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Split extension in Moufang loops

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Abstract. In this paper, we prove the following:

1. Let G be a Moufang loop of order $p^\alpha m$, $(p,m)=1$, $(p-1,p^\alpha m)=1$ and p is a prime. Suppose G has an element of order p^α. Then $G = P \rtimes K$, a split extension of a normal subloop K of order m with a subloop P order p^α.

2. Let G be a Moufang loop of odd order $p^2 m$, $(p,m)=1$, and p is the smallest prime dividing $|G|$. Then a similar result holds as in (1) with $\alpha = 2$.

I. Introduction

Let G be a group of order $p^\alpha m$, $(p,m)=1$, $(p-1,p^\alpha m)=1$ and suppose G has an element x of order p^α. Then $G = \langle x\rangle \rtimes K$, a split extension of a normal subgroup K of order m with the subgroup $\langle x\rangle$ [4]. To prove an analogous result on a Moufang loop G, we need a normal subloop K so that we can use induction on G/K. In other words, G has to be nonsimple. By Liebeck [8], every simple nonassociative Moufang loop is isomorphic to one of the Paige's loop $M^*(q)$. Considering the orders of elements in each of the conjugate classes of $M^*(q)$, as examined by Bannai and Song [2], we find that a Moufang loop G with the given properties stated above cannot be simple. Then G has a normal subloop K, and so induction will be possible.

Let G be a group of order, $p^2 m$, $(p,m)=1$, and p is the smallest prime dividing $|G|$. Then $G = P \rtimes K$, a split extension of a normal subgroup K of order m with the subgroup P of order p^2 [11]. We prove

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also an analogous result on a Moufang loop G by using repeatedly several theorems of GLAUBERMAN [5].

There exist nonassociative Moufang loops of order pq^3 with $q = 1 \pmod p$ [10]. Moufang loops of odd order p^2q^3 can be similarly constructed. Hence our two splitting theorems are very useful in studying the structure of such finite Moufang loops.

II. Definitions

1. A binary system $\langle G, \cdot \rangle$, in which specification of any two of the elements x, y, z in the equation $x \cdot y = z$ uniquely determines the third element, is called a quasigroup. If it further contains an identity element, then it is called a loop. Clearly, a group is a loop. But there are loops which are not associative.

2. A loop $\langle G, \cdot \rangle$ is a Moufang loop if $xy \cdot zx = (x \cdot yz)x$ for all x, y, z in G. From now on, G is defined as a finite Moufang loop.

3. Define $zR(x,y) = (zx \cdot y)(xy)^{-1}$,
$zL(x,y) = (yx)^{-1}(y \cdot xz)$ and
$zT(x) = x^{-1} \cdot zx$.

$I(G) = \langle R(x,y), L(x,y), T(x) \mid x, y \in G\rangle$ is called the inner mapping group of G.

4. Let x and y be elements of G. x and y are conjugate if there exists $\theta \in I(G)$ such that $x\theta = y$.

5. Let H be a subloop of G and π a set of primes.

(i) H is a normal subloop of G, $(H \triangleleft G)$, if $H\theta = H$ for all $\theta \in I(G)$ where $H\theta = \{h\theta \mid h \in H\}$.

(ii) H is a π-loop if the order of every element of H is a π-number. (A positive integer n is a π-number if every prime divisor of n lies in π).

(iii) H is a Hall π-subloop of G if $|H|$ is the largest π-number dividing $|L|$.

6. G_a, the associator subloop of G, is the subloop generated by all the associators (x,y,z) where $(x,y,z) = (x \cdot yz)^{-1} \cdot (xy \cdot z)$. Write $G_a = (G,G,G)$ also.

7. G_c, the commutator subloop of G, is the subloop generated by all the commutators $[x,y]$ where $[x,y]=(yx)^{-1}\cdot(xy)$.

8. $N=N(L)$, the nucleus of L, is the subloops generated by all n in L where $(n,x,y)=(x,n,y)=(x,y,n)=1$ for all x, y in L.

