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Qualitative Behavior of Integrodifferential Systems

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We shall associate Volterra integrodifferential equations

$$\text{(1)}\quad \frac{d}{dt}\left[x(t)-\int_0^t N(t-s)x(s)ds-g(t)\right]=(A+B(t))x(t)+ + \int_0^t K(t-s)x(s)ds+f(t)+F(x)(t)\,,\quad x(0)=x_0$$

$$\text{(2)}\quad \frac{d}{dt}\left[x(t)-\int_0^t N(t-s)x(s)ds-g(t)\right]=(A+B(t))x(t)+ + \int_0^t K(t-s)x(s)ds+f(t)\,,\quad x(0)=x_0$$

with

$$\text{(I)}\quad \frac{d}{dt}\left[x(t)-\int_0^t N(t-s)x(s)ds-g(t)\right]= = Ax(t)+\int_0^t K(t-s)x(s)ds+f(t)$$

via the resolvent equation

$$(3)\quad \frac{d}{dt}\left[Z(t)-\int_0^t N(t-s)Z(s)ds\right]=AZ(t)+\int_0^t K(t-s)Z(s)ds$$

$$Z(0)=I\,.$$

Here and hereafter, $0\le t<+\infty, B(t)$, $K(t)$, $N(t)$ are $n\times n$ matrices continuous for $0\le s\le t<+\infty$, A is a constant $n\times n$ matrix, I the $n\times n$ identity matrix, Z an $n\times n$ matrix, $x(t)$, $f(t)$, $g(t)$ are column vectors, and F is a "small" nonlinear functional.

In the analysis of of (1)–(I) and (2)–(I) we make use of the variation of constants formula given by JIANHONG WU in [3].

Equation (2) was studied by GROSSMAN and MILLER in [1, 2] in the case $N(t)=g(t)=0$.

Preliminaries

Let $\mathbf{R}^n$ denote the real n-dimensional Euclidean space of column vectors with Euclidean norm $|\bullet|$, R^+ the set of all t such that $0\le t<+\infty$, and C is the set of all continuous functions with domain R^+ and range $\mathbf{R}^n$. $BC(R^+)=\{u : u \text{ is bounded and continuous on } R^+\}$ with the sup norm $\|\bullet\|$. $BC_0(R^+)=\{u \text{ in } BC(R^+) : u(t)\to 0 \text{ as } t\to\infty\}$.
For p in the interval $1\le p<\infty$, L^p is the usual Lebesgue space of measurable functions f such that

$$\|f\|_p=\left(\int_0^\infty |f(t)|^p dt\right)^{\frac{1}{p}}<\infty\,.$$

LL^p is the set of all functions which are locally L^p on R^+.

The function $f:\langle 0,\infty)\to R^n$ is said to be interval bounded if it is measurable and

$$\sup_{t\ge o}\int_t^{t+1}|f(s)|ds<\infty\,.$$

We will denote the space of interval bounded functions by B_{IB} with norm

$$\|f\|_B=\sup_{t\ge 0}\int_t^{t+1}|f(s)|ds\,.$$

Notice that a measurable function $f(\bullet)$ is interval bounded if for example $f(s)$ is bounded or if for some $p \geq 1$

$$\int_0^\infty |f(s)|^p \, ds < \infty .$$

JIANHONG WU [3] has shown that (I) has a unique solution $x(t)$ which can be expressed as

$$(4) \quad x(t) = Z(t)[x(0) - g(0)] + g(t) + \int_0^t Z'(t-s)g(s)ds + \int_0^t Z(t-s)f(s)ds$$

where $Z(t)$ is an $n \times n$ continuously differentiable matrix satisfying equation (3) with initial value $Z(0) = I$.

The solution of (I) can for a given $x(0)$, $g(0)$, $f(t)$ and $g(t)$ be expressed in terms of $Z(t)$ and a map ϱ defined by

$$\varrho(f,g)(t) = \int_0^t Z'(t-s)g(s)ds + \int_0^t Z(t-s)f(s)ds, \quad t \geq 0 .$$

The following results concerning this map ϱ (for $g(t) = N(t,s) = 0$) may be found in GROSSMAN and MILLER [2].

