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Paracompact locally compact spaces as inverse limits of polyhedra

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Abstract. A topological space is paracompact locally compact (σ-compact locally compact), if and only if it admits a strictly canonical simplicial resolution consisting of metrizable (Polish) polyhedra with all the mappings proper. An analogous characterization of a proper mapping is also established.

1. Introduction

In 1981. S. Mardešić [7], [8] introduced the notions of resolutions of a space and of a mapping. He proved that every space (mapping) admits a polyhedral resolution. In recent years the theory of resolutions, in particular, the theory of approximate resolutions have been very successfully applied to solve various problems, where the classical inverse limit theory failed (see [8]–[13], [15]–[17], [19]).

Having in mind the basic idea, to represent a "bad" object as the limit or resolution of "nice" ones, authors have only rarely (exceptions are [5], [16] and few other references) taken care of appropriate conditions on bonding mappings of a *commutative* representation. For instance, if a space X has to be described in terms of a polyhedral system $\boldsymbol{X} = (X_a, p_{aa'}, A)$ and of a mapping system $\boldsymbol{p} = (p_a) : X \to \boldsymbol{X}$, one should endeavour to have the mappings $p_{aa'}$ and p_a as simple as possible. The same criterion should

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be applied in describing a mapping $f : X \to Y$ in terms of $\boldsymbol{p} : X \to \boldsymbol{X}$, $\boldsymbol{q} = (q_b) : Y \to \boldsymbol{Y}$ and $\boldsymbol{f} = (f, f_b) : \boldsymbol{X} \to \boldsymbol{Y}$.

In a recent paper, assuming that simple spaces are polyhedra, we succeeded in constructing polyhedral resolutions of spaces and mappings, such that all terms are geometric realizations of nerves of normal coverings, all projections are strictly canonical, while all mappings between the terms are piecewise linear or simplicial ([2], Theorems (4.3) and (4.5)). An application of that construction to paracompact locally compact spaces and proper mappings yields the main results of this paper:

(a) A topological space X is paracompact locally compact (σ-compact locally compact) if and only if it admits a strictly canonical simplicial resolution $\boldsymbol{p} = (p_a) : X \to \boldsymbol{X} = (X_a, p_{aa'}, A)$, where all X_a are metrizable (Polish) polyhedra and all mappings $p_{aa'}$ and p_a are proper. (See Theorem 3.3 and Corollaries 3.4 and 3.5.)

(b) A mapping $f : X \to Y$ of topological spaces is a proper mapping of paracompact locally compact (σ-compact locally compact) spaces if and only if it admits a strictly canonical simplicial resolution $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{f})$ consisting of metrizable (Polish) polyhedra and proper mappings. (See Theorem 4.2.)

In both (a) and (b), the word "resolution" can be replaced by "limit", which essentially strengthens the sufficiency parts of the statements.

Recall now some notions and notations. POL denotes the class of polyhedra (CW-topology). If (K, h) is a triangulation of a polyhedron P, we identify P with the geometric realization $|K|$. $M \subseteq K$ denotes a subcomplex, while $M \leq K$ denotes a subdivision; a subpolyhedron $Q \subseteq P$ implies existence of a corresponding subcomplex; if $F \subseteq P = |K|$ is a subset, then $|F|_K \subseteq P$ denotes the carrier of F (with respect to K), which is a subpolyhedron of P.

A mapping $f : |K| \to |L|$ is said to be PL (simplicial) provided it maps closed simplexes of K linearly into (onto) closed simplexes of L. A mapping of polyhedra, $f : P \to Q$, is said to be PL (simplicial), if it is PL (simplicial) with respect to some triangulations K and L of P and Q respectively [2], Definition (2.1)). Clearly, every simplicial mapping is PL, but not conversely. In the case of locally compact polyhedra these notions coincide with standard ones ([6], III. Theorem 3.6.B).

