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Finsler planes with a generalization of Ingarden–Tamássy's parabolic metric

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Dedicated to Professors Dr. R.S. Ingarden and L. Tamássy as the executive organizors of the Conference on Finsler Geometry and its Applications to Physics and Control Theory

Recently R.S. INGARDEN and L. TAMÁSSY ([3], [4]) consider a Minkowski plane having a parabola as the indicatrix and taking for its origin the vertex of the parabola. They come up with the interesting idea from a remarkable geometrical structure as a mathematical representation of the space–time of thermodynamical states given by INGARDEN [2]. This structure is regarded as a Finsler space with a special Kropina metric ([5], §16) and C. SHIBATA [9] has studied it from the standpoint of Finsler geometry.

In the present paper we shall consider two kinds of Finsler planes which are locally Ingarden–Tamássy's Minkowski plane. The metrics of these Finsler planes give remarkable examples of 1-form metrics [8] and of Finsler spaces having logarithmic spirals as geodesics [1].

In the two-dimensional case we have an important scalar I, called the main scalar, which may be regarded as the degree of Finslerian slippage from Riemannian space. In a previous paper [7] we have met with a little strange circumstances; this scalar has the upper limit $3/\sqrt{2}$ in some Finsler planes having typical curves as the indicatrices. We have the interesting fact that the Finsler planes in the present paper have the constant $I = 3/\sqrt{2}$.

§1. Parabolic Finsler plane of the first kind

Let R^2 be a euclidean plane having an orthonormal coordinate system

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(x, y) with the origin O and $\delta(x, y)$ a positive-valued function on $\pi = R^2 - \{O\}$.

We shall define a Finsler metric on π as follows. For an arbitrary point $P(x, y)$ of π (Fig.1) we take two points F_1, F_2 and two straight lines f_1, f_2 such that

(1) PF_1 and PF_2 are orthogonal to OP and their euclidean lengths are equal to the $\delta(x, y)$-fold of OP.

(2) f_1 (resp. f_2) is parallel to OP and through F_2 (resp. F_1).

We may choose F_1 and F_2 as

$$F_1(x - \delta(x, y)y,\ y + \delta(x, y)x), \quad F_2(x + \delta(x, y)y,\ y - \delta(x, y)x).$$

Then we have the equations

$$f_1 : yu - xv - \delta(x, y)(x^2 + y^2) = 0,$$
$$f_2 : yu - xv + \delta(x, y)(x^2 + y^2) = 0,$$

in the current coordinates (u, v).

Now we shall take indicatrix $I(P)$ at the point $P(x, y)$ as two parabolas c_i, $i = 1, 2$, having the focus F_i and the directrix f_i respectively, and the Finsler plane as thus obtained will be called the *parabolic Finsler plane of the first kind.*

Remark 1. In Ingarden–Tamássy's theory the "irreversibility" is emphasized from the standpoint of physics, so the indicatrix is only one parabola. But, from the standpoint of geometry, it will be better to take two symmetric parabolas as the indicatrix.

We shall find the fundamental function of the Finsler plane. For instance, c_1 is given by the equation

$$\sqrt{(u - x + \delta y)^2 + (v - y - \delta x)^2} = \{-yu + xv + \delta(x^2 + y^2)\}/\sqrt{x^2 + y^2},$$

paying attention to the fact that the term in $\{\dots\}$ is positive on c_1. By squaring this we get the equation of c_1:

$$c_1 : \{xu + yv - (x^2 + y^2)\}^2 + 4\delta(x^2 + y^2)(yu - xv) = 0.$$

Then, putting $u = x + \dot{x}$, $v = y + \dot{y}$ ([1]; [5], Example 16.3) the above is rewritten in the form

$$(x\dot{x} + y\dot{y})^2 + 4\delta(x^2 + y^2)(y\dot{x} - x\dot{y}) = 0,$$

and applying Okubo's method ([5], Example 16.4), that is, substituting $(\dot{x}/L,\ \dot{y}/L)$ for $(\dot{x}, \dot{y})$ in the above, we obtain the fundamental function $L(x, y; \dot{x}, \dot{y})$ as follows:

$$L(x, y; \dot{x}, \dot{y}) = (x\dot{x} + y\dot{y})^2/4\delta(x, y)(x^2 + y^2)(-y\dot{x} + x\dot{y}).$$

Let us remark that the term $(-y\dot{x} + x\dot{y})$ is positive on c_1.

