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On the oscillatory behavior of solutions of second order nonlinear differential equations

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Abstract. Let $\alpha > 0$ be a constant. We establish the oscillatory behavior of the second order nonlinear differential equation

$$(*)\qquad \big(a(t)\psi(x)|x'|^{\alpha-1}x'\big)' + q(t)f(x) = r(t), \quad t \ge t_0 > 0,$$

where a, q, $r \in C([t_0,\infty),\mathbb{R})$ and f, $\psi \in C(\mathbb{R},\mathbb{R})$, $a(t) > 0$, $\alpha > 0$ is a constant, $q(t) \not\equiv 0$ and $\psi(x) > 0$ for $x \ne 0$.

1. Introduction

Throughout this paper we consider the following second order nonlinear differential equation

$$\text{(E)}\qquad \big(a(t)\psi(x)|x'|^{\alpha-1}x'\big)' + q(t)f(x) = r(t), \quad t \ge t_0 > 0,$$

where a, q, $r \in C([t_0,\infty),\mathbb{R})$ and f, $\psi \in C(\mathbb{R},\mathbb{R})$, $a(t) > 0$, $\alpha > 0$ is a constant, $q(t) \not\equiv 0$ on $[t_0,\infty)$ and $\psi(x) > 0$ for $x \ne 0$.

In 1979, Elbert [2] established the existence and uniqueness of solutions to the initial value problem for equation (E) on $[t_0,\infty)$. By a solution of (E) we mean a function $x \in C^1[T_x,\infty)$, $T_x \ge t_0$, which has the property $|x'(t)|^{\alpha-1}x'(t) \in C^1[T_x,\infty)$ and satisfies (E) with $\psi(x) = 1$, $f(x) = |x|^{\alpha-1}x$ and $r(t) = 0$ on $[T_x,\infty)$. We consider only those solutions $x(t)$ of (E) which satisfy $\sup\{|x(t)| : t \ge T\} > 0$ for all $T \ge T_x$. We assume that (E) possesses such a solution. A nontrivial solution of (E) is

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called *oscillatory* if it has arbitrarily large zeros; otherwise it is said to be *nonoscillatory*. Equation (E) is *nonoscillatory* [resp. *oscillatory*] if all of its solutions are nonoscillatory [resp. oscillatory].

When $\alpha = 1$, equation (E) becomes

$$(\mathrm{E}_0) \qquad \big(a(t)\psi(x)x'\big)' + q(t)f(x) = r(t), \quad t \ge t_0 > 0.$$

Graef and Spikes [3] discussed the oscillatory behavior of solutions of (E_0). Now we generalize their results to the equation (E), where $q(t)$ is allowed to change signs and we do not require $\int^\infty q(s)ds = \infty$. In this paper we give sufficient conditions for any solution of (E) to be either oscillatory or satisfying $\liminf_{t\to\infty} |x(t)| = 0$. Three other results give sufficient conditions for all solutions of (E) to be oscillatory in the case when $r(t) \equiv 0$. Moreover, our results will cover all solutions not just the bounded ones, and some examples illustrating our results are also included. For other related results, we refer the reader to Rudolf Blaśko, John R. Graef, Miloš Haĉik and Paul W. Spikes [1], Hsu, Lian and Yeh [4], Kusano and Lalli [5], Kusano and Wang [6] and J. Yan [8].

2. Oscillatory and asymptotic behavior

In this paper the following conditions will be utilized as they are needed:

$$(1) \qquad \int_{t_0}^\infty \frac{1}{a^{\frac{1}{\alpha}}(s)} ds = \infty;$$

$$(2) \qquad xf(x) > 0 \quad \text{for all } x \ne 0;$$

$$(3) \qquad \int_{t_0}^\infty |r(s)| ds < \infty.$$

Also, to simplify notation we let $W(t) = \frac{a(t)\psi(x(t))|x'(t)|^{\alpha-1}x'(t)}{f(x(t))}$ for any nonoscillatory solution $x(t)$ of equation (E).

