorem 2 we shall give a sharpening of these results in the further special case when, in (5.2), both the subfields K_1, K_2 and the solutions x_1, x_2 are conjugate to each other.

Let K, S and H be as in Lemma 1, with the parameters specified there. Let K_1 be a subfield of K with degree d_1, unit rank r_1 and regulator R_{K_1}. Denote by S_1 the set of restrictions to K_1 of the places in S, and by R_{S_1} the S_1-regulator of K_1. Put $s_1 = \mathrm{Card}(S_1)$. Assume that for some $\mathbb{Q}$-isomorphism σ of K_1, $\sigma(K_1)$ is a subfield of K.

Lemma 3. *All solutions x_1, x_2, x_3 of (5.2) with $x_1 \in K_1$, $x_2 = \sigma(x_1)$ satisfy*

$$\max h(x_i/x_j) < \exp\left\{c_{21}\frac{P^d}{\log^* P}R_{S_1}\log^*(R_{S_1})\log H\log\left(\frac{\log h(x_1)}{\log H}\right)\right\} \tag{5.11}$$

provided that $h(x_1) \geq \exp\{c_{22}R_{S_1}\log H\}$, where $c_{21} = 3^{25}(9d^4/d_1)^{s_1+1}\cdot s_1^{5s_1+10}$ and $c_{22} = (ds_1^2)^{2s_1}$. Further, if in particular $S = S_\infty$, the bound in (5.11) can be replaced by

$$\exp\left\{c_{23}R_{K_1}\log H\log\left(\frac{\log h(x_1)}{\log H}\right)\right\}, \tag{5.12}$$

provided that $h(x_1) \geq \exp\{c_{24}\log H\}$, where $c_{23} = 3^{r_1+27}d^{2r_1+5}(r_1+1)^{5r_1+17}$ and $c_{24} = d^3(r_1+1)^{2(r_1+1)}$.

Proof. Let x_1, x_2, x_3 be an arbitrary but fixed solution of (5.2) with $x_2 = \sigma(x_1)$. The cases $x_2 = x_1$ and $s_1 = 1$ being trivial, we assume that $x_2 \neq x_1$ and $s_1 > 1$. Then $x_1/x_2, x_3/x_2$ is a solution of the equation

$$\left(-\frac{\alpha_1}{\alpha_2}\right)x + \left(-\frac{\alpha_3}{\alpha_2}\right)y = 1 \qquad \text{in} \quad x, y \in O_S^*. \tag{5.13}$$

We follow the proof of the Theorem of [6] concerning this equation. Only those steps will be detailed which differ from the corersponding arguments of [6].

Let $\{\varepsilon_1, \ldots, \varepsilon_{s_1-1}\}$ be a fundamental system of S_1-units in K_1 with the properties specified in Lemma 1 of [6]. Then $x_1 = \zeta\varepsilon_1^{b_1}\ldots\varepsilon_{s_1-1}^{b_{s_1-1}}$ with a root of unity ζ in K_1 and with rational integers $b_1, \ldots, b_{s_1-1}$. It follows as in [6] that

$$B \leq c_{25}\log h(x_1) \tag{5.14}$$

where $B = \max\{|b_1|, \dots, |b_{s_1-1}|, 3\}$ and $c_{25} = d_1((s_1-1)!)^2/(s^{s_1-3}\delta_{K_1})$. Put $S = \{v_1, \dots, v_s\}$, and let $v \in S$ for which $|x_3/x_2|_v$ is minimal. Setting $\beta_{s_1} = -\alpha_1\zeta/(\alpha_2\sigma(\zeta))$ and $\eta_i = \varepsilon_i/\sigma(\varepsilon_i)$ for $i = 1, \dots, s_1-1$ we deduce from (5.13) that

$$\left|\frac{\alpha_3 x_3}{\alpha_2 x_2}\right|_v = |\eta_1^{b_1} \dots \eta_{s_1-1}^{b_{s_1-1}}\beta_{s_1} - 1|_v. \tag{5.15}$$

We shall derive a lower bound for $|(\alpha_3x_3)/(\alpha_2x_2)|_v$. We have $h(\beta_{s_1}) \le H^2$. Further, by $h(\eta_i) \le h^2(\varepsilon_i)$, $i = 1, \dots, s_1-1$ and Lemma 1 of [6] we infer that

$$\log h(\eta_1) \dots \log h(\eta_{s_1-1}) \le c_{26} R_{S_1} \tag{5.16}$$

where $c_{26} = 2((s_1-1)!)^2/d_1^{s_1-1}$.

