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A polar-coordinate model of the hyperbolic plane

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K. Prazmowski in [2] introduced by central projection of Poincaré's half-sphere a model of the hyperbolic plane, in which the cycles of the hyperbolic plane are represented by the conics of the Euclidean plane. On the basis of this model he developed an axiomatic study of Štrambach planes with a coordinatization where the lines are hyperbolas (cf. [3]). The purpose of this article is the analytical definition of this model and the exact metrical description of the configurations of the hyperbolic plane. The model is suitable for further analytical examinations of the hyperbolic plane (cf. [4]).

We denote in the following by $\mathbb{H}^2$ and $\mathbb{E}^2$ the hyperbolic and Euclidean planes; the points of $\mathbb{H}^2$ are $A, B, C, \dots$; the points of $\mathbb{E}^2$ are $\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots$. Let $\overline{XY}$ (resp. $\overline{\mathcal{X}\mathcal{Y}}$) be the natural distance of the points $X, Y \in \mathbb{H}^2$ (resp. $\mathcal{X}, \mathcal{Y} \in \mathbb{E}^2$).

1. The analytical definition of the model

Let us take two oriented mutually perpendicular lines OX and OY in $\mathbb{H}^2$, let the positive sense of rotation be chosen according to $XOY\angle = \frac{\pi}{2}$, and let us take a rectangular coordinate system $\mathcal{O}\mathcal{X}\mathcal{Y}$ in $\mathbb{E}^2$.

Suppose that the point $O \in \mathbb{H}^2$ is represented by the Euclidean origin. Let us assign to a point $P \neq O \in \mathbb{H}^2$ with polar coordinates (φ, ρ) that point $\mathcal{P} \in \mathbb{E}^2$ whose polar coordinates are $(\varphi, \operatorname{sh}\rho)$. This assignment defines a bijective mapping

$$\xi : \mathbb{H}^2 \to \mathbb{E}^2, \quad P \mapsto \xi(P) = \mathcal{P}.$$

Our model is based on the bijection ξ. "Images" will mean "images by ξ" in the sequel (unless otherwise stated).

2. Transformation of coordinates

Theorem 1. *Let OXY and $O'X'Y'$ be orthogonal coordinate systems in $\mathbb{H}^2$.*

(a) *Let $O' = O$. Suppose that the axes OX' and OY' turn from the axis OX with angles α_1 and α_2 such that $\alpha_1 - \alpha_2 = \pm\frac{\pi}{2}$, according to the fact that the transformation preserves the orientation or not. Then the coordinate-transformation $OXY \to OX'Y'$ generates a rotation of $\mathbb{E}^2$ by angle $\omega := \alpha_1$.*

(b) *If the coordinate-transformation $OXY \to O'X'Y'$ is a translation by the oriented distance $t := \overline{O'O}$ along the axis OX, then it induces the transformation*

$$\mathbb{E}^2 \to \mathbb{E}^2, \quad \mathcal{P}(x,y) \mapsto \mathcal{P}'(x',y')$$

given by

$$x' = x \cdot \operatorname{ch} t + \sqrt{1 + x^2 + y^2} \cdot \operatorname{sh} t$$
$$y' = y.$$

($\mathcal{P}$ and $\mathcal{P}'$ are given in the same Euclidean coordinate system.)

PROOF. (a) is obvious by the definition of ξ.
Proof of (b): for the triangle $OO'P$ (see [1] 24.§,(I), (IV))

$$\frac{\sin(180^\circ - \varphi)}{\sin\varphi'} = \frac{\operatorname{sh}\rho'}{\operatorname{sh}\rho} \quad \text{(sine rule)}$$

$$\operatorname{sh} t \cdot \operatorname{cth}\rho = \sin(180^\circ - \varphi) \cdot \operatorname{ctg}\varphi' + \cos(180^\circ - \varphi) \cdot \operatorname{ch} t$$
$$\text{(cotangent rule)}$$

from which it follows:

$$y' = \operatorname{sh}\rho' \cdot \sin\varphi' = \operatorname{sh}\rho \cdot \sin\varphi = y$$
$$x' = \operatorname{sh}\rho' \cdot \cos\varphi' = \operatorname{sh}\rho' \cdot \sin\varphi' \cdot \operatorname{ctg}\varphi' =$$
$$= \operatorname{sh}\rho \cdot \cos\varphi \cdot \operatorname{ch} t + \operatorname{ch}\rho \cdot \operatorname{sh} t =$$
$$= x \cdot \operatorname{ch} t + \sqrt{1 + x^2 + y^2} \cdot \operatorname{sh} t. \qquad \square$$