We note that LL^1 is a Fréchet space.

Theorem. *Let X and Y be Fréchet subspaces of LL^1 both having a topology stronger than LL^1. If $\varrho(x) \subset Y$ then ϱ is continuous as a mapping from X into Y.*

Interesting examples of Fréchet subspaces of LL^1 are C, BC, BC_1 (of all functions in BC having a limit at infinity).

Perturbation Theorems

In this section we study the system (2) as a perturbation of (I). It is shown that (2) inherits much of the behavior (I) for an appropriate perturbation term $B(t)x$. The resolvent associated with (I) is denoted by Z_I.

Theorem A. *Assume*

(1^0) $\quad Z_I$ *is in* $L^1 \cap BC$, Z_I' *is in* L^1

(2^0) $\quad B$ *is in* BC_0

(3^0) $\quad f, g$ *are in* BC.

Then the solution $x(t)$ of (2) is in BC.

PROOF. From equation (4) the solution $x(t)$ of (2) satisfies

$$(5)\quad x(t) = Z_I(t)[x(0) - g(0)] + g(t) + \int_0^t Z_I'(t-s)g(s)ds + \\ + \int_0^t Z_I(t-s)[f(s) + B(s)x]ds\,.$$

We first prove the theorem for a "small" $B(t)$. By "small" we mean any $B(t)$ such that

$$\|B\| \le (\|Z_I\|_1)^{-1}\,.$$

Then from (5), for any $T > 0$

$$\|x\|_{\langle 0,T\rangle} = \sup_{t\in\langle 0,T\rangle} |x(t)| \le \|Z_I\|\,|x_0 - g_0| + \|g\|(1 + \|Z_I'\|_1) + \\ + \|f\|\,\|Z_I\|_1 + \|B\|\,\|Z_I\|_1\|x\|_{\langle 0,T\rangle}\,.$$

Hence

$$\|x\|_{\langle 0,T\rangle} \le (1 - \|B\|\,\|Z_I\|_1)^{-1}\big[\|Z_I\|\,|x(0) - g(0)| + \\ + \|g\|(1 + \|Z_I'\|_1) + \|f\|\,\|Z_I\|_1\big]\,.$$

Thus $x(t)$ is in BC and the theorem is true for "small" $B(t)$.

Now let $B(t)$ be an arbitrary matrix in BC_0. Then there is $T > 0$ such that $\|B_T(t)\| < (\|Z_I\|_1)^{-1}$ where $B_T(t) = B(t+T)$, $t \ge 0$. Since (2) has a continuous solution $x(t)$ on $\langle 0,T\rangle$, $x(t)$ is bounded on $\langle 0,T\rangle$. For $t > T$, $x(t)$ still solves (2) and this may be expressed by translating (2) and replacing t by $t+T$:

$$\frac{d}{dt}\left[x(t+T) - \int_0^{t+T} N(t+T-s)x(s)ds - g(t+T)\right] = \\ = (A + B(t+T))x(t+T) + \int_0^{t+T} K(t+T-s)x(s)ds + f(t+T);$$

$$x(0+T) = x(T)\,,$$

where now $t \geq 0$. Writing $x(t+T)$ as $x_T(t)$ we have

$$\frac{d}{dt}\left[x_T(t) - \int_0^{t+T} N(t+T-s)x(s)ds - g_T(t)\right] = (A + B_T(t))x_T(t) + \\ + \int_0^{t+T} K(t+T-s)x(s)ds + f_T(t);\ x_T(0) = X(T).$$

By performing the change of variables $u = s - T$ inside the integral we get

$$(6)\quad \frac{d}{dt}\left[x_T(t) - \int_0^{t} N(t-u)x_T(u)du - G(t)\right] = (A + B_T(t))x_T(t) + \\ + \int_0^{t} K(t-u)x_T(u)du + F(t);\ x_T(0) = x(T),$$

where

$$G(t) = \int_{-T}^{0} N(t-u)x_T(u)du + g_T(t)$$
$$F(t) = \int_{-T}^{0} K(t-u)x_T(u)du + f_T(t).$$

Now $F(t)$ and $G(t)$ are in BC. Then $x_T(t)$ solves the equation (6), and in this equation $B_T(t)$ is "small": $\|B_T(t)\| < (\|Z_I\|_1)^{-1}$. From the first portion of the proof, x_T is in BC, hence $x(t)$ is in BC.