First assume that v is infinite. Then applying (5.16) and an estimate of Waldschmidt [38] (cf. [6], Proposition 1), and working with the η_i in place of ε_i, we obtain as in [6] that

$$|\eta_1^{b_1} \dots \eta_{s_1-1}^{b_{s_1-1}}\beta_{s_1} - 1|_v > \exp\left\{-c_{27} R_{S_1} \log H \log\left(\frac{c_{28}B}{\log H}\right)\right\} \tag{5.17}$$

where $c_{27} = 2c_{29}(s_1)c_{26}d^{s_1+2}2^{-s_1+1}\delta_K^{-s_1}$, $c_{29}(s_1) = 1500 \cdot 38^{s_1+1}(s_1+1)^{3s_1+9}$ and $c_{28} = 2s\delta_K$. Since $|x_3/x_2|_v$ is minimal, we have $h(x_2/x_3) \le |x_2/x_3|^{s/d}$ and, by (5.17), (5.15), (5.14) and (5.1), we deduce first for $h(x_2/x_3)$ and then, by (5.13), for each $h(x_i/x_j)$ that

$$\max_{i,j} h(x_i/x_j) < \exp\left\{c_{30} R_{S_1} \log H \log\left(\frac{\log h(x_1)}{\log H}\right)\right\}, \tag{5.18}$$

provided that $h(x_1) \ge \exp\{c_{31}\log H\}$, where $c_{30} = 3^{s_1+26}d^{2s_1+3}s_1^{5s_1+12}$ and $c_{31} = d^3 s_1^{2s_1}$. For $S = S_\infty$, (5.18) proves the second part of our Lemma 3.

Next assume that v is finite. Putting

$$\log A_i = \delta_K^{-1}\log h(\eta_i) + \log^* P, \qquad i = 1, \dots, s_1-1,$$
$$\log A_{s_1} = \delta_K^{-1}\log H + \log^* P,$$

and using (5.16), we deduce as in [6] that

$$\log A_1 \dots \log A_{s_1-1} \le 2c_{32} R_{S_1} (\log^* P)^{s_1-2} \tag{5.19}$$

where $c_{32} = c_{26} s_1 d^{s_1-2} \delta_K^{-(s_1-1)}$. Let

$$\Phi = c_{33} \frac{P^d}{(\log^* P)^{s_1+1}} \log A_1 \dots \log A_{s_1} \log(10 s_1 d \log A) \tag{5.20}$$

with A defined later, and with $c_{33} = c_{34}(s_1)(d^2/\log 2)^{s_1+1}$, where $c_{34}(s_1) = 22000(9.5(s_1+1))^{2(s_1+1)}$.

We distinguish two cases. First assume that $\log H < c_{35} R_{S_1}$ where $c_{35} = c_{25}(\delta_K/d)^{2-s_1}$. Then we get as in [6] that

$$\log A := \max_{1 \le i \le s_1} \log A_i \le c_{36} R_{S_1} \tag{5.21}$$

with $c_{36} = c_{27}\delta_K^{-1} + (0,2 \cdot \log 2)^{-1}$. Further, using an estimate of Kunrui Yu [39] (cf. [6], Proposition 2) it follows that

$$h(x_2/x_3) \le \exp\{2s(\log^* P)\Phi \log(c_{37} \log h(x_1))\} \tag{5.22}$$

with $c_{37} = dc_{25}$.

Next assume that $\log H \ge c_{35} R_{S_1}$. Then we deduce again (5.21). Further, using again the estimate of [39] and following the argument of [6], we infer that

$$h(x_2/x_3) \le \exp\left\{2s(\log^* P)\Phi \log\left(\frac{c_{38} R_{S_1} \log h(x_1)}{\log^* P \log A_{s_1}}\right)\right\} \tag{5.23}$$

with $c_{38} = c_{25} \cdot c_{27}$.

It follows from (5.1), (5.2) and (5.19) to (5.23) that in both cases

$$h(x_i/x_j) < \exp\left\{c_{39} \frac{P^d}{\log^* P} R_{S_1} \log^*(R_{S_1}) \log H \log\left(\frac{\log h(x_1)}{\log H}\right)\right\} \tag{5.24}$$

for each $i \ne j$, subject to the condition that $h(x_1) \ge \exp\{c_{40} R_{S_1} \log H\}$ where $c_{39} = 3^{25}(9d^4/d_1)^{s_1+1} s_1^{5s_1+10}$ and $c_{40} = (ds_1^2)^{2s_1}$. We recall that by Lemma 3 of [6], $R_{S_1} \ge 0,1$. Now (5.18) and (5.24) prove the first part of Lemma 3. □