9. $Z=Z(L)$, the centre of L, is the subloop generated by all z in N such that $[z,x]=1$ for all x in L.

III. Known results with Mounfang loops

Let G be a finite Moufang loop.

$\boldsymbol{R}_1$. (a) $x\in G\Rightarrow |x|\,\big|\,|G|$. [1, p. 92, Thm. 1.2].

(b) G is disassociative. [1, p. 117, Moufang's Theorem].

(c) $x\in G$ and $\theta\in I(G)\Rightarrow x^n\theta=(x\theta)^n$ for any integer n. [1, p. 120, (4.1) and p. 117, Lemma 3.2].

$\boldsymbol{R}_2$. N and Z are normal subloops of G. [1, p. 114, Thm. 2.1 and p. 60, Lemma 1.1]. In fact N and Z are associative by their definitions.

$\boldsymbol{R}_3$. Let H be a normal subloop of G such that $H\subset N$. Then

(a) $G/C_G(H)<\operatorname{Aut}H$ where $C_G(H)=\{g\mid g\in G,\ gh=hg$ for all $h\in H\}$.

(b) $C_G(H)\cap H=Z(H)$, the centre of H. If $H=N$, then $G_a\subset C_G(N)$. [7, p. 33, Thm. 3].

$\boldsymbol{R}_4$. G is a 2-loop if and only if $|G|=2^m$ for some positive integer m. [6, p. 415, Thm.].

$\boldsymbol{R}_5$. Suppose $|G|$ is odd and K is a normal subloop of G.

(a) If K is minimal normal in G, then K is an elementary Abelian group and $(K,K,G)=1$. [5, p. 402, Thm. 7].

(b) If $(K,K,G)=1$ and $(|K|,|G/K|)=1$, then $K\subset N$. [5, p. 405, Thm. 10].

(c) G is solvable. [5, p. 413, Thm. 16].

(d) G contains a Hall π-subloop, π a set of primes. [5, p. 409, Thm. 12.]

(e) If $H<G$ then $|H|\,\big|\,|G|$. [5, p. 359, Thm. 2].

$\boldsymbol{R}_6$. If H is a subloop of G, $x \in G$ and d is the smallest positive integer such that $x^d \in H$, then $|\langle H, x\rangle| \geq |H|d$. [3, p. 5, Lemma 0].

$\boldsymbol{R}_7$. There exist simple nonassociative Moufang loops $M^*(p^n)$ with $|M^*(p^n)| = p^{3n}(p^{4n}-1)/d(p)$ where $d(2) = 1$ and $d(p) = 2$ if p is an odd prime. [9, p. 474, Thm. 4.1].

$\boldsymbol{R}_8$. G is a nonassociative simple Moufang loop $\iff$ G is isomorphic with $M^*(p^n)$ for some prime p. [8, p. 33, Theorem].

$\boldsymbol{R}_9$. The conjugacy classes of $M^*(p^n)$ contain elements whose orders are 1, p, divisors of $p^n - 1$ or divisors of $p^n + 1$. [2, p. 224, Thm. 2.1.1 and p. 227, Thm. 2.1.2].

IV. Moufang loops of order $p^\alpha m$

Lemma 1. *Let G be a simple nonassociative Moufang loop of order $2^\alpha m$, $(2, m) = 1$. Then G has no element of order 2^α.*

Proof. By R_7 and R_8, G is isomorphic to $M^*(q)$ for some q where $q = p^n$ and p is a prime. Let x be any 2-element of $M^*(q)$.

Case 1: $p \geq 3$. Let $q - 1 = 2^{\beta_1}m_1$ and $q + 1 = 2^{\beta_2}m_2$, where m_1 and m_2 are odd. Suppose $\beta = \max\{\beta_1, \beta_2\}$. By R_9, $|x| \bigm| 2^\beta$.

$$\begin{aligned}|M^*(q)| &= \frac{q^3(q^4-1)}{2} = \frac{q^3(q^2+1)}{2}(q-1)(q+1)\\ &= \frac{q^3(q^2+1)}{2}2^{(\beta_1+\beta_2)}m_1m_2 = 2^\alpha m.\end{aligned}$$

Since q is odd, $2 \mid (q^2+1)$. So $\beta_1 + \beta_2 \leq \alpha$. Thus $|x| \leq 2^\beta < 2^{\beta_1+\beta_2} \leq 2^\alpha$ as $\beta_1 > 0$ and $\beta_2 > 0$.