3. Lines

a. The image of a hyperbolic line passing through O is such a Euclidean line which passes through $\mathcal{O}$, and the Euclidean angle between the image line and the axis $\mathcal{OX}$ is equal to the hyperbolic angle between the original line and the axis OX.

b. Let l be a hyperbolic line non-passing through O. Let P_0 be the intersecting point of l and the line which is perpendicular to l and passes through O, and let

$$XOP_0\angle = \Phi_0, \quad \overline{OP_0} = a.$$

Then the parameters Φ_0 and a determine the line l uniquely and we get the

Theorem 2. *The image ℓ of the line l is that branch of a hyperbola which passes through $\mathcal{P}_0$. The real axis of this hyperbola is the line $\mathcal{OP}_0$. The asymptotes of the hyperbola pass through $\mathcal{O}$ and the angles between the asymptotes and the axis $\mathcal{OX}$ are $\Phi_0 \pm \alpha$, where α is the angle of parallelism belonging to the distance a. The half-lengths of the real and imaginary axes are $\operatorname{sh} a$ and 1, respectively.*

PROOF. Let $P \neq P_0$ be any point of l. For the triangle OP_0P in Fig. 1

$$\operatorname{th} a = \operatorname{th} \rho \cdot \cos(\Phi - \Phi_0)$$

from where

$$\operatorname{sh} a \cdot \frac{\operatorname{ch} \rho}{\operatorname{ch} a} = \operatorname{sh} \rho \cdot \cos(\Phi - \Phi_0).$$

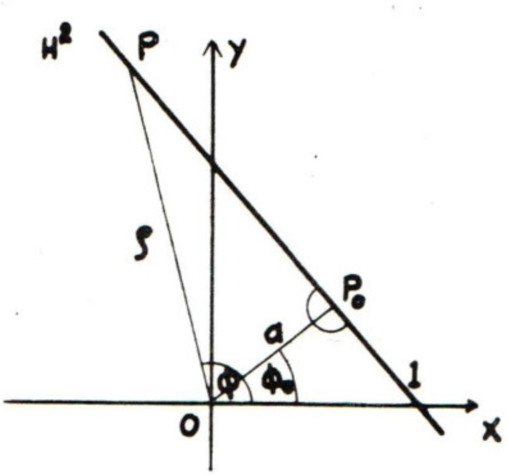

Figure 1

Applying the identities

$$(*) \qquad \begin{aligned} \operatorname{th} b &= \operatorname{th} c \cdot \cos \alpha \\ \operatorname{ch} c &= \operatorname{ch} a \cdot \operatorname{ch} b \\ \operatorname{th} a &= \operatorname{sh} b \cdot \operatorname{tg} \alpha, \end{aligned}$$

which are valid for any right triangle in $\mathbb{H}^2$ ($\gamma = \pi/2$, see [1] 17.§, (III), (IV), (VI)), we get in the triangle OP_0P:

$$\frac{\operatorname{ch} \rho}{\operatorname{ch} a} = \operatorname{ch} \overline{P_0P}, \quad \operatorname{sh} \rho = \frac{\operatorname{sh} a \cdot \operatorname{ch} \overline{P_0P}}{\cos(\Phi - \Phi_0)};$$

so the Euclidean coordinates of the image point $\mathcal{P}$ are

$$x = \operatorname{sh} \rho \cdot \cos \Big[(\Phi - \Phi_0) + \Phi_0 \Big] =$$
$$= \operatorname{sh} a \cdot \operatorname{ch} \overline{P_0P} \cdot \cos \Phi_0 - \operatorname{sh} a \cdot \operatorname{ch} \overline{P_0P} \cdot \operatorname{tg}(\Phi - \Phi_0) \cdot \sin \Phi_0,$$
$$y = \operatorname{sh} \rho \cdot \sin \Big[(\Phi - \Phi_0) + \Phi_0 \Big] =$$
$$= \operatorname{sh} a \cdot \operatorname{ch} \overline{P_0P} \cdot \sin \Phi_0 + \operatorname{sh} a \cdot \operatorname{ch} \overline{P_0P} \cdot \operatorname{tg}(\Phi - \Phi_0) \cdot \cos \Phi_0.$$