Proof of Theorem 2. We use the notation and follow the arguments of the proof of Theorem 1, and apply Lemma 3 instead of Lemma 1. We can write again (2.1) in the form (2.1'), where l_i has now integral coefficients in M_i with heights $\le A_1$, $\beta \in O_T$ and $h(\beta) \le B_1$. Denote by O_{V_i}, $O_{V_i}^*$ the ring of V_i-integers and its unit group in M_i for $i = 1, \dots, n$. Let $\mathbf{x} \in O_T^m$ be a solution of (2.1'), with $l(\mathbf{x}) \neq 0$ for $l \in \mathcal{L}'$ if $k > 1$. Putting $l_i(\mathbf{x}) = \beta_i$, we deduce that $\beta_i = \gamma_i \varepsilon_i$ with $\varepsilon_i \in O_{V_i}^*$, $\gamma_i \in O_{V_i}$ and

$$h(\gamma_i) \leq B_1 \exp\{c_{20} R + h \log Q\} = E_7$$

where c_{20} denotes the constant occurring in Lemma 2 with d replaced by d_3. Denote by ε that ε_i for which $h(\varepsilon_i)$ is maximal. First consider the case when $h(\varepsilon) \geq \exp\{c_{41} R_V \log H\}$ with $c_{41} = (d_3 v^2)^{2v}$ and with H defined below. Suppose that l_{i_1}, l_{i_2} is an edge in $\mathcal{G}_L(\mathcal{L}_F)$. If l_{i_1}, l_{i_2} are linearly independent then there is an $l_{i_{1,2}} \in \mathcal{L}_F$ such that two of $l_{i_1}, l_{i_2}, l_{i_{1,2}}$ are conjugate over L. Further denote by K the composite of $M_{i_1}, M_{i_2}, M_{i_{1,2}}$. There are non-zero integers $\lambda_{i_1}, \lambda_{i_2}, \lambda_{i_{1,2}}$ in K with heights at most A_2 such that putting $\alpha_i = \lambda_i \gamma_i$, we have $h(\alpha_i) \leq A_2 E_7 = H$ for $i \in \{i_1, i_2, i_{1,2}\}$ and (5.6) holds. On applying Lemma 3 to (5.6) we deduce that

$$h(\varepsilon_{i_1}/\varepsilon_{i_2}) < \exp\left\{c_{42} \frac{P^{d_3}}{\log^* P} R_V (\log^* R_V)(\log H) \log\left(\frac{\log h(\varepsilon)}{\log H}\right)\right\} = E_8,$$

where $c_{42} = 3^{25}(9 d_3^4/d)^{v+1} v^{5v+10}$. It is clear that this holds in that case, too, when l_{i_1}, l_{i_2} are linearly dependent. If now l_{i_2}, l_{i_3} is an edge in $\mathcal{G}_L(\mathcal{L}_F)$ then we obtain in the same way that $h(\varepsilon_{i_2}/\varepsilon_{i_3}) \leq E_8$, whence $h(\varepsilon_{i_1}/\varepsilon_{i_3}) \leq E_8^2$. By using assumption (ii') on $\mathcal{L}_F$ and repeating the above procedure we deduce that for each j with $1 \leq j \leq k$, there is an η_j (namely, one can choose $\eta_j = \varepsilon_i^{-1}$ for an $i \in \mathcal{I}_j$) such that

$$h(\eta_j \beta_i) \leq E_8^n \qquad \text{for } i \in \mathcal{I}_j. \tag{5.25}$$

We may assume that $\eta_1 = \varepsilon^{-1}$.

Suppose that $k > 1$. Then following the corresponding arguments of the proof of Theorem 1 we infer that

$$h(\eta_1 \beta_i) \leq E_9 \qquad \text{for } i = 1, \dots, n, \tag{5.26}$$

where

$$E_9 = E_8^n (m! A_1^m E_8^n)^{2m(k-1)}.$$

We deduce from (5.26) and $\eta_1^n = (\eta_1 \beta_1) \dots (\eta_1 \beta_n)/\beta$ that $h(\eta_1) \leq B_1^{1/n} E_9$ and so, using the shape of E_9 and H, we get

$$h(\eta_1) < \exp\{c_{43} P^{d_3} R_V (\log^* R_V)(\log^*(P R_V)/\log^* P) \log H\} = E_{10} \tag{5.27}$$

where $c_{43} = 4mnkc_{42}\log(2mnkc_{41})$. For $\eta_1{=}\varepsilon^{-1}$, this obviously holds for the case $h(\varepsilon){<}\exp\{c_{41}R_V \log H\}$, too. (5.26) and (5.27) imply that

$$h(\beta_i) \le E_{10}^2, \qquad i = 1, \dots, n. \tag{5.28}$$

Now we can follow again the corresponding arguments of the proof of Theorem 1, and after some calculations we arrive at (2.5) and (2.6). □