We first extend a result of Graef and Spikes [3]. For the proof we need the following lemma which can be found in [7, p. 14].

Lemma 1. *Let $k(t,s,z)$ be a real-valued function of t and s in $[T,C)$ and z in $[T_1,C_1]$ such that for fixed t and s, k is a nondecreasing function of z. Let $g(t)$ be a given function on $[T,C)$, and let u and v be two functions on $[T,C)$ satisfying $u(s)$ and $v(s)$ are in $[T_1,C_1]$ for all s in $[T,C)$, $k\big(t,s,v(s)\big)$ and $k\big(t,s,u(s)\big)$ are locally integrable in s for fixed t, and for all t in $[T,C)$*

$$v(t)=g(t)+\int_T^t k\big(t,s,v(s)\big)ds,$$

and

$$u(t)\geq g(t)+\int_T^t k\big(t,s,u(s)\big)ds.$$

Then $v(t)\leq u(t)$ for all t in $[T,C)$.

Lemma 2. *Suppose* (2) *holds and that*

$$f'(x)\geq 0 \quad \text{for } x\neq 0. \tag{4}$$

Let $x(t)$ be a positive (*negative*) *solution of* (E) *on $[T_1,C)$ for some T_1 such that $t_0\leq T_1<C\leq\infty$. If there exists T in $[T_1,C)$ and a positive constant A_1 such that*

$$\begin{aligned}&-W(T_1)+\int_{T_1}^t\Big(q(s)-\frac{r(s)}{f\big(x(s)\big)}\Big)ds\\&+\int_{T_1}^T\frac{f'\big(x(s)\big)|W(s)|^{\frac{\alpha+1}{\alpha}}}{\Big(a(s)\psi\big(x(s)\big)|f\big(x(s)\big)|^{\alpha-1}\Big)^{\frac{1}{\alpha}}}ds\geq A_1\end{aligned}\tag{5}$$

for all t in $[T_1,C)$, then $a(t)\psi\big(x(t)\big)|x'(t)|^{\alpha-1}x'(t)\leq -A_1f\big(x(T)\big)$ (respectively, $a(t)\psi\big(x(t)\big)|x'(t)|^{\alpha-1}x'(t)\geq -A_1f\big(x(T)\big)$) for all t in $[T,C)$.

Proof. Since

$$W'(t)=\frac{r(t)}{f\big(x(t)\big)}-q(t)-\frac{f'\big(x(t)\big)|W(t)|^{\frac{\alpha+1}{\alpha}}}{\Big(a(t)\psi\big(x(t)\big)|f\big(x(t)\big)|^{\alpha-1}\Big)^{\frac{1}{\alpha}}},$$

integrating it from T_1 to t, we have

$$-W(t) = -W(T_1) + \int_{T_1}^{t} \Big(q(s) - \frac{r(s)}{f\big(x(s)\big)} \Big) ds$$
$$+ \int_{T_1}^{t} \frac{f'\big(x(t)\big)|W(t)|^{\frac{\alpha+1}{\alpha}}}{\Big(a(s)\psi\big(x(s)\big)|f\big(x(s)\big)|^{\alpha-1}\Big)^{\frac{1}{\alpha}}} ds,$$

for $T_1 \leq t < C$, and thus from (5) we see that

$$-W(t) \geq A_1 + \int_{T}^{t} \frac{f'\big(x(s)\big)|W(s)|^{\frac{\alpha+1}{\alpha}}}{\Big(a(s)\psi\big(x(s)\big)|f\big(x(s)\big)|^{\alpha-1}\Big)^{\frac{1}{\alpha}}} ds, \tag{6}$$

for $T_1 \leq t < C$. Since the integral in (6) is nonnegative and by the definition of $W(t)$, we have $x(t)x'(t) < 0$ on $[T, C)$.