Similarly the equation of another parabola c_2 gives us

$$L(x, y; \dot{x}, \dot{y}) = (x\dot{x} + y\dot{y})^2 / 4\delta(x, y)(x^2 + y^2)(y\dot{x} - x\dot{y}).$$

Consequently we have

Proposition 1. *The metric function of the parabolic Finsler plane of the first kind is given by*

$$L_1(x, y; \dot{x}, \dot{y}) = (x\dot{x} + y\dot{y})^2 / 4\delta(x, y)(x^2 + y^2)\ |y\dot{x} - x\dot{y}|.$$

Remark 2. We find two differential 1-forms $x\dot{x} + y\dot{y}$, $y\dot{x} - x\dot{y}$ and the quadratic form $\dot{x}^2 + \dot{y}^2$ in all the 2-dimensional examples appearing in the paper [1]. Each of those examples contains $\dot{x}^2 + \dot{y}^2$, while our L_1 consists of $x\dot{x} + y\dot{y}$ and $y\dot{x} - x\dot{y}$ alone.

§2. Parabolic Finsler plane of the second kind

Next, we shall define another Finsler metric on π in a similar way. Take two points F_3, F_4 (Fig. 2) and two straight lines f_3, f_4 such that

(1) F_3 and F_4 are on the line OP and the euclidean lengths of PF_3 and PF_4 are equal to the $\delta(x, y)$-fold of OP.

(2) f_3 (resp. f_4) is orthogonal to OP and through F_4 (resp. F_3).

We may choose F_3 and F_4 as

$$F_3((1 - \delta)x, (1 - \delta)y), \quad F_4((1 + \delta)x, (1 + \delta)y).$$

Then we have the equations

$$f_3 : xu + yv - (1 + \delta)(x^2 + y^2) = 0,$$
$$f_4 : xu + yv - (1 - \delta)(x^2 + y^2) = 0.$$

Now we shall take the indicatrix $I(P)$ at P as two parabolas c_i, $i = 3, 4$, having the focus F_i and the directrix f_i respectively. The Finsler plane as thus obtained will be called the *parabolic Finsler plane of the second kind.* Then the equations of c_3 and c_4 are given by

$$(yu - xv)^2 \pm 4\delta(x^2 + y^2)(xu + yv - x^2 - y^2) = 0,$$

and similarly to the first kind we obtain the

Proposition 2. *The metric function of the parabolic Finsler plane of the second kind is given by*

$$L_2(x, y; \dot{x}, \dot{y}) = (y\dot{x} - x\dot{y})^2 / 4\delta(x, y)(x^2 + y^2)|x\dot{x} + y\dot{y}|.$$

Remark 3. We observe that the differential 1-forms $x\dot{x} + y\dot{y}$ and $y\dot{x} - x\dot{y}$ interchange with each other in L_1 and L_2. Hence these 1-forms may be said to be orthogonal to each other and one indicatrix is obtained from the other by a right-angle rotation around the point P.

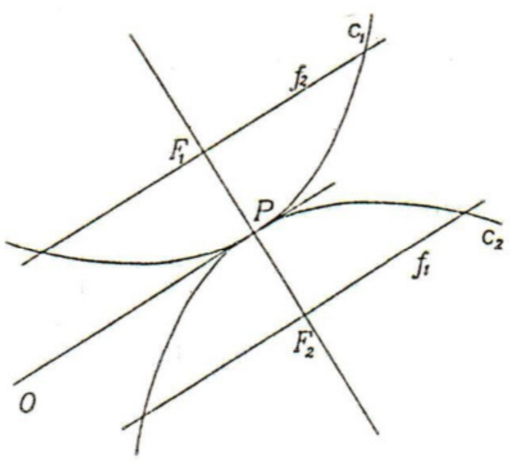

Figure 1.

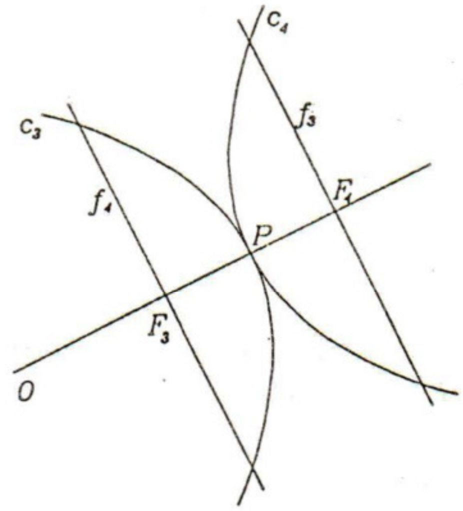

Figure 2.

