Publ. Math. Debrecen
52 / 1-2 (1998), 43–54

The rolling polyhedra

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Abstract. We consider a convex polyhedron P standing one of its faces on a fixed plane S. We rotate P into another similar position around any of its edges lying on S. We call the trace of P the set of all points of A of S for which A coincides with some vertex of P in some position of P.

We investigate the traces of the Archimedean polyhedra showing if P is not the (3,6,6) then its trace is everywhere dense in S.

1. Introduction

We consider a convex polyhedron P standing with one of its faces on a fixed plane σ. We rotate P into another similar position around any of its edges lying onσ (i.e. after the rotation another face of P will lean on σ). We shall call such a rotation a *P-rotation* and we perform all possible P-rotations indefinitely.

Let X be a vertex of P. We call the trace of X the set of all points in σ which are occupied by X in some position of P.

Further, the trace of P is the union of the traces of its vertices. The trace of P is denoted by T_P.

We assume throughout the paper that $0, 1 \in T_P$ and there is a position of P where adjacent vertices A_1 and A_2 are at 0 at 1. We call a polyhedron *trace-dense* if its trace is everywhere dense in σ.

In [3] the first author characterized all trace-dense regular polyhedra and rectangular parallelepipeds. He also established a sufficient condition for a general polyhedron to be trace-dense. It was

Mathematics Subject Classification: 52B05, 52B10.
*Research supported by CEE fund No. CIPA-CT92-4022.

Theorem A. *Let x be a vertex of the convex polyhedron P, and denote by $\Sigma(x)$ the sum of all angles which occur at x on the faces. If $\Sigma(x)/\pi$ is irrational for at least one vertex x, then P is trace-dense.*

An interesting class of polyhedra for which the condition of Theorem A is not true, i.e. $\forall x,\ x \in P, \Sigma(x)/\pi$ is rational, is the class of *semi-regular* or *Archimedean* polyhedra. In Section 3 we shall investigate the 13 semi-regular polyhedra. We prove

Theorem. *If P is an Archimedean polihedron and $P \neq (3,6,6)$ then P is trace-dense.*

We remark here that it is easy to see (and was mentioned in [3] as Example 1) that if $P = (3,6,6)$ then the trace of P is a lattice of equilateral triangles, so $(3,6,6)$ is not trace-dense.

We merely mention that similar questions were investigated in [2] and [3].

2. Notation

We shall identify σ with the complex plane $\mathcal{C}$. For α real, put $e(\alpha) = e^{2\pi i\alpha}$.

Let Q be the set of the rational and Q^* the set of the irrational numbers.

Let

$$U_\varepsilon(X) = \{Y : Y \in \mathcal{C};\ d(X,Y) < \varepsilon\}$$

be the ε-neighbourhood of the point X, where $d(X,Y)$ is the distance of X and Y.

Let x be a point of $\mathcal{C}$ and Y a subset of $\mathcal{C}$. Write $x + Y = \{x + y : y \in Y\}$. We shall follow the usual notation for Arcimedean polyhedra. Such a polyhedron may be denoted by a symbol giving the numbers of sides of the faces around one vertex, e.g. $(3,4,3,4)$ is the *cuboctahedron*, $(3,5,3,5)$ is the *icosidodecahedron* etc. The symbols of the Archimedean polyhedra are $(3,6,3)$, $(3,8,8)$, $(3,10,10)$, $(4,6,6)$, $(4,6,8)$, $(4,6,10)$, $(5,6,6)$, $(3,4,3,4)$, $(3,4,4,4)$, $(3,4,5,4)$, $(3,5,3,5)$, $(3,3,3,3,4)$, $(3,3,3,3,5)$.

Throughout this paper P denotes one of the Archimedean polyhedra.

3. The lemma on the density

In this section we prove a lemma that we shall use in investigating the Archimedean polyhedra.