If $x(t)$ is positive, let $u(t) = -a(t)\psi\big(x(t)\big)|x'(t)|^{\alpha-1}x'(t)$. Then (6) becomes

$$u(t) \geq A_1 f\big(x(t)\big) + \int_{T}^{t} \frac{f\big(x(t)\big)f'\big(x(s)\big)\big(-x'(s)\big)u(s)}{f^2\big(x(s)\big)} ds.$$

Define

$$k(t,s,z) = \frac{f\big(x(t)\big)f'\big(x(s)\big)\big(-x'(s)\big)z}{f^2\big(x(s)\big)},$$

for $t, s \in [T, C)$ and $z \in [0, \infty)$. It is easy to see that $k(t,s,z)$ is nondecreasing with respective to z for fixed t and s. Hence applying Lemma 1 with $g(t) = A_1 f\big(x(t)\big)$, we have that $u(t) \geq v(t)$, where $v(t)$ satisfies the equation

$$v(t) = A_1 f\big(x(t)\big) + f\big(x(t)\big) \int_{T}^{t} \frac{f'\big(x(s)\big)\big(-x'(s)\big)v(s)}{f^2\big(x(s)\big)} ds,$$

provided $v(s) \in [0, \infty)$, for each $s \in [T, C)$. Multiplying the last equation by $\frac{1}{f(x(t))}$ and differentiating, we obtain

$$\frac{v'(t)f\big(x(t)\big) - v(t)f'\big(x(t)\big)x'(t)}{f^2\big(x(t)\big)} = \frac{f'\big(x(t)\big)\big(-x'(t)\big)v(t)}{f^2\big(x(t)\big)},$$

then $\frac{v'(t)}{f(x(t))} \equiv 0$, so that $v'(t) \equiv 0$. Thus $v(t) = v(T) = A_1 f\big(x(T)\big)$, for all $t \in [T, C)$. Hence

$$a(t)\psi\big(x(t)\big)|x'(t)|^{\alpha-1}x'(t) \le -A_1 f\big(x(T)\big) \quad \text{for } T \le t < C.$$

The proof for $x(t)$ negative follows by a similar argument by taking

$$u(t) = a(t)\psi\big(x(t)\big)|x'(t)|^{\alpha-1}x'(t) \quad \text{and} \quad g(t) = -A_1 f\big(x(t)\big).$$

Lemma 3. *Suppose that* (1)–(4) *hold and that*

$$\int_{t_0}^{\infty} q(s)ds \quad \textit{converges} \tag{7}$$

and

$$|f(x)| \to \infty \quad \textit{as } |x| \to \infty. \tag{8}$$

If $x(t)$ is a solution of (E) *such that* $\liminf_{t\to\infty} |x(t)| > 0$, *then*

$$\int^{\infty} \frac{f'\big(x(s)\big)|W(s)|^{\frac{\alpha+1}{\alpha}}}{\Big(a(s)\psi\big(x(s)\big)|f\big(x(s)\big)|^{\alpha-1}\Big)^{\frac{1}{\alpha}}} ds < \infty, \tag{9}$$

$$W(t) \to 0 \quad \textit{as } t \to \infty, \tag{10}$$

and

$$W(t) = \int_t^{\infty} \left(q(s) - \frac{r(s)}{f\big(x(s)\big)}\right) ds + \int_t^{\infty} \frac{f'\big(x(s)\big)|W(s)|^{\frac{\alpha+1}{\alpha}}}{\Big(a(s)\psi\big(x(s)\big)|f\big(x(s)\big)|^{\alpha-1}\Big)^{\frac{1}{\alpha}}} ds, \tag{11}$$

for all sufficiently large t.

Proof. Since $x(t)$ is a solution of (E) satisfying $\liminf_{t\to\infty} |x(t)| > 0$, there exists $m > 0$, $M > 0$ and $t_1 > t_0$ such that $|x(t)| \ge m$ and $|f\big(x(t)\big)| \ge M$

for $t \geq t_1$. This, together with (3), implies that

$$\left|\int_{t_1}^{t} \frac{r(s)}{f\big(x(s)\big)} ds\right| \leq \int_{t_1}^{t} \left|\frac{r(s)}{f\big(x(s)\big)}\right| ds \leq N_1, \tag{12}$$

for some $N_1 > 0$ and for all $t \geq t_1$.