By a space we mean a topological space, and by a mapping, a continuous function. A mapping is proper if preimages of compact sets are compact. Cov(X) denotes the family of all normal (or numerable) coverings of a space X. If $\mathcal{U}$ and $\mathcal{V}$ are coverings of X, $\mathcal{U} \le \mathcal{V}$ means that $\mathcal{U}$ refines $\mathcal{V}$, while $\mathcal{U} \wedge \mathcal{V}$ denotes the covering $\{U \cap V \mid U \in \mathcal{U},\ V \in \mathcal{V},\ U \cap V \ne \emptyset\}$. $N(\mathcal{U})$ denotes the nerve of an open covering $\mathcal{U}$ of X, while $|N(\mathcal{U})|$ denotes its geometric realization. A mapping $p : X \to |N(\mathcal{U})|$ is said to be (strictly) canonical if $p^{-1}(\mathrm{st}\,(u)) \subseteq U\ (= U)$, $U \in \mathcal{U}$ ([2], Definition (3.1) and Lemma (3.3)), where $\mathrm{st}(u) \subseteq |N(\mathcal{U})|$ is the open star of the vertex u corresponding to U.

Basic definitions and facts on inverse systems, limits and resolutions can be found in [13], [7], [8] and [19] (for mappings). Here we recall only those important for our considerations. Let $\boldsymbol{X} = (X_a, p_{aa'}, A)$ be an inverse system of *normal* spaces X_a. A map $\boldsymbol{p} = (p_a) : X \to \boldsymbol{X}$ of a space X into the system $\boldsymbol{X}$, $p_{aa'} p_{a'} = p_{a'}$, $a \le a'$, is a resolution of X if and only if the following two conditions are satisfied:

(B1) $$(\forall \mathcal{U} \in \mathrm{Cov}(X))(\exists a \in A)(\exists \mathcal{V} \in \mathrm{Cov}(X_a)) \quad p_a^{-1}\mathcal{V} \le \mathcal{U};$$

(B2) $$(\forall a \in A)(\forall U \subseteq X_a \text{ open}, \\ \mathrm{Cl}(p_a(X)) \subseteq U)(\exists a' \ge a) \quad p_{aa'}(X_{a'}) \subseteq U.$$

A system $\boldsymbol{X}$ (a resolution $\boldsymbol{p} : X \to \boldsymbol{X}$) all of whose terms X_a are polyhedra is called a POL-system (POL-resolution). Such an $\boldsymbol{X}(\boldsymbol{p})$ is said to be PL or simplicial if all bonding mappings are PL or simplicial respectively. A POL-resolution, as well as a limit $\boldsymbol{p} : X \to \boldsymbol{X}$ is said to be (strictly) canonical if all the projections are (strictly) canonical mappings.

A polyhedral resolution (abbreviated as POL-resolution) of a mapping $f : X \to Y$ is a triple $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{f})$, where $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{q} : Y \to \boldsymbol{Y}$ are POL-resolutions of X and Y respectively, while $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a map of systems satisfying $\boldsymbol{q} = \boldsymbol{f}\boldsymbol{p}$. Similarly, one defines the notions of a PL, a simplicial and a (strictly) canonical resolution of a mapping.

2. Proper bonding mappings imply proper projections

Lemma 2.1. *Let $f : X \to Y$ be a proper mapping. If Y is σ-compact and X is Hausdorff, then X is σ-compact. If Y is locally compact (paracompact locally compact), then so is X.*

Proof. If $Y = \bigcup_{n\in\mathbb{N}} Y_n$, then $X = \bigcup_{n\in\mathbb{N}} f^{-1}(Y_n)$, so the first claim follows. Clearly, if Y is locally compact, then so is X. Finally, a space

Y is paracompact locally compact if and only if it is a disjoint union of σ-compact locally compact spaces ([3], XI. Theorem 7.3). The conclusion follows. □

One naturally asks whether proper bonding mappings imply proper projections. In the case of a limit this is indeed the case.