Proof of Corollary 2. Put $M' = L(\alpha_1, \dots, \alpha_m)$, and denote by $\mathcal{L}$ the set of the conjugates of the linear form $l(\mathbf{X}) = \alpha_1 X_1 + \cdots + \alpha_m X_m$ with respect to M'/L. Partition the linear forms in $\mathcal{L}$ into subsets so that l', l'' belong to the same subset if the coefficients of $X_1, \dots, X_{m-1}$ in l', l'' coincide. Then we get a partition $\mathcal{L}_1, \dots, \mathcal{L}_k$ with k denoting the degree of $L(\alpha_1, \dots, \alpha_{m-1})$ over L, and it is easily seen that each of the $\mathcal{L}_1, \dots, \mathcal{L}_k$ is triangularly connected over L. Further, $\mathcal{L}$ has the properties (i), (iii) with $\mathcal{L}' = \{X_m\}$. Now Corollary 2 immediately follows from Theorem 2. □

Proof of Corollary 3. Let $x_1, \dots, x_m$ be a solution of (2.8), and denote by m' the greatest integer with $x_{m'} \neq 0$. If $m' \ge 2$, Corollary 2 applies with m' instead of m, while for $m' = 1$ the assertion is trivial. □

6. Proofs of Theorem 3 and Corollaries 4, 5

Proof of Theorem 3. Suppose that $f \in O_T[X]$ is a monic polynomial with $\deg(f) = n$, $\deg(f_0) = m$, $D(f_0) = \beta$ and with roots in K. Denote by $\alpha_1, \dots, \alpha_n$ the roots of f. Assume, for convenience, that $\alpha_1, \dots, \alpha_m$ are the roots of f_0. The case $m = 1$ being trivial, we assume that $m \ge 2$. Denote by O_S the ring of S-integers in K. Putting $x_i = \alpha_i - \alpha_1$ for $i = 1, \dots, m$, we have $\alpha_i \in O_S$ and $x_i \in O_S$. Further, with the notation

$$F(X_2, \dots, X_m) = X_2 \cdots X_m \prod_{2 \le i < j \le m} (X_j - X_i)$$

$D(f_0) = \beta$ can be written in the form

$$F(x_2, \dots, x_m) = \pm\beta_0 \qquad \text{with } x_2, \dots, x_m \in O_S, \tag{6.1}$$

where $\beta_0 \in O_S \backslash \{0\}$ and $\beta_0^2 = \beta$. We have $h(\beta_0) \le B^{1/2}$. It is easy to verify that the decomposable form F satisfies the assumptions of Theorem 1 with

$k = 1$. Hence, by Theorem 1 we deduce from (6.1) that

$$\max_{2\le i\le m} h(x_i) < \exp\{c_1 m(m-1)P^d R_S(\log^* R_S)^2 \times (R_K + h_K\log Q + \log B + m^3)\} \tag{6.2}$$

with the c_1 specified in Theorem 1. Further, if $T = T_\infty$, this bound can be replaced by

$$\exp\{c_2 m(m-1)R_K(\log^* R_K)(R_K + \log B + m^3)\}$$

with the c_2 occurring in Theorem 1.

The sum $a_0 = \alpha_1 + \cdots + \alpha_m$ is contained in Q_T. Putting $\theta = -(x_1 + \cdots + x_m)$, it follows from (6.2) that $h(\theta) \le mE_{11}^{m-1}$ where E_{11} denotes the upper bound obtained in (6.2). Further, we have $m\alpha_1 - a_0 = \theta$. By Lemma 6 of [10], there is an $a_1 \in O_L$ such that $a_1 \equiv a_0 \pmod m$ in O_T and $h(a_1) \le lm|D_L|^{1/2}$. Set $\alpha_1^* = (\theta + a_1)/m$. Then $h(\alpha_1^*) \le 3lm^3|D_L|^{1/2}E_{11}^{m-1}$ and $\alpha_1 = a + \alpha_1^*$ with some $a \in O_T$. Further, we have $\alpha_1^* \in O_S$. Finally, with the notation $\alpha_i^* = x_i + \alpha_1^*$ it follows that

$$\alpha_i = a + \alpha_i^* \tag{6.3}$$

and

$$h(\alpha_i^*) \le 6lm^3|D_L|^{1/2}E_{11}^m, \qquad i = 1, \ldots, m. \tag{6.4}$$