Now suppose that (9) does not hold. Then, in view (8), there exists $A_1 > 0$ and $t_2 > t_1$ such that (5) holds for all $t \geq t_2$. If $x(t) > 0$ on $[t_1, \infty)$, it follows from Lemma 2 and its proof that $x'(t) < 0$ and $a(t)\psi\big(x(t)\big)|x'(t)|^{\alpha-1}x'(t) \leq -A_1 f\big(x(T)\big)$ for $t \geq t_2$. Since $x(t)$ is positive and decreasing on $[t_2, \infty)$, $0 < \psi\big(x(t)\big) \leq A_2$ on $[t_2, \infty)$ for some positive constant A_2. Thus

$$x'(t) \leq -\left\{\frac{A_1 f\big(x(t_2)\big)}{A_2 a(t)}\right\}^{\frac{1}{\alpha}}.$$

Integrating it from t_2 to t, we have

$$x(t) \leq x(t_2) - \left\{\frac{A_1 f\big(x(t_2)\big)}{A_2}\right\}^{\frac{1}{\alpha}} \int_{t_2}^{t} \frac{1}{a^{\frac{1}{\alpha}}(s)} ds.$$

Let $t \to \infty$ in the last equation, then (1) implies that $x(t) < 0$ for t large enough, this contradicts the assumption that $x(t)$ is positive on $[t_1, \infty)$. A similar argument handles the case when $x(t) < 0$ on $[t_1, \infty)$.

Since

$$W'(t) + \frac{f'\big(x(t)\big)|W(t)|^{\frac{\alpha+1}{\alpha}}}{\Big(a(t)\psi\big(x(t)\big)|f\big(x(t)\big)|^{\alpha-1}\Big)^{\frac{1}{\alpha}}} = \frac{r(t)}{f\big(x(t)\big)} - q(t),$$

integrating it from t to ξ, we have

$$\begin{aligned} W(\xi) &+ \int_{t}^{\xi} \frac{f'\big(x(s)\big)|W(s)|^{\frac{\alpha+1}{\alpha}}}{\Big(a(s)\psi\big(x(s)\big)|f\big(x(s)\big)|^{\alpha-1}\Big)^{\frac{1}{\alpha}}} ds \\ &= W(t) + \int_{t}^{\xi} \Big(\frac{r(s)}{f\big(x(s)\big)} - q(s)\Big) ds. \end{aligned} \tag{13}$$

From (7), (9), (12) and (13), we see that $\lim_{\xi\to\infty} W(\xi)$ exists, say $W(\xi) \to A_3$ as $\xi \to \infty$, so that from (13) we have

$$W(t) = A_3 + \int_t^\infty \Big(\frac{r(s)}{f\big(x(s)\big)} - q(s)\Big)ds$$

(14)
$$+ \int_t^\infty \frac{f'\big(x(s)\big)|W(s)|^{\frac{\alpha+1}{\alpha}}}{\Big(a(s)\psi\big(x(s)\big)|f\big(x(s)\big)|^{\alpha-1}\Big)^{\frac{1}{\alpha}}}ds,$$

for $t \geq t_1$. To show that (10) and (11) hold we have to show that $A_3 = 0$. Suppose first that $x(t) > 0$ on $[t_1, \infty)$. If $A_3 < 0$, then from (7), (9) and (12), there exists $T_1 > t_1$ such that

$$\Big|\int_{T_1}^t q(s)ds\Big| \leq -\frac{A_3}{8},$$

$$\Big|\int_{T_1}^t \frac{r(s)}{f\big(x(s)\big)}ds\Big| \leq -\frac{A_3}{8}, \quad \text{for } t \geq T_1, \text{ and}$$

$$\int_{T_1}^\infty \frac{f'\big(x(s)\big)|W(s)|^{\frac{\alpha+1}{\alpha}}}{\Big(a(s)\psi\big(x(s)\big)|f\big(x(s)\big)|^{\alpha-1}\Big)^{\frac{1}{\alpha}}}ds < -\frac{A_3}{8}.$$

From (14) we see that (5) holds on $[T_1, \infty)$ with $T = T_1$. But then, as argued above, Lemma 2 and its proof contradict the assumption that $x(t) > 0$ on $[t_1, \infty)$.