Also by $(*)$:

$$\operatorname{sh} a \cdot \operatorname{ch} \overline{P_0 P} \cdot \operatorname{tg}(\Phi - \Phi_0) = \operatorname{sh} \overline{P_0 P},$$

therefore

$$x = \operatorname{sh} a \cdot \operatorname{ch} \overline{P_0 P} \cdot \cos \Phi_0 - \operatorname{sh} \overline{P_0 P} \cdot \sin \Phi_0,$$

$$y = \operatorname{sh} a \cdot \operatorname{ch} \overline{P_0 P} \cdot \sin \Phi_0 + \operatorname{sh} \overline{P_0 P} \cdot \cos \Phi_0.$$

Rotate the coordinate system $\mathcal{OXY}$ by angle Φ_0. Then the "new" axis $\mathcal{OX}'$ coincides with the line $\mathcal{OP}_0$ (Fig. 2); the obtained coordinate system $\mathcal{OX}'\mathcal{Y}'$ will be mentioned as the *own system* of l. The coordinates of $\mathcal{P}$ are the following ones in this system:

$$(1) \qquad \begin{aligned} x' &= \operatorname{sh} a \cdot \operatorname{ch} \overline{P_0 P} > 0 \\ y' &= \operatorname{sh} \overline{P_0 P} \end{aligned}$$

from where

$$\frac{x'^2}{\operatorname{sh}^2 a} - y'^2 = 1, \quad x' > 0.$$

This is the equation of the branch $x' > 0$ of a hyperbola, whose real axis is the line $\mathcal{OX}'$. The equations of the asymptotes are

$$y' = \pm \frac{1}{\operatorname{sh} a} x' = \pm \operatorname{tg} \alpha \cdot x',$$

where α is the angle of parallelism belonging to the distance a (cf. [1] 18.§, (1)). The half-lengths of the real and imaginary axes are $\operatorname{sh} a$ and 1, respectively (Fig. 2). □

We note that the two half-asymptotes of the images of two different lines cannot coincide.

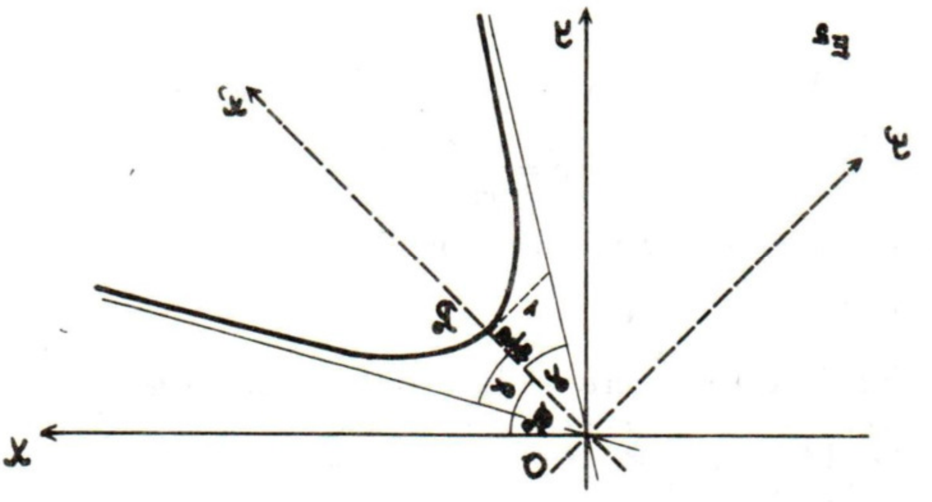

Figure 2

4. Parallelism

Theorem 3. *The images of the parallel lines in this Euclidean model are: an oriented line ℓ through $\mathcal{O}$ and the family of hyperbola branches whose half-asymptote is the half-line of ℓ with initial point $\mathcal{O}$ corresponding to the given orientation of ℓ.*