Lemma 2.2. *$\boldsymbol{p} = (p_a) : X \to \boldsymbol{X} = (X_a, p_{aa'}, A)$ be the limit of a system $\boldsymbol{X}$. If all bonding mappings $p_{aa'}$ are proper, then all projections p_a are proper mappings too.*

Proof. Choose any $a_0 \in A$ and any compact set $C \subseteq X_{a_0}$. Let $A_0 = \{a \in A \mid a \geq a_0\}$ carry the inherited preorder of A, let $X'_a = p^{-1}_{a_0 a}(C) \subseteq X_a$ and let $p'_{aa'} = p_{aa'} |_{X'_{a'}} : X'_{a'} \to X'_a$, $a' \geq a \geq a_0$. The mappings $p'_{aa'}$ are well defined and yield an inverse system $\boldsymbol{X}' = (X'_a, p'_{aa'}, A_0)$. Let $X' = p^{-1}_{a_o}(C) \subseteq X$ and let $p'_a = p_a |_{X'} : X' \to X'_a$, $a \in A_0$. The mappings p'_a are well defined and we obtain a mapping of X' to the system $\boldsymbol{X}'$, $\boldsymbol{p}' = (p'_a) : X' \to \boldsymbol{X}'$. Since A_0 is cofinal in A, $\boldsymbol{p}_0 = (p_a) : X \to \boldsymbol{X}_0 = (X_a, p_{aa'}, A_0)$ is a limit of $\boldsymbol{X}_0$. Without loss of generality, we may assume, that $\boldsymbol{p}_0 : X \to \boldsymbol{X}_0$ is the canonical limit. Then each point $x \in X$ is a unique thread (x_a) of $\boldsymbol{X}_0$, i.e., $x_a = p_a(x) = p_{aa'}(x_{a'})$, $a' \geq a \geq a_0$. By the construction of $\boldsymbol{p}' : X' \to \boldsymbol{X}'$, it is obvious that such a thread (x_a) belong to $\boldsymbol{X}'$ if and only if $x \in X'$. Therefore, $\boldsymbol{p}' : X' \to \boldsymbol{X}'$ is the limit of $\boldsymbol{X}'$. Since all mappings $p_{aa'}$ are proper, all the terms X'_a of $\boldsymbol{X}'$ are compact. Hence, the limit space $X' = p^{-1}_{a_0}(C)$ is compact ([3], App. Two (2.4) (2)), and the projection p_{a_0} is a proper mapping. □

Theorem 2.3. *Let $\boldsymbol{p} = (p_a) : X \to \boldsymbol{X} = (X_a, p_{aa'}, A)$ be a limit, where all X_a are paracompact locally compact (σ-compact locally compact) and all bonding mappings $p_{aa'}$ are proper. Then X is paracompact locally compact (σ-compact locally compact) and all projections p_a are proper mappings.*

Proof. The theorem is an immediate consequence of Lemmata 2.2 and 2.1. □

3. Polyhedral resolution of a paracompact locally compact space

In this section we will show how to construct, for a paracompact locally compact space X, a resolution $\boldsymbol{p} : X \to \boldsymbol{X}$ with proper bonding mappings and projections. Moreover, as in [2], Theorem (4.3), the resolution $\boldsymbol{p}$ will be cofinite, strictly canonical and simplicial, while the system $\boldsymbol{X}$ admits meshes ([15], Definition (1.1)).

Lemma 3.1. *Let $\mathcal{U}$ be an open covering of a space X and let $p : X \to |N(\mathcal{U})|$ be a canonical mapping with respect to $\mathcal{U}$. If all members of $\mathcal{U}$ are relatively compact, then p is a proper mapping. Conversely, if p is proper and $\mathcal{U}$ is star-finite, then $\mathcal{U}$ consists of relatively compact sets.*