If $A_3 > 0$, it follows from (7), (9), (12) and (14) that $W(t) \to A_3$ as $t \to \infty$, so there exists $T_2 > T_1$ such that $W(t) \geq \frac{A_3}{2}$ for $t \geq T_2$. Then

$$\int_{T_2}^t \frac{f'\big(x(s)\big)|W(s)|^{\frac{\alpha+1}{\alpha}}}{\Big(a(s)\psi\big(x(s)\big)|f\big(x(s)\big)|^{\alpha-1}\Big)^{\frac{1}{\alpha}}}ds$$

$$= \int_{T_2}^t \frac{a(s)\psi\big(x(s)\big)f'\big(x(s)\big)|x'(s)|^{\alpha+1}}{f^2\big(x(s)\big)}ds$$

$$\geq \frac{A_3}{2}\int_{T_2}^t \frac{f'\big(x(s)\big)x'(s)}{f\big(x(s)\big)}ds = \frac{A_3}{2}\ln\frac{f\big(x(t)\big)}{f\big(x(T)\big)}.$$

But this, together with (8) and (9), implies that $x(t)$ is bound above, hence, $0 < \psi\big(x(t)\big) \le A_4$ for some positive constant A_4.
Since $a(t)\psi\big(x(t)\big)|x'(t)|^{\alpha-1}x'(t) \ge \frac{A_3}{2} f\big(x(t)\big)$, then $x'(t) > 0$ for $t \ge T_2$ which, together with (4), implies that $f\big(x(t)\big) \ge f\big(x(T_2)\big)$ for $t \ge T_2$. Therefore

$$x'(t) \ge \left\{ \frac{A_3 f\big(x(T_2)\big)}{2A_4 a(t)} \right\}^{\frac{1}{\alpha}} \quad \text{on } [T_2, \infty).$$

Integrating it from T_2 to t, we have

$$x(t) \ge x(T_2) + \left\{ \frac{A_3 f\big(x(T_2)\big)}{2A_4} \right\}^{\frac{1}{\alpha}} \int_{T_2}^{t} \frac{1}{a^{\frac{1}{\alpha}}(s)} ds$$

for $t \ge T_2$. By (1), this contradicts the boundness of $x(t)$. Hence we obtain that $A_3 = 0$ for the case $x(t) > 0$ on $[t_1, \infty)$. The proof that $A_3 = 0$ when $x(t) < 0$ on $[t_1, \infty)$ is similar and will be omitted.

Before starting our first theorem we observe that if (3) and (7) hold, then

$$h_0(t) = \frac{1}{a^{\frac{1}{\alpha+1}}(t)} \int_t^\infty \Big[q(s) - P|r(s)| \Big] ds$$

is a well-defined function on $[t_0, \infty)$ for any positive constant P in the sense that the improper integrals involved converge, we can define

$$h_1(t) = \int_t^\infty h_0^{\frac{\alpha+1}{\alpha}}(s) ds$$

and

$$h_{n+1}(t) = \int_t^\infty \left[h_0(s) + \frac{L h_n(s)}{a^{\frac{1}{\alpha+1}}(s)} \right]^{\frac{\alpha+1}{\alpha}} ds,$$

for $n = 1, 2, 3, \ldots$, where L is any positive constant.

In the next two theorems we will need the condition that for every constant $L > 0$, there exists a positive integer N such that

(15) $\qquad h_n$ exists for $n = 0, 1, 2, \ldots, N-1$ and h_N does not exist.