Corollary. *In equation* (2) *suppose* $Z_I(t)$ *is in* $L^1 \cap BC_0$, $Z_I'(t)$ *is in* $L^1, B(t)$ *is in* BC_0, $g(t)$ *is in* BC_1 (BC_0), *and* $f(t)$ *is in* BC_1 (BC_0), *then*

$$x(\infty) = \lim_{t\to\infty} x(t) = g(\infty) + \int_0^{\infty} Z_I'(s)ds\ g(\infty) + \int_0^{\infty} Z_I(s)ds\ f(\infty).$$

PROOF. From equation (4) $x(t)$ of (2) satisfies

$$x(t) = Z_I(t)[x(0) - g(0)] + g(t) + \int_0^t Z_I'(t-s)g(s)ds +$$
$$+ \int_0^t Z_I(t-s)f(s)ds + \int_0^t Z_I(t-s)B(s)x(s)ds$$

and from Theorem A it is a bounded continuous function. By well-known theorems on the convolution product all the terms on the right-hand side have limits.

Hence

$$\lim_{t\to\infty} x(t) = g(\infty) + \int_0^\infty Z_I'(s)ds\ g(\infty) + \int_0^\infty Z_I(s)ds\ f(\infty)$$

where

$$f(\infty) = \lim_{t\to\infty} f(t), \quad g(\infty) = \lim_{t\to\infty} g(t).$$

Thus $x(t)$ is in BC_1 (BC_0). □

Corollary. *In equation* (2) *suppose $Z_I(t)$ is in $L^1 \cap BC_0$, $Z_I'(t)$ is in $L^1 \cap BC_0$. If for some fixed $p \in \langle 1, \infty)$, $B(t)$ is in $L^p \cap BC_0$, g is in $L^p \cap BC_0$ and f is in $L^p \cap BC_0$, then $x(t)$, the solution of* (2) *is in $L^p \cap BC_0$.*

PROOF. From equation (4) $x(t)$ of (2) satisfies

$$x(t) = Z_I(t)[x(0) - g(0)] + g(t) + \int_0^t Z_I'(t-s)g(s)ds +$$
$$+ \int_0^t Z_I(t-s)f(s)ds + \int_0^t Z_I(t-s)B(s)x(s)ds.$$

Let p be a fixed element of $\langle 1, \infty)$.

Now $Z_I(t)[x(0) - g(0)]$ is in both L^1 and BC_0 so $Z_I(t)[x(0) - g(0)]$ is in L^p. Hence $Z_I(t)[x(0) - g(0)]$ is also in $L^p \cap BC_0$. Analogously $Z_I'(t)$ is in $L^p \cap BC_0$. We known from Theorem A that $x(t)$ is bounded. Thus $B(t)x(t)$ is in $L^p \cap BC_0$.

Now since the convolution of an L^p function with an L^1 function is an L^p function, the three remaining terms on the right-hand side are in $L^p \cap BC_0$. It follows that $x(t)$ is in $L^p \cap BC_0$. □

Theorem B. *Let p be a fixed number satisfying $1 \le p < \infty$. Suppose that $Z_I(t)$ is a function in $L^1 \cap BC$, $Z_I'(t)$ is a function in $L^1 \cap BC$. If $B(t)$ is in BC_0 and $f(t)$ is in $L^p \cap BC$, $g(t)$ is in $L^p \cap BC$, then $x(t)$, the solution of (2) is in $L^p \cap BC$.*

PROOF. The method of proof is the same as that of Theorem A.