§3. Examples of 1-form metrics

We shall introduce two differential 1-forms

$$(3.1) \quad \begin{aligned} \lambda &= (xdx + ydy)/(x^2 + y^2) = d(\log \sqrt{x^2 + y^2}\,), \\ \mu &= (xdy - ydx)/(x^2 + y^2) = d(\operatorname{Arctan}(y/x)). \end{aligned}$$

In our parabolic Finsler planes we are concerned with a positive-valued function $\delta(x, y)$, which will be called the *density at the point* $P(x, y)$. If we put $a^1 = a^1_i(x)\,dx^i = \lambda/4\delta$ and $a^2 = a^2_i(x)\,dx^i = \mu/4\delta$, then we have ([7], [8])

Proposition 3. *The metrics L_1 and L_2 of the parabolic Finsler planes are 1-form metrics of the forms*

$$L_1 = (a^1)^2 / |a^2|, \quad L_2 = (a^2)^2 / |a^1|,$$

where we put $a^1 = \lambda/4\delta$ and $a^2 = \mu/4\delta$.

As is well-known, the *main scalar* I ([5], §28; [7]) is an important scalar of a two-dimensional Finsler space. The scalar of a Finsler space with 1-form metric has been completely studied by the author with H. SHIMADA [8]. In particular we observe that the metrics of the parabolic Finsler planes, as given in Proposition 3, belong to one of the special classes of 1-form metrics whose main scalars are constant ([5], (28.28)(i)), as originally given by L. BERWALD. Thus we have

Theorem 1. *The square of the main scalar I of the parabolic Finsler planes is identically equal to 9/2.*

Remark 4. In the paper [4] the main scalar is taken as $-3/\sqrt{2}$, but its algebraic sign is not essential from the standpoint of geometry ([5], Theorem 28.4). Next, in Ingarden–Tamássy's papers, the parabola as the indicatrix is given by $y = x^2$ in [3], while it is $y = x^2/2$ or $y = (K/2)x^2$ in [4]. In the present paper we introduced the notion of density $\delta(x, y)$, paying attention to the difference above, but it makes no difference to the main scalar. (Cf. Theorem 2.)

For a Finsler space with 1-form metric $L(a^\alpha)$, $a^\alpha = a^\alpha_i(x)y^i$, the *1-form Finsler connection* $F1 = (\Gamma_j{}^i{}_k(x),\ \Gamma_0{}^i{}_j,\ C_j{}^i{}_k)$ is essential [8]. The connection coefficients $\Gamma_j{}^i{}_k(x)$ are defined by

$$\Gamma_j{}^i{}_k(x) = b^i_\alpha(\partial a^\alpha_j/\partial x^k),$$

where (b^i_α) is the inverse matrix of (a^α_i). In case of our parabolic planes we have from (3.1)

$$(a^\alpha_i) = \begin{pmatrix} xz & yz \\ -yz & xz \end{pmatrix}, \quad (b^i_\alpha) = \begin{pmatrix} 4\delta x & -4\delta y \\ 4\delta y & 4\delta x \end{pmatrix} \qquad (3.2)$$
$$z = 1/4\delta(x^2+y^2),$$

and then the $\Gamma_j{}^i{}_k$ are given by

$$\Gamma_1{}^1{}_1 = \Gamma_2{}^2{}_1 = -x/(x^2+y^2) - \delta_x/\delta, \quad \Gamma_2{}^1{}_2 = -\Gamma_1{}^2{}_2 = x/(x^2+y^2),$$
$$\Gamma_1{}^1{}_2 = \Gamma_2{}^2{}_2 = -y/(x^2+y^2) - \delta_y/\delta, \quad \Gamma_1{}^2{}_1 = -\Gamma_2{}^1{}_1 = y/(x^2+y^2).$$

Thus the connection $F1$ has the $(h)h$-torsion tensor $T_j{}^i{}_k = \Gamma_j{}^i{}_k - \Gamma_k{}^i{}_j$:

$$T_1{}^1{}_2 = -\delta_y/\delta, \quad T_1{}^2{}_2 = \delta_x/\delta.$$

If the $(h)h$-torsion tensor T vanishes, then the space belongs to the class of locally Minkowski spaces [6], called *T-Minkowski*. In a locally Minkowski space we have an *adapted coordinate system* (x^i) in which the fundamental function is written in $(\dot{x}^i)$ alone ([5], Definition 24.1). Therefore we have