Proof. For the necessity, it suffices to prove that, for every simplex $\sigma \subseteq |N(\mathcal{U})|$, $p^{-1}(\sigma)$ is a compact subset of X. Let $\sigma = [u_0, \ldots, u_n]$. Since p is canonical for $\mathcal{U}$,

$$p^{-1}(\sigma) \subseteq p^{-1}\Big(\bigcup_{i=0}^{n} \mathrm{st}(u_i)\Big) = \bigcup_{i=0}^{n} p^{-1}(\mathrm{st}(u_i)) \subseteq \bigcup_{i=0}^{n} U_i \subseteq \bigcup_{i=0}^{n} \mathrm{Cl}(U_i).$$

Note that $p^{-1}(\sigma)$ is closed, and $\bigcup_{i=0}^{n} \mathrm{Cl}(U_i)$ is compact. Therefore, $p^{-1}(\sigma)$ is compact. Conversely, for every $U \in \mathcal{U}$, $p(\mathrm{Cl}(U)) \subseteq |\,\mathrm{st}(u)|$ (see [2], Lemma (2.2) (Proof)). Hence, $\mathrm{Cl}(U) \subseteq p^{-1}p(\mathrm{Cl}(U)) \subseteq p^{-1}|\,\mathrm{st}(u)|$. Since $\mathcal{U}$ is star-finite, each closed star $|\,\mathrm{st}(u)| \subseteq N(\mathcal{U})|$ is compact. Therefore, if p is proper then $\mathrm{Cl}(U)$ is compact, $U \in \mathcal{U}$. □

Let us recall the construction of the canonical and simplicial mappings in the proof of [8], Theorem 11. For a normal covering $\mathcal{U}$ of a space X, choose a locally finite partition of unity $(\varphi_U, U \in \mathcal{U})$ subordinated to $\mathcal{U}$. Then $(\varphi_U, U \in \mathcal{U})$ determines a mapping $p_{\mathcal{U}} : X \to |N(\mathcal{U})|$ which sends the point $x \in X$ to the point $p_{\mathcal{U}}(x)$,whose barycentric coordinate with respect to the vertex $u \in N(\mathcal{U})$, $U \in \mathcal{U}$, equals $\varphi_U(x)$. $p_{\mathcal{U}}$ is called the canonical mapping of the partition $(\varphi_U, U \in \mathcal{U})$. Obviously, $p_{\mathcal{U}}$ is a canonical mapping with respect to $\mathcal{U}$. Let $p_{\mathcal{V}} : X \to |N(\mathcal{V})|$ be another such canonical mapping determined by $(\psi_V, V \in \mathcal{V})$. Then

$$(\chi_{U\cap V}, U \cap V \in \mathcal{U} \wedge \mathcal{V}) = (\varphi_U \cdot \psi_V, (U, V) \in \mathcal{U} \times \mathcal{V}, U \cap V \neq \emptyset)$$

is a partition of unity subordinated to $\mathcal{U} \wedge \mathcal{V}$. It determines the canonical mapping $p_{\mathcal{U}\wedge\mathcal{V}} : X \to |N(\mathcal{U} \wedge \mathcal{V})|$. Furthermore, by sending the vertex $(u, v) \in N(\mathcal{U} \wedge \mathcal{V})$ to the vertices $u \in N(\mathcal{U})$ and $v \in N(\mathcal{V})$, one obtains two simplicial mappings $f : |N(\mathcal{U} \wedge \mathcal{V})| \to N(\mathcal{U})|$ and $g : |N(\mathcal{U} \wedge \mathcal{V})| \to |N(\mathcal{V})|$, which satisfy $fp_{\mathcal{U}\wedge\mathcal{V}} = p_{\mathcal{U}}$ and $gp_{\mathcal{U}\wedge\mathcal{V}} = p_{\mathcal{V}}$.

Let $\mathcal{U}$ and $\mathcal{V}$ be coverings of a space X. $\mathcal{V}$ is said to be star-finite with respect to $\mathcal{U}$ if each $U \in \mathcal{U}$ meets at most finitely many members of $\mathcal{V}$.