Case 2: $p = 2$. As $q-1$ and $q+1$ are odd, $2 \nmid (q-1)(q+1)$. So by R_9, $|x| = 2$. Now $|M^*(2^n)| = 2^{3n}(2^{4n}-1) = 2^\alpha m$. So $2^\alpha = 2^{3n} \geq 2^3 > 2 = |x|$. Thus G has no element of order 2^α.

Lemma 2. *Let G be a Moufang loop and M a normal subloop of G. Suppose H is a normal Hall π-subloop of M. Then H is normal in G in each of the following two cases:*

(a) *G is of odd order;*

(b) $|M| = 2^r m$ *where* m *is odd,* $|H| = m$ *and there exists an element of order* 2^r *in* M*.*

PROOF. Suppose $H \ntriangleleft G$. Then there exists $\theta \in I(G)$ such that $H\theta \neq H$. Since any inner mapping θ is a permutation of G, $H\theta - H \neq \emptyset$.

Let $h\theta \in H\theta - H$. Since $H \triangleleft M \triangleleft G$, $H\theta \subset M\theta = M$. Since H and $\langle h\theta\rangle$ are both subloops of M with $H \triangleleft M$, clearly $H\langle h\theta\rangle$ is a subloop of M. Now by $R_1(c)$, $(h\theta)^{|h|} = \left(h^{|h|}\right)\theta = 1\theta = 1$. So $|h\theta| \bigm| |h|$. By $R_1(a)$, $|h| \bigm| |H|$.

So $|h\theta| \bigm| |H|$. Also $|H\langle h\theta\rangle| = \frac{|H|\,|\langle h\theta\rangle|}{|H\cap\langle h\theta\rangle|}$.

(a) Suppose G is of odd order. Since $|h\theta| \bigm| |H|$, $H\langle h\theta\rangle$ is a π-loop in M strictly containing the Hall π-subloop H. So $|H\langle h\theta\rangle| \nmid |M|$. This is a contradicition by $R_5(e)$.

So $H \triangleleft G$ if G is of odd order.

(b) Suppose $|M| = 2^r m$ where m is odd, $|H| = m$ and there exists an element x of order 2^r in M. Since $|h\theta| \bigm| |H|$, $|H\langle h\theta\rangle|$ is odd. Also $|H\langle h\theta\rangle| > m$ as $h\theta \notin H$. Now by $R_1(a)$, $x^d \notin H\langle h\theta\rangle$ for each $0 < d < 2^r$. But $x^{2^r} = 1 \in H\langle h\theta\rangle$. Thus $|\langle H\langle h\theta\rangle, x\rangle| \geq |H\langle h\theta\rangle| 2^r$ by $R_6 > m2^r = |M|$.

This is a contradiction as $\langle H\langle h\theta\rangle, x\rangle \subset M$. Hence $H \triangleleft G$ in this case also.

Lemma 3. *Suppose* G *is a Moufang loop of order* $p^\alpha m$*,* $(p,m) = 1$*;* K *is a normal subloop of* G *such that* $|G/K| = p^\beta m_0$*,* $m_0 \mid m$*. Suppose there exists an element* x *of order* p^α *in* G*. Then* xK *is an element of order* p^β *in* G/K*.*

PROOF. $(xK)^{p^\alpha} = x^{p^\alpha}K = 1K \Rightarrow |xK| \bigm| p^\alpha \Rightarrow xK$ is a p-element in $G/K \Rightarrow |xK| \bigm| p^\beta$ by $R_1(a)$. So $|xK| = p^\gamma$, $\gamma \leq \beta$. Then $(xK)^{p^\gamma} = x^{p^\gamma}K = 1K$ and $x^{p^\gamma} \in K$. Since $|K| = p^{\alpha-\beta}m/m_0$ and x^{p^γ} is a p-element in K, $\left|x^{p^\gamma}\right| \Bigm| p^{\alpha-\beta}$ by $R_1(a)$. Thus $\left(x^{p^\gamma}\right)^{p^{\alpha-\beta}} = 1$ or $x^{p^{\alpha+\gamma-\beta}} = 1$. So $\alpha + \gamma - \beta \geq \alpha$. Then $\gamma \geq \beta$. Hence $\gamma = \beta$. So $|xK| = p^\beta$.