PROOF. Among the lines parallel to a given line there exists a line l which passes through O. Let ψ be the angle between l and the axis OX and let f be an arbitrary line parallel to l. The line OP is perpendicular to the line f (Fig. 3a). The angle between the lines OP and l is the angle of parallelism α belonging to the distance $\overline{OP}$. The angles between the asymptotes of the image of f and the axis $\mathcal{OX}$ are $(\alpha+\psi)+\alpha$ and $(\alpha+\psi)-\alpha=\psi$ (cf. Theorem 2). Thus one of the half-asymptotes of the image of f is a half-line of ℓ. This half-line is just the image of the half-line of l in the direction of parallelism (Fig. 3b). □

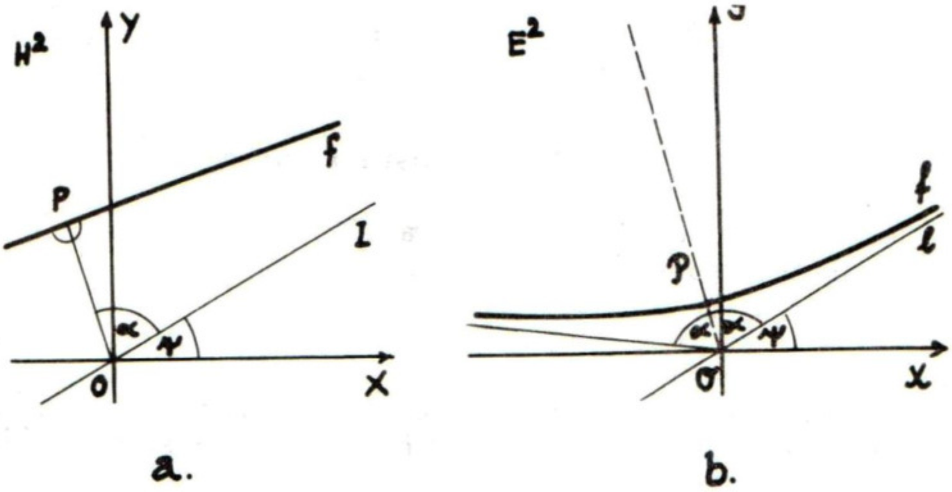

Figure 3

5. Formulas of the distance

We express the signed distance $\overline{P_1P_2}$ with the help of the Euclidean coordinates of the points $\mathcal{P}_i=\xi(P_i)$ $(i=1,2)$. There are two cases.

(a) *The line P_1P_2 passes through O*

Let $\overline{OP_1}=\rho_1$, $\overline{OP_2}=\rho_2$, $\overline{\mathcal{OP}_1}=r_1$, $\overline{\mathcal{OP}_2}=r_2$, Then $\overline{P_1P_2}=\overline{P_1O}+\overline{OP_2}=-\rho_1+\rho_2=\operatorname{area\,sh} r_2-\operatorname{area\,sh} r_1=\ln(r_2+\sqrt{{r_2}^2+1})-\ln(r_1+\sqrt{{r_1}^2+1})=\ln\dfrac{r_2+\sqrt{{r_2}^2+1}}{r_1+\sqrt{{r_1}^2+1}}$.

(b) The line P_1P_2 does not pass through O

Let a and Φ be the parameters of the line P_1P_2. The coordinates of the image points $\mathcal{P}_1, \mathcal{P}_2$ are in the own system of the line P_1P_2 (cf. 3§.):

$$x_1{}' = \operatorname{sh} a \cdot \operatorname{ch} \overline{P_0P_1}, \quad y_1{}' = \operatorname{sh} \overline{P_0P_1},$$
$$x_2{}' = \operatorname{sh} a \cdot \operatorname{ch} \overline{P_0P_2}, \quad y_2{}' = \operatorname{sh} \overline{P_0P_2};$$