Lemma 3.2. *Let $\mathcal{U}$ and $\mathcal{V}$ be normal coverings of a space X, and let $p_{\mathcal{U}} : X \to |N(\mathcal{U})|$ and $p_{\mathcal{U}\wedge\mathcal{V}} : X \to |N(\mathcal{U} \wedge \mathcal{V})|$ be the above described canonical mappings. Then the corresponding simplicial mapping $f : |N(\mathcal{U} \wedge \mathcal{V})| \to |N(\mathcal{U})|$, satisfying $fp_{\mathcal{U}\wedge\mathcal{V}} = p_{\mathcal{U}}$, is proper if and only if $\mathcal{V}$ is star-finite with respect to $\mathcal{U}$.*

Proof. Note that f is proper if and only if, for every $U \in \mathcal{U}$,

$$(f^{-1}(u))^0 = \{(u, v_1), \dots (u, v_n)\} \subseteq |N(\mathcal{U} \wedge \mathcal{V})^0|$$

is a finite set of vertices. This is equivalent to the condition that U meets only finitely many $V_1, \dots, V_n \in \mathcal{V}$, i.e., that $\mathcal{V}$ is star-finite with respect to $\mathcal{U}$. □

We can now prove the (strong) converse of Theorem 2.3.

Theorem 3.3. *Every paracompact locally compact (σ-compact locally compact) space X admits a cofinite strictly canonical simplicial resolution $\boldsymbol{p} = (p_a) : X \to \boldsymbol{X} = (X_a, p_{aa'}, A)$, where all polyhedra X_a are locally compact (separable locally compact) and all mappings $p_{aa'}$ and p_a are proper. $\boldsymbol{p} : X \to \boldsymbol{X}$ is also a limit.*

Proof. Let $\boldsymbol{C}$ be the family of all open star-finite coverings of X, which consist of relatively compact members. Then $\boldsymbol{C}$ is a cofinal subfamily of $\mathrm{Cov}(X)$. Furthermore, if $n \in \mathbb{N}$ and $\mathcal{U}_1, \dots, \mathcal{U}_n \in \boldsymbol{C}$, then $\mathcal{U}_1 \wedge \dots \wedge \mathcal{U}_n \in \boldsymbol{C}$. Therefore, we may apply the construction of [8], Theorem 11 (the first part of the proof), to the family $\boldsymbol{C}$. This yields a map $\boldsymbol{p}' = (p'_\lambda) : X \to \boldsymbol{X}' = (X'_\lambda, p'_{\lambda\lambda'}, \Lambda)$ of the space X to the inverse system $\boldsymbol{X}'$ such that:

(1) $\Lambda = (\Lambda, \leq)$ is cofinite, i.e., each $\lambda \in \Lambda$ has at most finitely many predecessors;

(2) $(\forall \lambda \in \Lambda)$ X'_λ is the realized nerve of a member of $\boldsymbol{C}$;

(3) $(\forall \lambda \in \Lambda)$ p'_λ is a corresponding canonical mapping;

(4) $(\forall \lambda \leq \lambda')$ $p_{\lambda\lambda'}$ is the simplicial mapping induced by p'_λ and $p'_{\lambda'}$;

(5) $\boldsymbol{p}'$ satisfies condition (B1);

Recall that every polyhedron is paracompact. Furthermore, the choice of $\boldsymbol{C}$ and Lemmata 3.1 and 3.2 imply

(6) $(\forall \lambda \in \Lambda)$ X'_λ is locally compact, i.e., metrizable;

(7) $(\forall\lambda \in \Lambda)$ p'_λ is a proper mapping;

(8) $(\forall\lambda \leq \lambda')$ $p_{\lambda\lambda'}$ is a proper mapping.