Lemma. *Let P be an arbitrary semi-regular polyhedron. Assume there are lines e and f $(e \neq f)$ for which*

$$e \cap f = 0 \in \mathcal{C}$$

and the sets

$$e \cap T_P, \quad f \cap T_P \tag{1}$$

are everywhere dense in e, resp. f.

Then P is trace-dense.

Proof of the Lemma. Let $U_\varepsilon(X) \subset \mathcal{C}$, where ε is fixed positive real and X is a point in the plane. We seek a point of the trace of P in this neighbourhood of X. Define the complex numbers u and v by

$$u + v = X \quad \text{and} \quad u \parallel e; \quad v \parallel f.$$

By (1) there is a point Z of the trace of P for which

$$Z \in e \cap U_{\varepsilon/2}(u).$$

Since P is a semi-regular polyhedron we see that there is a line f' for which $f' \parallel f$; $Z \in f'$ and $T_P \cap f'$ is everywhere dense in f' (indeed we only have to repeat the rotations starting from Z which were used starting from 0 and clearly the position of P at Z and at 0 differ by a translation). This implies that there is a point V in the trace of P for which

$$V \in f' \cap U_{\varepsilon/2}(Z + v).$$

But then

$$V \in U_\varepsilon(u + v)$$

as we wanted.

Remark. Let us note if $u, v, w \in T_P$ then $u - v + w \in T_P$ holds. This means that T_P is a free abelian group. Thus by the Lemma we conclude that if there are lines e, f

$$0, A, B \in e \quad \text{and} \quad d(0, A) \in Q, \quad d(0, B) \in Q^*$$

(and $0, A', B' \in f$ and the same conlusions hold) then T_P is everywhere dense.

4. The Archimedean polyhedra $(4,6,6)$; $(4,6,8)$; $(4,6,10)$

4.1 The case $(4,6,6)$, *truncated octahedron*

In this case if A_1 is a vertex of $(4,6,6)$ then

$$\Sigma(A_1) = \frac{\pi}{2} + 2\frac{2\pi}{3} = \frac{11}{6}\pi.$$

Thus considering those P-rotations that leave A_1 fixed we get

$$\{e(k/12) : k = 0, \ldots, 11\} \subset T_{(4,6,6)}.$$

On the other hand there is a regular hexagon containing the vertex A_1 and A_1A_3 is a diagonal with $d(A_1, A_3) = \sqrt{3}$ and $\angle A_2A_1A_3 = \frac{\pi}{6}$. T_P is invariant under those P-rotations that leave A_1 fixed. The existence of a dense line implies the existence of another one. So $(4,6,6)$ is trace-dense.

In the next two cases, if A is a vertex of P then there is a regular hexagon containing A and so, as we have seen in the above mentioned case, we only have to check

$$\Sigma(A) = \frac{m}{6q}\pi.$$

4.2 The case $(4,6,8)$, *the great rhombicuboctahedron and the case* $(4,6,10)$, *the great rhombicosidodecahedron*

For every vertex A of $(4,6,8)$

$$\Sigma(A) = \frac{\pi}{2} + \frac{2\pi}{3} + \frac{3\pi}{4} = \frac{23}{12}\pi.$$

For every vertex A of $(4,6,10)$

$$\Sigma(A) = \frac{\pi}{2} + \frac{2\pi}{3} + \frac{4\pi}{5} = \frac{59}{30}\pi.$$

Thus $(4,6,8)$ and $(4,6,10)$ are trace-dense.

5. The Archimedean polyhedra $(3,8,8)$; $(3,4,3,4)$; $(3,3,3,3,4)$; $(3,4,4,4)$ (see Fig. 2)

5.1. The case $(3,8,8)$, *the truncated cube*

The situation in this case is slightly different.