therefore

$$\begin{aligned}
\operatorname{ch} \overline{P_1P_2} &= \operatorname{ch}(\overline{P_1P_0} + \overline{P_0P_2}) = \operatorname{ch} \overline{P_1P_0} \cdot \operatorname{ch} \overline{P_0P_2} + \operatorname{sh} \overline{P_1P_0} \cdot \operatorname{sh} \overline{P_0P_2} = \\
&= \operatorname{ch} \overline{P_0P_1} \cdot \operatorname{ch} \overline{P_0P_2} - \operatorname{sh} \overline{P_0P_1} \cdot \operatorname{sh} \overline{P_0P_2} = \\
&= \frac{x_1{}'}{\operatorname{sh} a} \cdot \frac{x_2{}'}{\operatorname{sh} a} - y_1{}'y_2{}', \qquad (2)\\
\operatorname{sh} \overline{P_1P_2} &= \operatorname{sh}(\overline{P_1P_0} + \overline{P_0P_2}) = \operatorname{sh} \overline{P_1P_0} \cdot \operatorname{ch} \overline{P_0P_2} + \operatorname{ch} \overline{P_1P_0} \cdot \operatorname{sh} \overline{P_0P_2} = \\
&= -\operatorname{sh} \overline{P_0P_1} \cdot \operatorname{ch} \overline{P_0P_2} + \operatorname{ch} \overline{P_0P_1} \cdot \operatorname{sh} \overline{P_0P_2} = \\
&= -y_1{}' \cdot \frac{x_2{}'}{\operatorname{sh} a} + \frac{x_1{}'}{\operatorname{sh} a} \cdot y_2{}' = \frac{1}{\operatorname{sh} a} \cdot (x_1{}'y_2{}' - x_2{}'y_1{}'). \qquad (3)
\end{aligned}$$

6. Equidistant curves

Let $\mathfrak{e}$ be an equidistant curve and l be its base line. The distance of the points of $\mathfrak{e}$ from l is constant: d.

a. The base line passes through O

Theorem 4. *The image of the equidistant curve $\mathfrak{e}$ is a line e which is parallel to the image line of l and their distance is* $\operatorname{sh} d$, *if the base line l passes through O.*

Proof. Let P be any point of $\mathfrak{e}$ and let f be the line through P perpendicular to l. Let T be the intersecting point of l and f (Fig. 4). The same own system belongs to each f in $\mathbb{E}^2$ (cf. 3§.): namely the axis $\mathcal{O}\mathcal{X}'$ is the line ℓ (the image of l). In this system the coordinates of $\mathcal{P}$ are (x', y'), the coordinates of $\mathcal{T}$ are $(\operatorname{sh} a, O)$, where $a = \overline{OT}$ is the parameter of f.

Applying (2) to the distance $\overline{TP} = d$, we get

$$\operatorname{ch} \overline{TP} = \operatorname{ch} d = \frac{x' \cdot \operatorname{sh} a}{\operatorname{sh}^2 a},$$

from where

$$x' = \operatorname{ch} d \cdot \operatorname{sh} a.$$

The point P lies on f, thus for its coordinates

$$\frac{x'^2}{\mathrm{sh}^2\, a} - y'^2 = 1.$$

Replacing x' we get: $y'^2 = \mathrm{sh}^2\, d$, from where $y' = \mathrm{sh}\, d$ or $y' = -\,\mathrm{sh}\, d$, according to the fact that $\mathfrak{e}$ is in one or in the other half-plane of l (Fig. 4). □