Lemma 4. *Let* G *be a Moufang loop of order* $2^\alpha m$*,* $(2,m) = 1$*. Suppose* G *has an element* x *of order* 2^α*. Then* $G = \langle x\rangle \rtimes K$*, i.e.,* G *is a split extension of a cyclic group* $\langle x\rangle$ *of order* 2^α *with a normal subloop* K *of order* m*.*

PROOF. If G is a group, we are through by [4, p. 14, Problem 2.16]. So we assume that G is nonassociative. By Lemma 1, we know that G is nonsimple. Let K be a maximal normal subloop of G. Let $|G/K| = 2^{\beta} m_0$, $0 \leq \beta \leq \alpha$, $m_0 \mid m$.

Case 1: $1 < m_0 < m : |G/K| = 2^{\beta} m_0$.

1$(a) : \beta = 0$. Then $|G/K| = m_0$ and $|K| = 2^{\alpha}(m/m_0)$. By Lemma 3, $|xK| = 1$. Hence $x \in K$. By induction, there exists a subloop K_0 of order m/m_0 normal in K. By Lemma 2, $K_0 \lhd G$. Now $|G/K_0| = 2^{\alpha} m_0$ and xK_0 is an element of order 2^{α} in G/K_0 by Lemma 3. By induction, there exists a subloop K_1/K_0 of order m_0 normal in G/K_0. Then $K_1 \lhd G$ and $|K_1| = |K_0| m_0 = m$. So $G = \langle x \rangle \rtimes K_1$.

1$(b) : \beta \geq 1$. By Lemma 3, xK is an element of order 2^{β} in G/K. By induction, there exists a subloop K_1/K of order m_0 normal in G/K. Thus $K_1 \lhd G$ and $|K_1| = |K| m_0 > |K|$, contradicting the maximality of K.

Case 2: $m_0 = 1 : |G/K| = 2^{\beta}$.

2$(a) : \beta = 0$. $|G/K| = 1 \Rightarrow G = K$, a contradiction.

2$(b) : 0 < \beta < \alpha$. $|K| = 2^{\alpha - \beta} m$. Since $xK \in G/K$, $(xK)^{2^{\beta}} = 1K$ and $x^{2^{\beta}} \in K$. Clearly $\left|x^{2^{\beta}}\right| = 2^{\alpha - \beta}$. By induction, K has a normal subloop K_0 of order m. Thus $K_0 \lhd G$ by Lemma 2(b). So $G = \langle x \rangle \rtimes K_0$.

2$(c) : \beta = \alpha$. $|K| = m$ and $G = \langle x \rangle \rtimes K$.

Case 3: $m_0 = m : |G/K| = 2^{\beta} m$.

3$(a) : \beta = 0$. $|G/K| = m$. Suppose m is not a prime. Then G/K is solvable by $R_5(c)$. So it has proper normal subloop K_1/K. Then $K_1 \lhd G$ and $|K| < |K_1| < |G|$. This contradicts that K is a maximal normal subloop of L. So $m = p$, an odd prime. Now $|G| = 2^{\alpha} p$. By $R_1(a)$, there exists $w \in G$ sucht that $|w| = p$ as otherwise, G would be a 2-loop, which is impossible by R_4. Now by R_6, $G = \langle x, w \rangle$ is a group by diassociativity, a contradiction.

3$(b) : 0 < \beta < \alpha$. By Lemma 3, xK is an element of order 2^{β} in G/K. By induction, there exists a subloop K_1/K of order m normal in G/K. Then $K_1 \lhd G$ and $|K_1| = m|K| > |K|$, a contradiction.