Since each α_i, $i > m$, is equal to one of the α_i, $1 \le i \le m$, (6.3) and (6.4) are valid for each i with $1 \le i \le n$. Now (3.1) and (3.2) easily follow from (6.4). □

Proof of Corollary 4. Let $f \in O_L[X]$ be a monic polynomial with $\deg(f) = n$, $\deg(f_0) = m$, $D(f_0) \in \beta\mathcal{T}$ and with roots is K. Let S denote the subset of places on K which consists of the infinite places and of those finite places which are extensions of the places on L, associated to the prime ideals $\wp_1, \ldots, \wp_t$. Let s and R_S be as in Section 3. Then it follows from Theorem 3 in the same way as Theorem 9 was deduced from Theorem 7 in [24] that f is O_L-equivalent to a polynomial of the form $\eta^n f^*(\eta^{-1}X)$, where $\eta \in \mathcal{T}$, $f^* \in O_L[X]$ and

$$h(f^*) < |D_L|^{n/2}\exp\{m^6 n(c_{44}s)^{5s}P^d R_s(\log^* R_S)^2 \times (R_K + h_K\log Q + \log^* b + R_L + h_L\log Q)\}. \tag{6.5}$$

Here R_L, h_L denote the regulator and class number of L, and c_{44} is an effectively computable positive constant which depend only on d.

We have $ld_0 = d$ and $s \leq d_0(l+t)$. Further, $R_L \leq c_{45} R_K$, $h_L \leq c_{46} h_K$, where c_{45}, c_{46} are effectively computable positive constants which depend only on l (see e.g. [7], Lemma 4). Finally, in view of (2.4) we have $R_S \leq R_K h_K (d^t W)^{d_0}$. Using these estimates, (3.3) follows at once from (6.5). □

Proof of Corollary 5. Corollary 5 follows from Corollary 4 in the same way as we deduced Theorem 2 from Theorem 1 in [21], except the following modification. The above-mentioned deduction in [21] involved some arguments from the proof of Theorem 1 of ([15], p. 188). At the end of the proof of this theorem of [15] we can work with $p_1, \ldots, p_s$ in place of P^s and hence, in (21) of [21], $P^{s(lk-1)}$ can also be replaced, with our present notation, by Q^{lk-1}. Now, after some calculations, the estimate (3.4) of the present paper follows from (3.3) in our Corollary 4. □

7. Proofs of Theorem 4 and Corollaries 6, 7, 8

We keep the notation of Section 4. Let $\sigma_1 = \mathrm{id}, \sigma_2, \ldots, \sigma_n$ be the distinct L-isomorphisms of M, and denote by $M^{(i)} = \sigma_i(M)$ and by $l^{(i)}(\mathbf{X}) = \sigma_i(l(\mathbf{X})) = X_0 + \sigma_i(\alpha_1)X_1 + \cdots + \sigma_i(\alpha_m)X_m$ the conjugates of M and $l(\mathbf{X}) = X_0 + \alpha_1 X_1 + \cdots + \alpha_m X_m$, respectively, with respect to M/L. Because of the assumption $M = L(\alpha_1, \ldots, \alpha_m)$, the linear forms $l^{(i)}$ are pairwise non-proportional. Putting $l_{ij}(\mathbf{X}) = l^{(i)}(\mathbf{X}) - l^{(j)}(\mathbf{X})$ we have

$$D_{M/L}(\alpha_1 X_1 + \cdots + \alpha_m X_m) = \prod_{\substack{1 \leq i,j \leq n \\ i \neq j}} l_{ij}(\mathbf{X}). \tag{7.1}$$

Denote by $\mathcal{L}$ the system of linear forms l_{ij}. For distinct $u, v, w \in \{1, \ldots, n\}$, we have $l_{uv} + l_{vw} + l_{wu} = 0$, hence any two of l_{uv}, l_{vw}, l_{wu} are connected by edge in the graph $\mathcal{G}(\mathcal{L})$. This implies that $\mathcal{G}(\mathcal{L})$ is triangularly connected. In order to apply Theorem 2 to discriminant form equation (4.1), we must prove that if the normal closure of M over L, denoted by K, is "large" with respect to M then $\mathcal{L}$ satisfies the assumptions (i), (ii'), (iii) made in Theorem 2. It is known that $\operatorname{rank} \mathcal{L} = m$. Hence there remain the cases (ii'), (iii).