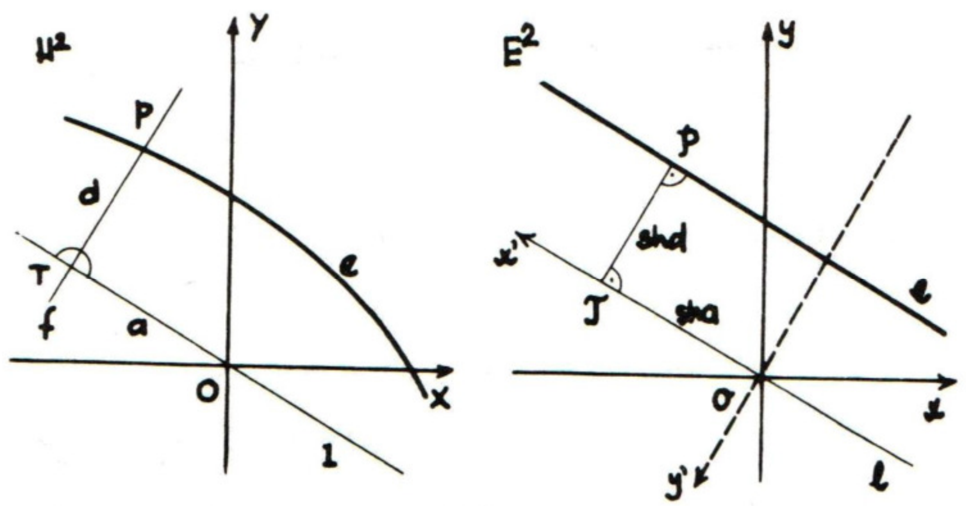

Figure 4

b. The base line does not pass through O

Theorem 5. *The image of the equidistant curve $\mathfrak{e}$ is a branch e of a hyperbola if the base line l does not pass through O. The asymptotes of e are parallel to the asymptotes of the image hyperbola ℓ of l, the distance between the parallel asymptotes is $\mathrm{sh}\, d$. One of the two angle-domains containing e and ℓ is within the other (Fig. 6). The half-lengths of the real and imaginary axes are $\mathrm{ch}\, d \cdot |\mathrm{sh}\, t|$ and $\mathrm{ch}\, d$ respectively, where t is the distance of O from l.*

PROOF. In consequence of Theorem 1(a) it is sufficient to deal with the case when the base line l is perpendicular to the axis OX.

Let OXY be such a coordinate system in $\mathbb{H}^2$ whose OY axis coincides with the base line l. Then the image of the equidistant curve $\mathfrak{e}$ is the line with the equation $x = \mathrm{sh}\, d$ (cf. Theorem 4).

We consider the coordinate transformation in $\mathbb{H}^2$ described in Theorem 1(b). In this coordinate system $O'XY'$ the line l is perpendicular to the axis $O'X$. The image of l corresponding to $O'XY'$ is a hyperbola ℓ; the angles between the asymptotes of ℓ and the axis $\mathcal{OX}$ are $\pm\tau$, where τ is the angle of parallelism belonging to the distance t (Fig. 5).

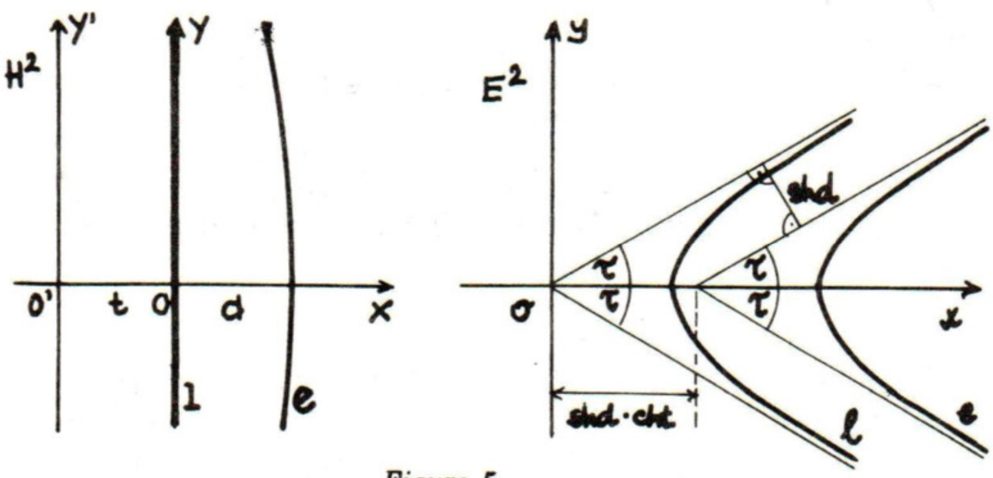

Figure 5

The Euclidean coordinates of the image of any point $P \in \mathfrak{e}$ corresponding to $O'XY'$ are

$$x' = \operatorname{sh} d \cdot \operatorname{ch} t + \sqrt{1 + \operatorname{sh}^2 d + y^2} \cdot \operatorname{sh} t,$$
$$y' = y$$

from where the equation of the image e of the equidistant curve corresponding to $O'XY'$ is

$$\frac{(x' - \operatorname{sh} d \cdot \operatorname{ch} t)^2}{\operatorname{ch}^2 d \cdot \operatorname{sh}^2 t} - \frac{y'^2}{ch^2 d} = 1, \tag{4}$$
$$x' - \operatorname{sh} d \cdot \operatorname{ch} t \gtrless 0, \quad \text{when} \quad t \gtrless 0.$$