3$(c) : \beta = \alpha$. Then $|K| = 1$, a contradiction since K is a maximal normal subloop of G.

Theorem 1. *Let G be a finite Moufang loop of order $p^\alpha m$, $(p,m)=1$, $(p-1,p^\alpha m)=1$. Suppose G has an element x of order p^α. Then $G=\langle x\rangle \rtimes K$, i.e., G is a split extension of a cyclic group $\langle x\rangle$ of order p^α and a normal subloop K of order m.*

Proof. By Lemma 4, we can assume that p is an odd prime. Since $(p-1,p^\alpha m)=1$, G is of odd order. By $R_5(c)$, G is solvable. Let K be a minimal normal subloop of G. By $R_5(a)$, K is an elementary abelian q-group (where q is a prime).

Case 1: $q=p$. $K<\langle x\rangle$. Otherwise, $K\langle x\rangle$ is a p-subloop of G whose order is bigger than p^α, contradicting $R_5(e)$. As $\langle x\rangle$ is cyclic, K is cyclic. So $K=C_p$ as it is an elementary abelian group.

1(a) : $K \nleq \langle x\rangle$. Then $\alpha\geq 2$, $|G/K|=p^{\alpha-1}m$ and xK is an element of order $p^{\alpha-1}$ by Lemma 3. By induction, there exists a subloop K_1/K of order m normal in G/K. Then $K_1 \lhd G$ and $|K_1|=pm$. Now $x^{p^{\alpha-1}}$ is an element of order p in K_1. By induction, there exists a subloop K_2 of order m normal in K_1. Now K_2 is a normal Hall subloop in K_1 and $K_1 \lhd G$ implies that $K_2\lhd G$ by Lemma 2(a). Thus $G=\langle x\rangle \rtimes K_2$.

1(b) : $K=\langle x\rangle = C_p$. Now (K,K,G)=1 by $R_5(a)$ and $(|K|,|G/K|)$=1 $\Rightarrow K\subset N$, the nucleus of G, by $R_5(b)$. By $R_3(a)$, $G/C_G(K)\leq \operatorname{Aut} K$. As the order of the group of automorphisms of C_p is $p-1$, $\left|\frac{G}{C_G(K)}\right| \,\Big|\, p-1$. As $(p-1,|G|)=(p-1,p^\alpha m)=1$, $G=C_G(K)$. Thus $K\subset Z$, the centre of G. By $R_5(d)$, there exists a Hall subloop H of order m in G. Then $G=HZ$.

Now
$$G_a=(G,G,G)=(HZ,HZ,HZ)=(H,H,H)\subset H; \text{ and}$$
$$G_c=[G,G]=[HZ,HZ]=[H,H]\subset H.$$

Let $h\in H$, $x,y\in G$.

Then
$$hT(x)=x^{-1}hx=hh^{-1}x^{-1}hx=h[h,x] \text{ and}$$
$$\begin{aligned} hL(x,y)&=hR(x^{-1},y^{-1}), && \text{by [1, p. 124, Lemma 5.4, (5.13)]}\\ &=h(h,y,x)^{-1}, && \text{by [1, p. 124, Lemma 5.4, (5.16)].}\end{aligned}$$

Since $G_a\subset H$ and $G_c\subset H$, $h\theta\in H$ for all $\theta\in I(G)$. Thus $H\lhd G$ and $G=\langle x\rangle \lhd H$.

Case 2: $q \neq p$. Let $|K| = q^\gamma$. Then $|G/K| = p^\alpha \frac{m}{q^\gamma}$ where $q^\gamma \mid m$.

2(a) : $m > q^\gamma$. By Lemma 3, xK is an element of order p^α in G/K. By induction, there exists a normal subloop K_1/K of order m/q^γ in G/K. Therefore $K_1 \lhd G$ and $|K_1| = \frac{|K|m}{q^\gamma} = m$. Thus $G = \langle x \rangle \rtimes K_1$.