Define the subgraph $\mathcal{G}_L^*(\mathcal{L})$ of $\mathcal{G}_L(\mathcal{L})$ in the following way. $\mathcal{G}_L^*(\mathcal{L})$ has vertex set $\mathcal{L}$, and any two of the above l_{uv}, l_{vw}, l_{wu} are connected by edge in $\mathcal{G}_L^*(\mathcal{L})$ if there is a permutation i_u, i_v, i_w of u, v, w and $\sigma \in \mathrm{Gal}(K/L)$, such that $\sigma(l^{(i_u)}) = l^{(i_u)}, \sigma(l^{(i_v)}) = l^{(i_w)}$. In this case $\sigma(l_{i_u i_v}) = l_{i_u i_w}$, hence any two of l_{uv}, l_{vw}, l_{wu} form an edge in $\mathcal{G}_L(\mathcal{L})$.

Lemma 4. *Suppose that $M^{(i)}M^{(j)}/L$ is not normal for any i, j with $1 \le i,\ j \le n$, and that $\mathcal{G}_L^*(\mathcal{L})$ is not connected. Let $\mathcal{L}_1, \dots, \mathcal{L}_k$ denote the vertex sets of the connected components of $\mathcal{G}_L^*(\mathcal{L})$. Then the graph $\mathcal{H}_{\mathcal{L}}(\mathcal{L}_1, \dots, \mathcal{L}_k)$ is connected.*

This implies that if $M^{(i)}M^{(J)}/L$ is not normal for any i, j then $\mathcal{L}$ satisfies the assumptions made in Theorem 2.

Proof. It suffices to show that if $l_{u,v} \in \mathcal{L}_i$, $l_{u',v'} \in \mathcal{L}_j$ and $u = u'$ or $v = v'$ then $\mathcal{L}_i$ and $\mathcal{L}_j$ is connected in $\mathcal{H} = \mathcal{H}_{\mathcal{L}}(\mathcal{L}_1, \dots, \mathcal{L}_k)$ by a path. For simplicity we consider $\mathcal{L}_1$ and $\mathcal{L}_2$, and we assume that $l_{1,2} \in \mathcal{L}_1$ and $l_{1,v} \in \mathcal{L}_2$ for some $v > 2$.

Any $\sigma \in \mathrm{Gal}(K/L)$ permutes the linear factors $l^{(1)}, \dots, l^{(n)}$. Denote by $\sigma(u)$ the index u', $1 \le u' \le n$, for which $\sigma(l^{(u)}) = l^{(u')}$, $u = 1, \dots, n$. By assumption $M^{(1)}M^{(2)}/L$ is not normal, hence there is a $\sigma \in \mathrm{Gal}(K/L)$, $\sigma \ne \mathrm{id}$, such that $\sigma(1) = 1$ and $\sigma(2) = 2$. If $\sigma(v) \ne v$, then $\sigma(v) \ne 1, 2$ and $l_{1,v} + l_{v,\sigma(v)} - l_{1,\sigma(v)} = 0$, $l_{2,v} + l_{v,\sigma(v)} - l_{2,\sigma(v)} = 0$. Since $\sigma(l_{1,v}) = l_{1,\sigma(v)}$ and $\sigma(l_{2,v}) = l_{2,\sigma(v)}$, if follows that $l_{1,v}$ and $l_{2,v}$ are connected in $\mathcal{G}_L^*(\mathcal{L})$ by a path, i.e. $l_{2,v} \in \mathcal{L}_2$. Further, $l_{1,2} = l_{1,v} - l_{2,v}$ which proves that $\mathcal{L}_1$ and $\mathcal{L}_2$ is connected in $\mathcal{H}$ by an edge.

Next consider the case when $\sigma(v) = v$. By the assumption made on $M^{(1)}M^{(2)}$ there is a w, $1 \le w \le n$, different from $1, 2, v$ for which $\sigma(w) \ne w$. If $l_{1,w} \in \mathcal{L}_i$ for some i then it follows as above that $\mathcal{L}_1$ and $\mathcal{L}_i$ are connected in $\mathcal{H}$ by an edge. Further,

$$l_{1,w} + l_{w,\sigma(w)} - l_{1,\sigma(w)} = 0, \qquad l_{v,w} + l_{w,\sigma(w)} - l_{v,\sigma(w)} = 0$$

where $\sigma(l_{1,w}) = l_{1,\sigma(w)}$ and $\sigma(l_{v,w}) = l_{v,\sigma(w)}$. This implies that $l_{v,w} \in \mathcal{L}_i$. However, we have $l_{1,w} + l_{w,v} = l_{1,v}$, hence $\mathcal{L}_i$ and $\mathcal{L}_2$ are connected in $\mathcal{H}$ by an edge. This proves that $\mathcal{L}_1$ and $\mathcal{L}_2$ are connected in $\mathcal{H}$ by a path which completes the proof. □

We shall also need the following two lemmas.