Thus e is a branch of hyperbola for which

– the half real-axis is $\operatorname{ch} d \cdot |\operatorname{sh} t|$, the half imaginary-axis is $\operatorname{ch} d$;

– the intersecting point of the asymptotes is the point $(\operatorname{sh} d \cdot \operatorname{ch} t, 0)$ which lies on the axis $\mathcal{OX}$; the slopes of the asymptotes are $\pm \frac{1}{\operatorname{sh} t} = \pm \operatorname{tg} \tau$, that is the asymptotes are parallel with the asymptotes of ℓ and their distance is $\operatorname{sh} d$.

Condition (4) means that one of the two angle-domains including e and ℓ is within the other (Fig. 5).

The general case is illustrated by Fig. 6. □

The distance d between an equidistant curve and its base line gives the distance $\operatorname{sh} d$ between the asymptotes of their images.

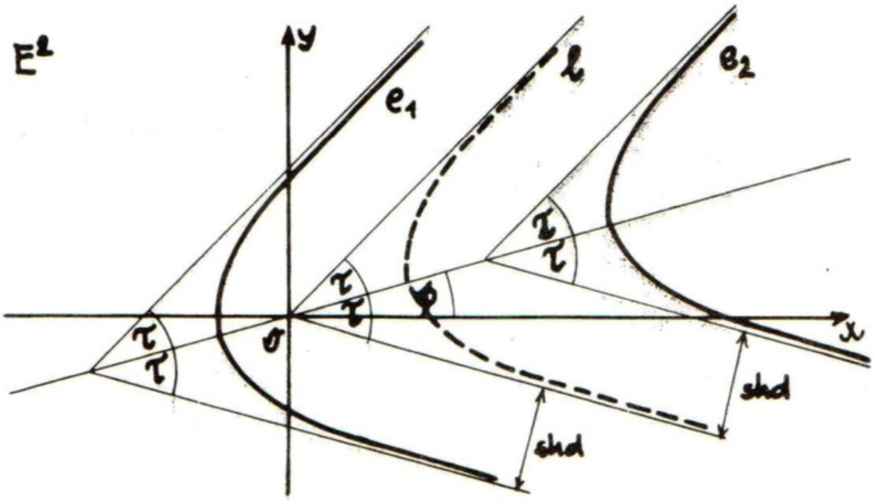

Figure 6.

7. Circles

Theorem 6. *The image of a circle k is an ellipse whose major axis is the line $\mathcal{O}\mathcal{C}$, where C is the centre of the circle. The half-lengths of the major and minor axes are $\operatorname{sh} r \cdot \operatorname{ch} t$ and $\operatorname{sh} r$, where r is the radius of k, $t = \overline{OC}$.*

PROOF. The image of the circle with origin O and radius r is the circle with origin $\mathcal{O}$ and radius $\operatorname{sh} r$. If $C \neq O$ then it is sufficient to deal with the case when C lies on the axis OX.

Let us take first a coordinate system with origin C in $\mathbb{H}^2$, then the coordinates of the image of any point $P \in k$ are

$$x = \operatorname{sh} r \cdot \cos\theta,$$
$$y = \operatorname{sh} r \cdot \sin\theta \quad (0 \leq \theta < 2\pi).$$

We consider the transformation of coordinates in $\mathbb{H}^2$ given in Theorem 1(b). In this coordinate system $O'XY'$ the image point of C is the point $(\operatorname{sh} t, 0)$. For the Euclidean coordinates of the image of P corresponding to $O'XY'$ we get

$$x' = \operatorname{sh} r \cdot \cos\theta \cdot \operatorname{ch} t + \sqrt{1 + \operatorname{sh}^2 r} \cdot \operatorname{sh} t, \tag{5}$$

$$y' = \operatorname{sh} r \cdot \sin\theta. \tag{6}$$

From (5)

$$x' - \operatorname{ch} r \cdot \operatorname{sh} t = \operatorname{sh} r \cdot \cos\theta \cdot \operatorname{ch} t.$$