By [2], Proposition (4.2), $\boldsymbol{p'}$ admits an extension up to a cofinite, strictly canonical, PL resolution $\boldsymbol{p} = (p_a) : X \to \boldsymbol{X} = (X_a, p_{aa'}, A)$, such that

(9) $(\forall a \in A)(\exists\lambda \in \Lambda)$ $X_a \subseteq X'_\lambda$ is a subpolyhedron;

(10) $(\forall a \leq a')(\exists\lambda \leq \lambda')$ $p_{aa'} : X_{a'} \to X_a$ is either the inclusion mapping of the subpolyhedra of X'_λ or the restriction mapping of $p'_{\lambda\lambda'}$ to the corresponding subpolyhedra;

(11) $(\forall a \in A)(\exists\lambda \in \Lambda)$ $p_a = p'_\lambda : X \to X_a \subseteq X'_\lambda$.

Since every proper PL mapping is simplicial ([6], III. Theorem 3.6.C; [2], Lemma (2.2) (ii)), all the requirements on the resolution $\boldsymbol{p} : X \to \boldsymbol{X}$ are fulfilled. Note that in the case of a σ-compact locally compact space X, the nerves which appear are locally finite and countable. Therefore, all polyhedra X'_λ and the subpolyhedra $X_a \subseteq X'_\lambda$ are locally compact and separable, i.e. Polish (see [17], Propositions (2.1) and (2.2)). Finally, the last statement follows by [15], Theorem (3.1). □

Corollary 3.4. *Let X be a topological space. The following statements are equivalent:*

(i) *X is paracompact locally compact;*

(ii) *X admits a proper mapping into a paracompact locally compact space;*

(iii) *X is a limit of a system of metrizable polyhedra with proper bonding mappings;*

(iv) *X admits a resolution $\boldsymbol{p} = (p_a) : X \to \boldsymbol{X} = (X_a, p_{aa'}, A)$, where all X_a are metrizable polyhedra and all bonding mappings $p_{aa'}$ (and all projections p_a) are proper.*

Proof. (ii) implies (i) by Lemma 2.1, (i) implies (iv) by Theorem 3.3, (iv) implies (iii) by [15] Theorem (3.1), while (iii) implies (ii) by Lemma 2.2 □

Similarly to the above, the following corollary holds:

Corollary 3.5. *Let X be a topological space. The following statements are equivalent:*

(i) *X is σ-compact locally compact;*

(ii) *X admits a proper mapping into a σ-compact locally compact space;*

(iii) *X is a limit of a system of Polish polyhedra with proper bonding mappings;*
(iv) *X admits a resolution $\boldsymbol{p} = (p_a) : X \to \boldsymbol{X} = (X_a, p_{aa'}, A)$, where all X_a are Polish polyhedra and all bonding mappings $p_{aa'}$ (and all projections p_a) are proper.*

Remark. The systems $\boldsymbol{X}$ in the above statements admit meshes, i.e., they satisfy condition (A3) (see [15], Definition (1.1)). Indeed, the system $\boldsymbol{X}'$ in the proof of Theorem 3.3 satisfies condition (C) ([18], Theorem (2.3); [14], Theorem (2.8)). Furthermore, the resolutions in the necessity parts of both corollaries are cofinite strictly canonical and simplicial.

4. Characterization of a proper mapping

Analogously to the previous characterization of paracompact locally compact spaces, one can characterize proper mappings of such spaces. First recall a few elementary facts:

Lemma 4.1. *Let $f : X \to Y$ be a mapping and $\mathcal{V}$ an open covering of Y.*

(i) *If $\mathcal{V}$ is star-(locally, point-) finite, then so is $f^{-1}(\mathcal{V})$;*
(ii) *If $\mathcal{V}$ is countable, then so is $f^{-1}(\mathcal{V})$;*
(iii) *If $\mathcal{V}$ consists of relatively compact sets and f is proper, then $f^{-1}(\mathcal{V})$ consists of relatively compact sets.*