Theorem 4. *Suppose that* (1)–(4), (7)–(8) *and* (15) *hold, and for any* $\lambda_1 > 0$ *there exists* $\lambda_2 > 0$ *such that*

$$\frac{f'(x)}{\left(\psi(x)|f(x)|^{\alpha-1}\right)^{\frac{1}{\alpha}}} \geq \lambda_2 \quad \textit{for all } |x| \geq \lambda_1. \tag{16}$$

Suppose, furthermore, that for any $P > 0$,

$$h_0(t) \geq 0 \textit{ for all sufficiently large } t. \tag{17}$$

Then any solution $x(t)$ *of* (E) *is either oscillatory or satisfies* $\liminf_{t\to\infty} |x(t)| = 0$.

Proof. Assume the conclusion is false. Then there is a solution $x(t)$ of (E) such that $\liminf_{t\to\infty} |x(t)| > 0$. It follows from (4) that $|f\big(x(t)\big)| \geq M$ for some $M > 0$ and all $t \geq t_1$ for some $t_1 \geq t_0$. From (11) and (16) we then have

$$W(t) \geq a^{\frac{1}{\alpha+1}}(t)h_0(t) + L\int_t^\infty \frac{|W(s)|^{\frac{\alpha+1}{\alpha}}}{a^{\frac{1}{\alpha}}(s)}ds, \tag{18}$$

for $t \geq t_1$ and some $L > 0$. Now (9) implies that

$$\int_{t_1}^\infty \frac{|W(s)|^{\frac{\alpha+1}{\alpha}}}{a^{\frac{1}{\alpha}}(s)}ds < \infty, \tag{19}$$

together with (17) and (18), imply that $W(t) \geq a^{\frac{1}{\alpha+1}}(t)h_0(t) \geq 0$. Thus

$$\frac{W^{\alpha+1}(t)}{a(t)} \geq h_0^{\alpha+1}(t). \tag{20}$$

If $N = 1$, then (19) and (20) imply that

$$h_1(t) = \int_t^\infty h_0^{\frac{\alpha+1}{\alpha}}(s)ds < \infty,$$

which contradicts the nonexistence of $h_N(t) = h_1(t)$. If $N = 2$, then (18) and (20) yield

$$\begin{aligned} W(t) &\geq a^{\frac{1}{\alpha+1}}(t)h_0(t) + L\int_t^\infty h_0^{\frac{\alpha+1}{\alpha}}(s)ds \\ &= a^{\frac{1}{\alpha+1}}(t)h_0(t) + Lh_1(t). \end{aligned} \tag{21}$$

So

$$\frac{W^{\alpha+1}(t)}{a(t)} \geq \left[h_0(t) + \frac{Lh_1(t)}{a^{\frac{1}{\alpha+1}}(t)}\right]^{\alpha+1}.$$

From (19), an integration of the above inequality would give a contradiction to the nonexistence of $h_N(t) = h_2(t)$. A similar arguments leads to a contradiction for any integer $N > 2$. Hence we complete the proof.

Example 5. Consider the equation

$$(\mathrm{E}_1) \qquad \begin{aligned} &\left(|x|^{3-\alpha}|x'|x'\right)' + \frac{1}{2t^{\frac{3}{2}}}(2+\sin t - 2t\cos t)x^3 \\ &= \frac{3+\alpha}{t^{4+\alpha}} + \frac{1}{2t^{\frac{9}{2}}}(2+\sin t - 2t\cos t), \quad t \geq 1. \end{aligned}$$

Then any solution of (E_1) is either oscillatory or satisfying $\liminf_{t\to\infty} |x(t)|=0$.