2(b) : $m = q^\gamma$. Then $G = \langle x \rangle \rtimes K$ as required.

Corollary 1. *Let G be a Moufang loop of order $p^\alpha m$, $(p, m) = 1$, $(p-1, p^\alpha m) = 1$ and suppose G has an element of order p^α. Then G is solvable.*

Proof. *Case 1:* $p = 2$. Then by Theorem 1, $G = C_{2^\alpha} \rtimes K$ with $|K| = m$ which is odd. So G/K is isomorphic to C_{2^α} which is solvable. By $R_5(c)$, K is solvable. Thus G is solvable.

Case 2: $p \neq 2$. Then $|G|$ is odd as $(p-1, p^\alpha m) = 1$. Thus G is solvable by $R_5(c)$.

V. Moufang loops of odd order $p^2 m$

Theorem 2. *Let G be a Moufang loop of odd order $p^2 m$, $(p, m) = 1$, p the smallest prime dividing $|G|$. Then there exist subloops M and P in G with $|P| = p^2$, $|M| = m$, $M \lhd G$ such that $G = P \rtimes M$.*

Proof. If G is a group, we are through by [10, p. 141, 6.3.16]. By $R_5(c)$, G is solvable. Let K be a minimal normal subloop of G. By $R_5(a)$, K is elementary abelian. Let $|K| = q^\alpha$. Existence of P is guaranteed by $R_5(d)$.

Case 1: $q \neq p$. If $|K| = m$, then $K = M$ and we are through. If $|K| < m$, then $|G/K| = p^2 \left({}^{m}\!/_{q^\alpha}\right)$. By induction, there exists a normal subloop M/K in G/K with $|M/K| = \frac{m}{q^\alpha}$. Then $M \lhd G$ and $|M| = \frac{m}{q^\alpha} |K| = m$.

Case 2: $q = p$. Then by $R_5(e)$, $\alpha = 1$ or 2.

2(a): $\alpha = 1 : |K| = p$. By $R_5(d)$, we can get an element xK of order p in G/K. $|G/K| = pm$. So by Theorem 1, there exists a normal subloop $\widehat{M}/K$ of order m in G/K. Then $\widehat{M} \lhd G$ and $|\widehat{M}| = pm$. Similarly by $R_5(d)$ and by Theorem 1, there exists a subloop M of order m normal in $\widehat{M}$. By Lemma 2(a), $M \lhd G$.

2(b): $\alpha = 2 : |K| = p^2$. By $R_5(a)$ and $R_5(b)$, $K \subset N$. Since K is an elementary abelian group, $K = C_p \times C_p$.

Now by $R_3(a)$, $|G/C_G(K)| \bigm| |\operatorname{Aut} K| = (p+1)p(p-1)^2$ using [10, p. 141, 6.3.15]. Since $K \subset C_G(K)$, and p is the smallest prime dividing $|G|$, $|G/C_G(K)| \bigm| (p+1)$. As p is odd and 2 does not divide the order of G, $G = C_G(K)$. Thus $K \subset Z$.

By $R_5(d)$, there exists a subloop M of order m in G. As $G = KM = ZM$, it can be shown in a similar way as before (see the proof of Theorem 1, Case 1(b)) that $M \lhd G$.

Corollary 2. *Let G be a Moufang loop of odd order $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$ where $p_1 < p_2 < \cdots < p_m$ and $1 \le \alpha_i \le 2$. Then there exists a subloop of order $p_m^{\alpha_m}$ normal in G.*

Proof. For $\alpha_1 = 1$, $R_5(d)$ guarantees the existence of an element of order p_1 in G. So by Theorem 1 or Theorem 2, there exists M_1, a normal subloop in G with $|M_1| = p_2^{\alpha_2} \dots p_m^{\alpha_m}$. Again there exists a subloop M_2 of order $p_3^{\alpha_3} \dots p_m^{\alpha_m}$ normal in M_1. By Lemma 2(a), $M_2 \lhd G$. By this process, we get a subloop M_{m-1} of order $p_m^{\alpha_m}$ normal in G.

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