Let $f : X \to Y$ be a proper mapping of paracompact locally compact spaces. Recall the construction of [8], Theorem 11 (the first part of the proof), which now let be based on the families $\boldsymbol{D} \subseteq \mathrm{Cov}(Y)$ and $\boldsymbol{C} \subseteq \mathrm{Cov}(X)$ consisting of all locally finite open coverings with relatively compact members. Clearly, $\boldsymbol{D}$ and $\boldsymbol{C}$ are cofinal subfamilies, and any two of their members are mutually star-finite. Furthermore, by Lemma 4.1, $f^{-1}\mathcal{V} \in \boldsymbol{C}$ whenever $\mathcal{V} \in \boldsymbol{D}$. The above mentioned construction, according to Lemma 3.1, 3.2 and 4.1, yields now the POL-systems $\boldsymbol{X}' = (X'_\lambda, p'_{\lambda\lambda'}, \Lambda)$, $\boldsymbol{Y}' = (Y'_\mu, q'_{\mu\mu'}, M)$ and maps $\boldsymbol{p}' = (p'_\lambda) : X \to \boldsymbol{X}'$, $\boldsymbol{q}' = (q'_\mu) : Y \to \boldsymbol{Y}'$, $\boldsymbol{f}' = \left\{f', f'_\mu \mid \mu \in M\right\} : \boldsymbol{X}' \to \boldsymbol{Y}'$, such that

(1) $\boldsymbol{X}'$ and $\boldsymbol{Y}'$ are cofinite satisfying condition (C), i.e., (for $\boldsymbol{X}'$) $(\forall \lambda \in \Lambda)$ $\mathrm{cw}(X'_\lambda) \leq \mathrm{card}(\Lambda)$ (cw denotes *covering weight*; see [14]);
(2) $\boldsymbol{p}'$ and $\boldsymbol{q}'$ satisfy condition (B1);

(3) $\boldsymbol{q}'f = \boldsymbol{f}'\boldsymbol{p}'$;
(4) X'_λ and Y'_μ are locally compact realized nerves;
(5) p'_λ and q'_μ are canonical mappings;
(6) $p'_{\lambda\lambda'}$, $q'_{\mu\mu'}$ and f'_μ are simplicial mappings (with respect to the unique triangulations on the terms given by the nerves), and f'_μ are embeddings;
(7) $f' : M \to \Lambda$ is an increasing injection.

We can now construct, in the same way as in [2], Proposition (4.2) and Theorem (4.5), a desired resolution $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{f})$ of f. Consequently, $\boldsymbol{p} = (p_a) : X \to \boldsymbol{X} = (X_a, p_{aa'}, A)$ and $\boldsymbol{q} = (q_b) : Y \to \boldsymbol{Y} = (Y_b, q_{bb'}, B)$ are cofinite, strictly canonical, PL resolutions of X and Y respectively, which admit meshes, while $\boldsymbol{f} = \{f, f_b \mid b \in B\} : \boldsymbol{X} \to \boldsymbol{Y}$ is a map of systems satisfying $\boldsymbol{q}f = \boldsymbol{f}\boldsymbol{p}$ with all mappings f_b simplicial embeddings (hence, proper and closed). Moreover, by Lemmata 3.1, 3.2 and 4.1, all polyhedra X_a, Y_b are metrizable, all projections p_a, q_b are proper and strictly canonical mappings and all bonding mappings are proper and simplicial. Of course, in the special case of σ-compact locally compact spaces all polyhedra X_a, Y_b are Polish. Therefore, we can state the following characterization:

Theorem 4.2. *Let $f : X \to Y$ be a mapping of topological spaces. Then f is a proper mapping of paracompact locally compact (σ-compact locally compact) spaces if and only if it admits a strictly canonical simplicial resolution $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{f})$, where the systems are cofinite, admit meshes and consist of metrizable (Polish) polyhedra, and all the within appearing mappings are proper.*

PROOF. The necessity follows by the preceding construction. For the sufficiency, X and Y are paracompact locally compact (σ-compact locally compact) by Corallary 3.4 (Corollary 3.5), while f is proper since the composition $q_b f = f_b p_{f(b)}$ is a proper mapping. □

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