Theorem 2. *The parabolic Finsler planes are T-Minkowski, if and only if the density $\delta(x, y)$ is constant. Then their metrics are written as*

$$L_1 = \dot{\xi}^2/4\delta|\dot{\eta}|, \qquad L_2 = \dot{\eta}^2/4\delta|\dot{\xi}|,$$

where the adapted coordinate system (ξ, η) is obtained from (x, y) by the transformation $\xi = \log\sqrt{x^2 + y^2}$, $\eta = \mathrm{Arctan}(y/x)$.

For general L_1 and L_2 we get T-Minkowski metrics δL_1 and δL_2 which have $\delta = 1$. Therefore we have the following

Corollary. *A parabolic Finsler plane is conformal to a T-Minkowski parabolic Finsler plane of the same kind having a constant density δ.*

Remark 5. These forms of L_1 and L_2 in Theorem 2 are remarkable, because the metric of Ingarden–Tamássy's Minkowski plane is just given in the above forms.

We shall continue to be concerned with the T-Minkowski parabolic Finsler planes. If we refer to the polar coordinates (ρ, θ) defined by $x = \rho\cos\theta$, $y = \rho\sin\theta$, then we get $(\xi, \eta) = (\log\rho, \theta)$ and $(\dot{\xi}, \dot{\eta}) = (\dot{\rho}/\rho, \dot{\theta})$. Thus we recall Hōjō's Theorem [1]: If a two-dimensional Finsler metric L on π is given by $L = f(\rho\dot{\theta}, \dot{\rho})/\rho$, then its geodesics are represented by logarithmic spirals. Since L is, of course, assumed to be positively homogeneous of degree one, the above condition is rewritten as $L = f(\dot{\theta}, \dot{\rho}/\rho)$. Consequently we have

Theorem 3. *Any geodesic of the T-Minkowski parabolic Finsler planes is a logarithmic spiral with the pole O.*

§4. Right angle

We shall find the right angle in our parabolic metric $L = \beta^2/\gamma$, where $a^1 = \beta$ and $a^2 = \gamma$. To do so, we consider the Berwald frame (ℓ, m) ([5], §28). If we put $L_\alpha = \partial L/\partial a^\alpha$, we have $\ell^i = y^i/L = (\gamma/\beta^2)(\dot{x}, \dot{y})$ and $\ell_i = L_\alpha a_i^\alpha$, that is, from (3.1)

$$(\ell_i) = (\beta z/\gamma^2)(\gamma x + \dot{y}/4\delta, \gamma y - \dot{x}/4\delta). \tag{4.1}$$

Next, the fundamental tensors $g_{ij} = (\partial^2 L^2/\partial\dot{x}^i\partial\dot{x}^j)/2$ are given by

$$\begin{aligned} g_{11} &= (\beta^2 z^2/\gamma^4)(6\gamma^2 x^2 + 8\beta\gamma xy + 3\beta^2 y^2), \\ g_{22} &= (\beta^2 z^2/\gamma^4)(6\gamma^2 y^2 - 8\beta\gamma xy + 3\beta^2 x^2), \end{aligned} \tag{4.2}$$

where g_{12} is not needed in the sequel. Then the equations

$$(m_1)^2 = g_{11} - (\ell_1)^2, \quad (m_2)^2 = g_{22} - (\ell_2)^2,$$

and $\ell^i m_i = 0$ give

$$(m_i) = (\sqrt{2}\,\beta z/4\delta\gamma^2)(-\dot{y}, \dot{x}). \tag{4.3}$$

Further $\ell_i m^i = 0$ and $m_i m^i = 1$ lead us to

$$(m^i) = (4\delta\gamma/\sqrt{2}\,\beta^2)(-\gamma y + \dot{x}/4\delta,\ \gamma x + \dot{y}/4\delta). \tag{4.4}$$

Consequently this (m^i) is orthogonal to $(\ell^i) = (\gamma/\beta^2)(\dot{x}, \dot{y})$. If we use the vector $b_2 = (-4\delta y, 4\delta x)$ in (3.2), we have another expression of (m^i):

$$m^i = \{\ell^i + (\gamma/\beta)^2 b_2^i\}/\sqrt{2}. \tag{4.4'}$$

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