Lemma 5. *Let N be an algebraic number field of degree $k \geq 2$ with discriminant D_N, regulator R_N and class number h_N and with q complex places on N. Then*

$$R_N h_N \leq \Delta(\log \Delta)^{k-1-q}(k-1+\log \Delta)^q/(k-1)!$$

where

$$\Delta = (2/\pi)^q |D_N|^{1/2}.$$

Proof. This is an explicit version of a well-known theorem. For the proof of Lemma 5, see [28]. □

Lemma 6. *Let $N_1, \dots, N_p$ and $N = N_1 \dots N_p$ be algebraic number fields with degrees $k_1, \dots, k_p$ and k, and discriminants $D_{N_1}, \dots, D_{N_p}$ and D_N, respectively, over $\mathbb{Q}$. Then*

$$D_N \mid \prod_{i=1}^{p} D_{N_i}^{k/k_i}.$$

Proof. See [35]. □

Proof of Theorem 4. Using the above notation and relation (7.1), equation (4.1) can be written in the form

$$\prod_{\substack{1 \leq i < j \leq n \\ i \neq j}} l_{ij}(\mathbf{x}) = \beta \qquad \text{in } \mathbf{x} \in O_T^m. \tag{7.2}$$

We distinguish two cases. Assume first that for some $i \neq j$, $M^{(i)}M^{(j)}/L$ is a normal extension. Putting $K = M^{(i)}M^{(j)}$, K is the normal closure of M over L. We apply Theorem 1 to (7.2). For this purpose we have to introduce some further notation.

Let d, r, R_K, h_K and O_K denote the degree, unit rank, regulator, class number and the ring of integers of K, respectively. Let S be the set of extensions to K of the places of T on L, let $s = \mathrm{Card}(S)$ and R_S the S-regulator of K. The system $\mathcal{L}$ of linear forms l_{ij} considered above satisfies the assumptions (i), (ii) of Theorem 1 with $k = 1$ (see e.g. [26]). Hence, by Theorem 1, all solutions $\mathbf{x} = (x_1, \dots, x_m) \in O_T^m$ of (7.2) satisfy

$$\max_{1 \leq i \leq m} h(x_i) < E_{11} \tag{7.3}$$

and, if $T = T_\infty$,

$$\max_{1 \le i \le m} h(x_i) < E_{12}, \tag{7.4}$$

where E_{11} and E_{12} denote the upper bounds occurring in (2.2) and (2.3), respectively, with n, A, Q replaced by n_2^2, $2A^2$, Q^{n_2}.

We now estimate from above the paremeters of K and S under consideration in terms of the parameters involved in Theorem 4. We have $d = ln_2$, $r \le ln_2 - 1$, $s \le tn_2$. Using (2.4) and the definition of W, one can easily see that

$$R_S \le R_K h_K (ln_2)^{t_0 n_2} W^{n_2},$$

where $t_0 = \mathrm{Card}(T \setminus T_\infty)$. By Lemma 5 and $R_K > 0{,}2$ (cf. [11]), $R_K h_K$, R_K and h_K can be estimated from above in terms of $|D_K|$ and ln_2, where D_K denotes the discriminant of K. Further, by Lemma 6, $|D_K| \le |D_M|^{2n_2/n}$. After some calculations we deduce from (7.3) and (7.4) the estimates (4.2) and (4.3).

Next consider the case when $M^{(i)}M^{(j)}/L$ is not normal for any i, j, $1 \le i, j \le n$. Denote by $\mathcal{L}_1, \dots, \mathcal{L}_k$, $k \ge 1$, the vertex sets of the connected components of $\mathcal{G}_L^*(\mathcal{L})$. Then $\mathcal{L}_1, \dots, \mathcal{L}_k$ are triangularly connected over L. By Lemma 4 the graph $\mathcal{H}_{\mathcal{L}}(\mathcal{L}_1, \dots, \mathcal{L}_k)$ is connected. Further, $l(\mathbf{x}) \ne 0$ for all $l \in \mathcal{L}$ and all solutions $\mathbf{x}$ of (7.2). Hence Theorem 2 can be applied to equation (7.2). Put $M_{ij} = M^{(i)}M^{(j)}$ and denote by $R_{M_{ij}}, h_{M_{ij}}$ the regulator and class number of M_{ij}. Let V_{ij} denote the set of extensions to M_{ij} of the places in T, and $R_{V_{ij}}$ the V_{ij}-regulator of M_{ij}. Denote by v the maximum of the cardinalities of the V_{ij}, and by r, R, h and R_V the maximums of the unit ranks, regulators, class numbers and V_{ij}-regulators, respectively, of the number fields M_{ij}. Then by Theorem 2, it follows that all solutions of (7.2) satisfy

$$\max_{1 \le i \le m} h(x_i) < E_{13} \tag{7.5}$$

and, if $T = T_\infty$,

$$\max_{1 \le i \le m} h(x_i) < E_{14} \tag{7.6}$$

where E_{13}, E_{14} denote the upper bounds occurring in (2.5) and (2.6), respectively, with k, n, A, Q replaced by n, n_2^2, $2A^2$ and Q^{n_2}.