We now prove similar results for a C_h space.

Definition. Let h be a continuous n by n matrix such that $n^{-1}(t)$ exists for all $t \in R^+$. A C_h–space is the set of all functions x in C such that

$$\|x\|_h = \sup_{t \in R^+} |h^{-1}(t)x(t)| \le M \text{ for some } M > 0 .$$

Definition. A C_h–space is translation invariant if for each $x \in C$ and $T > 0$, $x \in C_h$ implies $x_T \in C_h$ ($x_T(t) = x(t+T), t \in R^+$) and $x_T \in C_h$ implies $x \in C_h$.

Definition. Let $B(t)$ be an $n \times n$ matrix in C and C_G, C_H be C_h spaces. $B(t)$ is "eventually small" with respect to the pair (C_G, C_H) if for every $\varepsilon > 0$ there is a $T > 0$ such that $\|G^{-1}B_T H\| < \varepsilon$ where $B_T(t) = B(t+T)$.

Theorem C. *Let C_h and C_G be translation invariant. Suppose:*

(i) *$Z_I(t)$ is in C_G, $g(t)$ is in C_G*

(ii) *$f(t)$ is in C_h*

(iii) *$B(t)$ is "eventually small" with respect to (C_h, C_G)*

(iv) *There is a $K_1 > 0$ such that*

$$\int_0^t |G^{-1}(t)Z_I'(t-s)G(s)|ds < K_1 \quad \text{for} \quad t \in R^+$$

(v) *There is a $K_2 > 0$ such that*

$$\int_0^t |G^{-1}(t)Z_I(t-s)h(s)|ds < K_2 \quad \text{for} \quad t \in R^+$$

(vi) *For any $T > 0$ and any $v \in C$,*

$$\int_0^T N(t+s)v(s)ds \in C_G \ ; \quad \int_0^T K(t+s)v(s)ds \in C_h .$$

Then $x(t)$, the solution of (2) is in C_G.

PROOF. The theorem is proved first in the case where $B(t)$ is small, that is $\|h^{-1}BG\| < K_2^{-1}$.

We known from equation (5) that $x(t)$ of (2) satisfies

$$x(t) = Z_I(t)[x(0) - g(0)] + g(t) + \int_0^t Z_I'(t-s)g(s)ds +$$
$$+ \int_0^t Z_I(t-s)f(s)ds + \int_0^t Z_I(t-s)B(s)x(s)ds .$$

So for any $M > 0$

$$\|G^{-1}x\|_{\langle 0,M\rangle} = \sup_{t\in\langle 0,M\rangle} |G^{-1}(t)x(t)| \leq \|Z_I\|_G |x(0) - g(0)| + \|g\|_G(1 + K_1) +$$
$$+ K_2\|f\|_h + K_2\|h^{-1}Bx\|_{\langle 0,M\rangle}$$

and

$$\|h^{-1}Bx\|_{\langle 0,M\rangle} = \|h^{-1}BGG^{-1}x\|_{\langle 0,M\rangle} \leq \|h^{-1}BG\|\,\|G^{-1}x\|_{\langle 0,M\rangle} .$$

Thus

$$\|G^{-1}x\|_{\langle 0,M\rangle} \leq (1 - \|h^{-1}BG\|K_2)^{-1} \cdot \Big[\|Z_I\|_G|x(0) - g(0)| +$$
$$+ \|g\|_G(1 + K_1) + K_2\|f\|_h\Big] ,$$

hence $x \in C_G$.