First let us note if $A_1, A_2, \dots, A_8$ is the set of consecutive vertices of a regular octagon then, with $d(A_1, A_2) = 1$,

$$\begin{aligned} d(A_1, A_4) &= 1 + \sqrt{2} \\ \angle(A_2 A_1 A_4) &= \pi/4. \end{aligned} \tag{5.1}$$

Furthermore, for every vertex A of $(3,8,8)$ $\Sigma(A) = 11\pi/6$ holds. Thus

$$\{e(k/12) : k = 0, \dots, 11\} \subset T_{(3,8,8)}. \tag{5.2}$$

Also by (5.1) and (5.2) $\{e(k/24) : k = 0, \dots, 23\} \subset T_{(3,8,8)}$ and thus $0, 1, 1+\sqrt{2} \in T_{(3,8,8)}$.

T_P is invariant under those P-rotations that leave A_1 fixed. The existence of a dense line implies the existence of another one. So $(3,8,8)$ is trace-dense.

Thus by the Lemma we get that $(3,8,8)$ is trace-dense.

5.2 The case $(3,4,3,4)$*, the cuboctahedron*

We claim that

$$\{e(k/12) : k = 0, \dots, 11\} \subset T_{(3,4,3,4)}. \tag{5.3}$$

Let us consider a face of $(3,4,3,4)$ which is the square $A_1A_2A_3A_4$, and assume A_1, A_2 occupy the points 0 and 1. Since the four faces adjacent to the vertex A_1 are *square-triangle-square-triangle* we get

$$\left\{ e\left(n \left(\frac{\pi}{2} + \frac{\pi}{3} \right) \right) \right\} \subset T_{(3,4,3,4)}$$

which implies (5.3).

Now let us consider the equilateral triangle $A_1A_4B_1$. Then $A_4B_1C_1C_2$ is a square and $C_1C_2C_3$ is an equilateral triangle with faces of $(3,4,3,4)$ (see Fig. 2, here we use the following notation: if A is a vertex of P then $\tilde{A}$ is the trace of it).

Thus if A_4 and B_1 occupy the points $e\left(\frac{k}{12}\right)$ and $e\left(\frac{k+4}{12}\right)$ then rotating $(3,4,3,4)$ around A_4B_1 and thereafter around C_1C_2 we get that C_3 occupies the point $\left(1+\sqrt{3}\right) e\left(\frac{k+2}{12}\right)$. So we get

$$0, 1, 1+\sqrt{3} \in T_{(3,4,3,4)}.$$

Thus $(3,4,3,4)$ is trace dense.

5.3 The cases $(3,3,3,3,4)$*, the snub cube, and* $(3,4,4,4)$*, the small rhombicuboctahedron*

We can treat these two cases simultaneously. If A is a vertex of $(3,3,3,3,4)$ or $(3,4,4,4)$ then

$$\Sigma(A) = 4\frac{\pi}{3} + \frac{\pi}{2} = \frac{11}{6}\pi$$

or

$$\Sigma(A) = \frac{\pi}{3} + 3\frac{\pi}{2} = \frac{11}{6}\pi.$$

As we have seen in the previous cases we get

$$\{e(k/12) : k = 0, \dots, 11\} \subset T_P. \tag{5.6}$$

if P is $(3,3,3,3,4)$ or $(3,4,4,4)$.

If $A_1A_2A_3$ is an equilateral triangle (one face of $(3,3,3,3,4)$ or $(3,4,4,4)$) and A_1, resp. A_2 occupies 0, resp. 1 then A_3 occupies the point $e(1/6)$. Assume that $A_1A_2A_3$ is a face of $(3,3,3,3,4)$ (or $(3,4,4,4)$) for which there is a vertex A_4 of P and $A_2A_4A_3$ is an equilateral face of P. So we get that $\sqrt{3}e(1/12)$ is in the trace of A_4 hence by (5.6) and the Lemma the polyhedra $(3,3,3,3,4)$ and $(3,4,4,4)$ are trace-dense.