We have to estimate from above some parameters in E_{13}, E_{14}. We have $v \le tn_2$. Further,

$$R_{V_{ij}} \le R_{M_{ij}} h_{M_{ij}} (ln_2)^{t_0 n_2} W^{n_2}.$$

Using again Lemma 5, $R_{M_{ij}} h_{M_{ij}}, R_{M_{ij}}$ and $h_{M_{ij}}$, and hence R_V, R and h can be estimated from above in terms of $\max_{i,j} |D_{M_{ij}}|$ and ln_2, where $D_{M_{ij}}$ denotes the discriminant of M_{ij}. Further, by Lemma 6, $\max_{i,j} |D_{M_{ij}}| \le |D_M|^{2n_2/n}$. After some calculations it is easy to deduce now from (7.5) and (7.6) the estimates (4.2) and (4.3), respectively. □

Proof of Corollary 6. There is a primitive integral element γ in M such that $\overline{|\gamma|} \le |D_M|^{1/2}$. Then $M = L(\gamma)$ and

$$h(D_{M/L}(\gamma)) \le (2|D_M|^{1/2})^{n(n-1)}.$$

Let α be a solution of (4.4). Then it can be represented in the form

$$\alpha = y_0 + y_1\gamma + \cdots + y_{n-1}\gamma^{n-1}$$

with $y_i \in L$ for $i = 0, \ldots, n-1$. By taking conjugates with respect to M/L and using Cramer's rule we deduce that $y_i^2 = \kappa_i / D_{M/L}(\gamma)$, where $\kappa_i \in O_L$, $i = 0, \ldots, n-1$. Since $D_{M/L}(\gamma) \in O_L$, it follows that $(y_i D_{M/L}(\gamma))^2 \in O_L$. Hence, putting $y_i' = y_i D_{M/L}(\gamma)$ we infer that $y_i' \in O_L$ for $i = 0, \ldots, n-1$. Set $\beta' = \beta(D_{M/L}(\gamma))^{n(n-1)}$. Then we deduce from (4.4) that

$$D_{M/L}(y_1'\gamma + \cdots + y_{n-1}'\gamma^{n-1}) = \beta'. \tag{7.7}$$

By Theorem 4, we obtain

$$\max_{1 \le i \le n-1} h(y_i') < \exp\{c_{14} |D_M|^{n_2/n} (\log |D_M|)^{2ln_2 - 1} \times (|D_M|^{n_2/n} + n^4 \log |D_M| + \log B)\} = E_{15} \tag{7.8}$$

with the c_{14} occurring in Theorem 4.

We infer that

$$\alpha D_{M/L}(\gamma) = y_0' + \tau, \tag{7.9}$$

where $\tau \in O_M$ with $h(\tau) \le 2n|D_M|^{n^2/4}E_{15}^{n-1}$. By Lemma 6 of [10] there are ρ, $a \in O_L$ such that

$$y_0' = \rho + aD_{M/L}(\gamma) \tag{7.10}$$

and

$$h(\rho) \le l|D_L|^{1/2} \cdot |N_{L/\mathbb{Q}}(D_{M/L}(\gamma))|^{1/l}. \tag{7.11}$$

With the notation $\alpha^* = (\rho+\tau)/D_{M/L}(\gamma)$, (7.9), (7.10), (7.11) and $|D_L| \le |D_M|$ imply that $\alpha = a + \alpha^*$, where $a \in O_L$, $h(\alpha^*) \le E_{15}^n$ and (4.5) immediately follows. □

Proof of Corollary 7. Corollary 7 can be deduced from Corollary 6 in the same way as we obtained Corollaries 3.2 and 3.3 in [14] from Theorem 3B of [14]. □

Proof of Corollary 8. Let $x_1, \dots, x_n$ be a solution of (4.7). Then it follows from (4.6) that

$$D_{M/\mathbb{Q}}(w_2x_2 + \cdots + w_nx_n)a^2D_M.$$

By applying now Theorem 4 to this equation, the assertion immediately follows. □

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