Now let $B(t)$ be any matrix satisfying (iii). Then there is $T > 0$ such that $\|h^{-1}B_T G\| < K_2^{-1}$. Now $x(t)$ the solution of (2) is bounded on $\langle 0, T\rangle$ and from (5) $x_T(t) = x(t + T)$ solves (6).
From the hypothesis (vi) and from (iii) it follows that F is in C_h and G is in C_G. Since B_T is small with respect to (C_h, C_G), it follows from above that $x_T \in C_G$. Hence x is in C_G. □

Theorem D. *Assume*

(1^0) $$Z_I \text{ is in } BC,\ g \text{ is in } BC$$

(2^0) $$f \text{ is in } L^1, Z_I' \text{ is in } L^2$$

(3^0) $$\sup_{t\geq o} \int_0^t \max_{s\leq r\leq s+1} |Z_I(t-r)B(r)|ds < 1$$

(recall that all functions vanish for negative arguments).
Then the solution $x(t)$ of (2) is in B_{IB}.

PROOF. From (4) the solution $x(t)$ of (2) satisfies (5). Then for all $t \geq 0$

$$\begin{aligned}
|x(t)| \leq& \|Z_I\|\,|x(0)-g(0)| + \|g\|(1+\|Z_I'\|)_1 + \|Z_I\|\,\|f\|_1 + \\
&+ \int_0^t \int_{r-1}^{r} |Z_I(t-r)B(r)|\,|x(r)|ds dr = \\
=& \|Z_I\|\,|x(0)-g(0)| + \|g\|(1+\|Z_I'\|_1) + \|Z_I\|\,\|f\|_1 + \\
&+ \int_{-1}^{t} \int_{s}^{s+1} |Z_I(t-r)B(r)|\,|x(r)|dr ds - \\
&- \int_{-1}^{0} \int_{-1}^{r} |Z_I(t-r)B(r)|\,|x(r)|ds dr - \\
&- \int_{t}^{t+1} \int_{r-1}^{t} |Z_I(t-r)B(r)|\,|x(r)|ds dr \leq \\
\leq& \|Z_I\|\,|x(0)-g(0)| + \|g\|(1+\|Z_I'\|_1 + \|Z_I\|\,\|f\|_1 + \\
&+ \|x\|_B \sup_{t\geq o} \int_0^t \max_{s\leq r\leq s+1} |Z_I(t-r)B(r)|ds\,.
\end{aligned}$$

Hence

$$\begin{aligned}
\|x\|_B \leq& \left(1 - \sup_{t\geq 0} \int_0^t \max_{s\leq r\leq s+1} |Z_I(t-r)B(r)|ds\right)^{-1} \cdot \\
&\cdot \left\{\|Z_I\|\,|x(0)-g(0)| + \|g\|(1+\|Z_I'\|_1) + \|Z_I\|\,\|f\|_1\right\}.
\end{aligned}$$

Thus $x(t)$ is in B_{IB} and the theorem is true.

In this section we shall establish stability results for (1) by use of the variation of constants formula given in (4) above.

MILLER and GROSSMAN in [1] consider such a system where $N(t) = g(t) = 0$, and $F(x)$ is a "higher order" functional. Here we prove a similar theorem but allow $F(x)$ to be the sum of a "higher order" term and a small Lipschitz term.

Definition. A functional F is of "higher order" in the Banach subspace X of LL^1 if F maps X into $X, F(0) = 0$, and for each $\varepsilon > 0$ there exists a

$\delta > 0$ such that
$$\|F(u) - F(v)\|_X < \|u - v\|_X$$
where $\| \bullet \|_X$ is the norm defined on X and u, v are in X and satisfy $\|u\|_X < \delta$, $\|v\|_X < \delta$.

Theorem E. *Suppose that in equation* (1) $F(x) = F_1(x) + F_2(x)$ *where* $F_1(x)$ *is of "higher order" with respect to* X, *a Banach subspace of* LL^1, *and* F_2 *maps* X *into* X *and satisfies* $F_2(0) = 0$, $\|F_2(x) - F_2(y)\|_X \leq L\|x - y\|_X$ *for* $x, y \in X$ *and* $L > 0$. *If* f, g *are in* X, $Z_I(t)$ *is in* X *for any* $u \in X$, $\rho_1(u) = \int\limits_0^t Z_I'(t-s)u(s)ds \in X$, $\rho_2(u) = \int\limits_0^t Z_I(t-s)u(s)ds \in X$ *and there exist* $M > 0$, $M_1 > 0$ *such that* $\|\rho_1(u)\|_X \leq M\|u\|_X$, $\|\rho_2(u)\|_X \leq M_1\|u\|_X$, $B(t)$ *is in* BC, *then for each* $\varepsilon > 0$ *there is an* $\eta > 0$ *such that if* $|x_0| < \eta/2$, $|g(0)| < \eta/2$, $\|g\|_X < \eta$, $\|f\|_X < \eta$, $L < \eta$, *then equation* (1) *has a unique solution* $X(t)$ *in* x *with* $\|x\|_X < \varepsilon$.