6. The Archimedean polyhedra $(3,3,3,3,5)$, $(3,4,5,4)$, $(3,5,3,5)$, $(5,6,6)$ and $(3,10,10)$ (see Fig. 3)

In the next five cases we shall use the following idea: we are going to show that

$$A_0 = \{e(k/5) : k = 0, \dots, 4\} \subset T_P. \tag{6.1}$$

Furthermore it is easy to see that

$$A_0 + (e(0) - e(1/5)) \subset T_P \tag{6.2}$$

and by symmetry

$$A_0 + \{e(0) - e(1/5)\} + \{e(1/5) - e(4/5)\} \subset T_P \tag{6.3}$$

is also true. Let

$$A_k = A_{k-1} + (e(0) - e(4/5)) \qquad (k \in Z).$$

Now using (6.2) and (6.3) we get

$$A_k \subset T_P \tag{6.4}$$

and so for every integer k

$$A_0 + k \cdot (e(0) - e(4/5)) \subset T_P. \tag{6.5}$$

Furthermore by symmetry using a similar idea we get that for every integer m

$$P_{(m)} := A_0 + m \cdot (e(1) - e(3/5)) \subset T_P. \tag{6.6}$$

(see Fig. 4)

Thus by (6.5) and (6.6)

$$A_0 + k \cdot (e(0) - e(4/5)) + m \cdot (e(1) - e(3/5)) \subset T_P.$$

Let us observe that the vectors $(e(0) - e(4/5)$ and $e(1/5) - e(3/5)$ are parallel and the ratio of their lengths is $\left(\sqrt{5}+1\right)/2$.

Thus if e is a line containig the points $e(3/5)$ and $e(1/5)$ then we get that $e \cap T_P$ is everywhere dense in e. Clearly we can find a line f, $e \neq f$; $e \cap f = 0$, for which $e \cap T_P$ is everywhere dense in f. So by the Lemma we get that P is trace-dense.

6.1 The case $(3,3,3,3,5)$*, the snub dodecahedron*

In this case for every vertex A

$$\Sigma(A) = 4\frac{\pi}{3} + \frac{3\pi}{5} = \frac{29}{15}\pi$$

so

$$\{e(k/30) : k = 0, \ldots, 29\} \subset T_{(3,3,3,3,5)}.$$

6.2 The case $(3,4,5,4)$*, the small rhombicosadodecahedron*

For every vertex A

$$\Sigma(A) = \frac{\pi}{3} + 2\frac{\pi}{2} + \frac{3\pi}{5} = \frac{29}{15}\pi$$

so

$$\{e(k/30) : k = 0, \ldots, 29\} \subset T_{(3,4,5,4)}.$$

6.3 The case $(3,5,3,5)$*, the icosidodecahedron*

In this case

$$\Sigma(A) = 2\frac{\pi}{3} + 2\frac{3\pi}{5} = \frac{28}{15}\pi$$

and so

$$\{e(k/30) : k = 0, \ldots, 29\} \subset T_{(3,5,3,5)}.$$

6.4 The case $(5,6,6)$*, the truncated icosahedron*

In this case

$$\Sigma(A) = \frac{3\pi}{5} + 2\frac{4\pi}{6} = \frac{29}{15}\pi$$

as we stated.

6.5 The case $(3,10,10)$*, the truncated dodecahedron*

For every vertex A

$$\Sigma(A) = \frac{\pi}{3} + 2\frac{8\pi}{10} = \frac{29}{15}\pi$$

In the Cases 4–6 we discussed all Archimedean polyhedra $\neq (3,6,6)$.

This completes the theorem.

Figure 1.

Figure 2.

Figure 3.

Figure 4.

Acknowledgement. We wish to thank the referee for the helpful comments and suggestions.

References

[1] S. Wagon, The Banach-Tarski Paradox, Chap. 5, *Cambridge University Press*, 1986, 69.

[2] T. J. Dekker, On reflections in Euclidean spaces generating free products, *Nieuw Arc. Wisk* (3) **7** (1959), 57–60.

[3] N. Hegyvári, On the Trace of Polyhedra, *Geometriae Deadicata* **55** (1995), 71–74.

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(Received November 30, 1995; revised May 14, 1997)