Equation (E_1) has a nonoscillatory solution $x(t) = \frac{1}{t}$. Here $a(t) = 1$, $\psi(x) = |x|^{3-\alpha}$, $q(t) = \frac{1}{2t^{\frac{3}{2}}}(2+\sin t - 2t\cos t)$, $f(x) = x^3$ and $r(t) = \frac{3+\alpha}{t^{4+\alpha}} + \frac{1}{2t^{\frac{9}{2}}}(2+\sin t - 2t\cos t)$. It is easy to see that (1)–(4) hold and

$$\frac{f'(x)}{\left(\psi(x)|f(x)|^{\alpha-1}\right)^{\frac{1}{\alpha}}} = 3,$$

$$\int_t^\infty q(s)ds = \frac{2+\sin t}{\sqrt{t}} \geq \frac{1}{\sqrt{t}}.$$

Now $|r(t)| \leq \frac{6+\alpha}{t^3}$, for $t \geq 1$, so

$$\int_t^\infty |r(s)|ds \leq \frac{6+\alpha}{2t^2}.$$

Hence $h_0(t) = \frac{1}{a^{\frac{1}{\alpha+1}}(t)} \int_t^\infty \Big[q(s) - P|r(s)|\Big] ds \geq 0$ for all sufficiently large t. Since

$$\int_t^\infty h_0^{\frac{\alpha+1}{\alpha}}(s)ds \geq \int_t^\infty \left[\frac{1}{\sqrt{s}} - \frac{P}{2s^2}\right]^{\frac{\alpha+1}{\alpha}} ds = \infty,$$

we have that $N = 1$ and thus (15)–(17) hold. Then by Theorem 4 we have that any solution of (E_1) is either oscillatory or satisfying $\liminf_{t\to\infty} |x(t)| = 0$.

Our next three theorems are oscillation results for the case when $r(t) \equiv 0$. Observe that in this case equation (E) becomes

$$(\mathrm{E}_2) \qquad \big(a(t)\psi(x)|x'|^{\alpha-1}x'\big)' + q(t)f(x) = 0, \quad t \geq t_0 > 0,$$

and

$$h_0(t) = \frac{1}{a^{\frac{1}{\alpha+1}}(t)} \int_t^\infty q(s)ds.$$

Theorem 6. *Suppose that conditions* (1)–(4), (7), (8) *and* (17) *hold. If there exists* $\lambda > 0$ *such that*

$$(22) \qquad \frac{f'(x)}{\big(\psi(x)|f(x)|^{\alpha-1}\big)^{\frac{1}{\alpha}}} \geq \lambda \quad \text{for all } x \neq 0.$$

Then equation (E_2) *is oscillatory.*

Proof. Suppose, to the contrary, that $x(t)$ is a nonocsillatory solution of (E_2). Then there exists $t_1 \geq t_0$ such that $|x(t)| > 0$ for $t \geq t_1$. Since (22) implies that $f'(x) \geq 0$ for $x \neq 0$, we have $|f\big(x(t)\big)| > 0$ for $t \geq t_1$. It is easy to see that Lemma 2 is valid for equation (E_2) whith condition (5) replaced by

$$(5') \quad -W(T_1) + \int_{T_1}^t q(s)ds + \int_{T_1}^T \frac{f'\big(x(s)\big)|W(s)|^{\frac{\alpha+1}{\alpha}}}{\Big(a(s)\psi\big(x(s)\big)|f\big(x(s)\big)|^{\alpha-1}\Big)^{\frac{1}{\alpha}}} ds \geq A_1.$$

Proceeding as in the proof of Lemma 3, we again obtain (10), i.e.,

$$(10') \qquad \int_{T_1}^T \frac{f'\big(x(s)\big)|W(s)|^{\frac{\alpha+1}{\alpha}}}{\Big(a(s)\psi\big(x(s)\big)|f\big(x(s)\big)|^{\alpha-1}\Big)^{\frac{1}{\alpha}}} ds < \infty,$$

since (12) obviously holds. Using (14) with $r(t) \equiv 0$ and continuing as in the proof of Lemma 3, we again obtain

$$(10) \qquad W(t) \to 0 \quad \text{as } t \to \infty,$$

and (11) with $r(t) \equiv 0$, i.e.,

$$(11') \qquad W(t) = \int_t^\infty q(s)ds + \int_t^\infty \frac{f'\big(x(s)\big)|W(s)|^{\frac{\alpha+1}{\alpha}}}{\Big(a(s)\psi\big(x(s)\big)|f\big(x(s)\big)|^{\alpha-1}\Big)^{\frac{1}{\alpha}}} ds,$$

for sufficiently large t. The remainder proof of this theorem is similar to that of Theorem 4 and will be omitted.