PROOF. For any y in X define
$$\begin{aligned} T(y)(t) =& Z_I(t)[x(0) - g(0)] + g(t) + \int\limits_0^t Z_I'(t-s)g(s)ds + \\ &+ \int\limits_0^t Z_I(t-s)f(s)ds + \int\limits_0^t Z_I(t-s)B(s)y(s)ds + \\ &+ \int\limits_0^t Z_I(t-s)\left[F_1(y)(s) + F_2(y)(s)\right]ds \quad \text{for} \quad t \in \langle 0, \infty) \end{aligned}$$

Clearly T maps X into X. Now F_1 is of "higher order" in X so there exists a $\delta > 0$ such that
$$\|F_1(y_1) - F_1(y_2)\|_X \leq \frac{1}{8M}\|y_1 - y_2\|_X \quad \text{if} \quad \|y_1\|_X, \; \|y_2\|_X < \delta .$$
Given $\varepsilon > 0$ define $\varepsilon_0 = \min(\delta, \varepsilon, 1)$ and let
$$\eta = \min\left\{\delta, \frac{\varepsilon_0}{8\|Z_I\|_X}, \frac{\varepsilon_0}{8(1+M)}, \frac{\varepsilon_0}{8M_1}, \frac{\varepsilon_0}{8M_1(1+\|B\|)}, \right\}.$$
Also, define $Q(0, \varepsilon_0) = \{u \in X : \|u\|_X < \varepsilon_0\}$.
For any $y \in Q(0, \varepsilon_0)$
$$\begin{aligned} \|T(y)\|_X \leq& \|Z_I\|_X(|x(0)| + |g(0)|) + M(2+M)\|g\|_X + M_1\|f\|_X + \\ &+ M_1\|B\| \, \|y\|_X + M_1\|F_1(y)\|_X + M_1\|F_2(y)\|_X . \end{aligned}$$

So if $|x_0| < \eta/2$, $|g(0)| < \eta/2$, $\|g\|_X < \eta$, $\|f\|_X < \eta$, $L < \eta$, then $\|T(y)\|_X < \frac{6}{8}\varepsilon_0 < \varepsilon_0$. Hence T maps $Q(0,\varepsilon_0)$ into itself and for $y_1, y_2 \in Q(0,\varepsilon_0)$

$$\begin{aligned}\|T(y_1) - T(y_2)\|_X \leq &\|\rho_1(F_1(y_1)) - \rho_1(F_1(y_2))\|_X + \\ &+ \|\rho_2(F_2(y_1)) - \rho_2(F_2(y_2))\|_X \\ \leq &M\|F_1(y_1) - F_1(y_2)\|_X + M_1\|F_2(y_1) - F_2(y_2)\|_X < \\ < &\frac{1}{4}\|y_1 - y_2\|_X .\end{aligned}$$

Thus T is a contraction which maps on $Q(0,\varepsilon_0)$ and has a unique fixed point $x(t)$. We see from (4) that this is also a solution of (1) with $\|x(t)\|_X < \varepsilon_0$. □

Remark. Analogously we can prove the results given in Theorems A, B for the system

$$\frac{d}{dt}\left[x(t) - \int_0^t N(t-s)x(s)ds - g(t)\right] = (A + B(t))x(t) + \\ + \int_0^t [K(t-s) + Q(t,s)]x(s)ds + f(t).$$

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