Squaring and replacing (6), we obtain

$$(x' - \operatorname{ch} r \cdot \operatorname{sh} t)^2 = \operatorname{sh}^2 r \cdot \operatorname{ch}^2 t - y'^2 \cdot \operatorname{ch}^2 t$$

$$\frac{(x' - \operatorname{ch} r \cdot \operatorname{sh} t)^2}{\operatorname{sh}^2 r \cdot \operatorname{ch}^2 t} + \frac{y'^2}{\operatorname{sh}^2 r} = 1.$$

This is the equation of an ellipse with centre $(\operatorname{ch} r \cdot \operatorname{sh} t, 0)$ for which the line of the major-axis is the axis $\mathcal{OX}$ and the half-lengths of the major and minor axes are $\operatorname{sh} r \cdot \operatorname{ch} t$ and $\operatorname{sh} r$. □

8. Horocycles

Theorem 7. *Let h be a horocycle, and let l be the axis of h which passes through O. The image of h is a parabola whose axis is the image line of l.*

PROOF. In consequence of Theorem 1(a) it is sufficient to deal with the case when the axis l is the axis OX and the direction of the parallelism is the positive direction of the axis OX.

Let us take first such a coordinate system in $\mathbb{H}^2$ where the origin O lies on h (Fig. 7) and let P be any point of h different from O. The

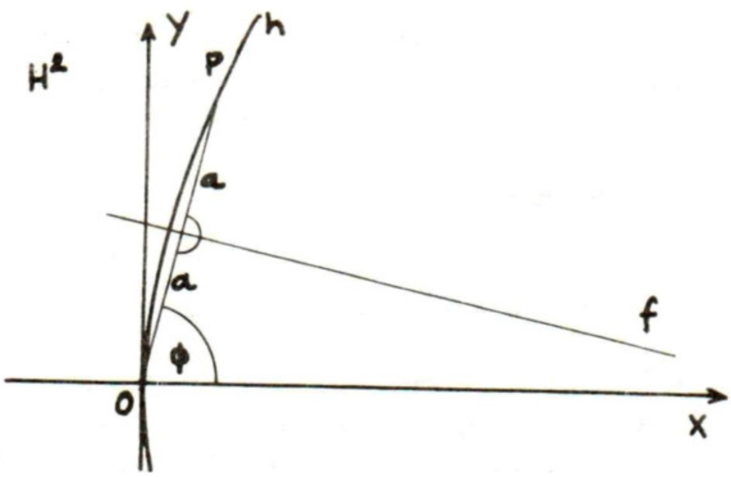

Figure 7.

perpendicular bisector f of the line segment OP is parallel to the axis OX in the positive direction of OX, because the perpendicular bisectors of every two points of h are axes of h. The equation of the asymptotes of the image of f according to Fig. 7 (cf. Theorem 2.) is

$$y = \operatorname{tg}(\phi \pm \alpha) \cdot x,$$

where α is the angle of parallelism belonging to the distance a. One of these asymptotes is the axis $\mathcal{OX}$ because f is parallel to the axis OX (cf. 4§.), thus e.g. $\phi = \alpha$ (the proof is similar in the case $\phi = -\alpha$). The coordinates of the image of P are

$$x = \operatorname{sh} 2a \cdot \cos\alpha = \operatorname{sh} 2a \cdot \operatorname{th} a = 2\operatorname{sh}^2 a,$$
$$y = \operatorname{sh} 2a \cdot \sin\alpha = \operatorname{sh} 2a \cdot \frac{1}{\operatorname{ch} a} = 2\operatorname{sh} a;$$

from where

$$x = \frac{y^2}{2}.$$

We consider again the transformation of coordinates in $\mathbb{H}^2$ from Theorem 1(b). The coordinates of the image of P corresponding to $O'XY'$ are

$$x' = x \cdot \operatorname{ch} t + \sqrt{1 + x^2 + y^2} \cdot \operatorname{sh} t,$$
$$y' = y.$$

Replacing here $x = \dfrac{y^2}{2}$ we get

$$x' = \frac{y'^2}{2} \cdot (\operatorname{sh} t + \operatorname{ch} t) + \operatorname{sh} t;$$

this is the equation of a parabola with the axis $\mathcal{OX}$. □

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