Example 7. Consider the equation

$$(\mathrm{E}_3) \qquad \big(|x'|x'\big)' + \frac{1}{2t^{\frac{3}{2}}}(2+\sin t - 2t\cos t)(x + x^{\frac{1}{3}}) = 0.$$

It is easy to see that (E_3) satisfies the hypotheses of Theorem 5, hence (E_3) is oscillatory.

The following theorem removes the condition that h_n must fail to exist for some $n = N$ (see (15)). It is an extension of part (ii) of Theorem 3 in [8] to nonlinear ordinary equation.

Theorem 8. *Assume that the conditions* (1)–(4), (7), (8) *and* (16) *hold. If* h_n *exists for all* $n = 1, 2, \ldots$ *and there exists an increasing sequence* $s_m \to \infty$ *as* $m \to \infty$ *such that* $h_n(s_m) \to \infty$ *as* $n \to \infty$ *for each* m*, then equation* (E_2) *is oscillatory.*

Proof. Suppose, to the contrary, that $x(t)$ is a nonocsillatory solution of (E_2). Then there exists $t_1 \geq t_0$ such that $|x(t)| > 0$ for $t \geq t_1$. Proceeding as in the proof of Theorem 6, we again obtain (10), (11'), so (18) and (20) hold. Since $h_n$exists for each n, an argument similar to one use in Theorem 4 shows that

$$W(t) \geq a^{\frac{1}{\alpha+1}}(t)h_0(t) + Lh_n(t), \quad \text{for each } n \geq 1.$$

Hence there exists $s_M > t_1$ such that

$$W(s_M) \geq a^{\frac{1}{\alpha+1}}(s_M)h_0(s_M) + Lh_n(s_M) \to \infty, \quad \text{as } n \to \infty,$$

which contradicts (10).

Theorem 9. *Suppose that condition* (1)–(4), (7), (17) *and* (22) *hold. Let*

$$(23) \qquad \int_{t_0}^{\infty} \frac{\psi\big(x(s)\big)|x'(s)|x'(s)}{f\big(x(s)\big)} ds < \infty,$$

and for some positive integer N *such that the functions* h_n *exist for* $n = 0, 1, 2, \ldots, N$*. If for every* $B > 0$

$$\int^{\infty} \Big[a^{-\frac{\alpha}{\alpha+1}}(s)h_0(s) + Ba^{-1}(s)h_N(s)\Big] ds = \infty,$$

then equation (E_2) *is oscillatory.*

Proof. Suppose, to the contrary, that $x(t)$ is a nonocsillatory solution of (E_2) and proceeding as in the proof of Theorem 6 (also see (21) in the proof of Theorem 4), we eventually obtain

$$W(t) \geq a^{\frac{1}{\alpha+1}}(t)h_0(t) + Bh_N(t),$$

for $t \geq t_1$, for some $t_1 \geq t_0$ and $B > 0$. Hence

$$\frac{\psi\big(x(t)\big)|x'(t)|^{\alpha-1}x'(t)}{f\big(x(t)\big)} \geq a^{-\frac{\alpha}{\alpha+1}}(t)h_0(t) + Ba^{-1}(t)h_N(t).$$

Integrating it from t_1 to t, we obtain

$$\int_{t_1}^{t} \frac{\psi\big(x(s)\big)|x'(s)|^{\alpha-1}x'(s)}{f\big(x(s)\big)}ds \geq \int_{t_1}^{t}\Big[a^{-\frac{\alpha}{\alpha+1}}(t)h_0(t) + Ba^{-1}(t)h_N(t)\Big] \to \infty$$

as $t \to \infty$ which contradicts (23).

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