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An asymptotic approach to the multiple machine interference problem with Markovian environments

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Abstract. This paper is concerned with a queueing model to analyse the asymptotic behaviour of the machine interference problem with N heterogeneous machines and one operative. The machines and the repair facility are assumed to operate in independent random environments governed by ergodic Markov chains. The running and repair times of a machine are supposed to be exponentially distributed random variables with parameter depending on the index of the machine and state of the corresponding random environment. Assuming that the repair rates are many times greater than the corresponding failure rates ("fast" service), it is shown that the time until the number of stopped machines reaches a certain level converges weakly, under appropriate norming, to an exponentially distributed random variable. Furthermore, some numerical examples illustrate the problem in question in the field of textile winding.

Introduction

The problem of calculating the running efficiency and the operative utilization in situations where a group of identical machines, subject to random breakd)wns, are maintained by one or more operatives, has been treated by a nuımber of authors. They have used a variety of approaches and have made different assumptions about the statistical distributions of running time between breakdowns and repair time. For an extensive bibliography on the basic homogeneous finite-source models, reference may be made to CARMICHAEL [6], STECKE and ARONSON [16]. In recent years the machine interference model has been used, for example, for the mathematical description of computer terminal systems, cf. TAKAGI [21], or for modelling production systems in textile winding, see BUNDAY [4]. More recently several authors have tackled the problem for non-identical set

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of machines. The major problem when considering different types of machines is that it is neccesary to keep track of where each individual machine is in the system. Among those contributing are BUNDAY and KHORRAM [5], SZTRIK [18–19], TOSIRISUK and CHANDRA [22] in which an up-to-date bibliography can be found on this topic. In these papers the main aim has been to predict the steady-state operational measures, such as machine availability, operative utilization, mean waiting time, average queue length.

In this study an asymptotic approach (when the repair rates are many times greater than the corresponding failure rates) is presented to analyse the distribution of the time until the number of stopped machines reaches a certain level. This method is quite common in reliability theory; see among others ANISIMOV and SZTRIK [3], SZTRIK [20], GERTSBAKH [9–10], KEILSON [11]. Realistic consideration of certain stochastic systems, however, often requires the introduction of a random environment where system parameters are subject to randomly occuring fluctiations. This situation may be attribute to certain changes in the physical environment, or sudden personnel changes and work load alterations. Computational problems of birth- and-death models in random environments, sometimes called Markov-modulated processes, have been the subject of several works (c.f., GAVER *et al.* [7], NEUTS [12–13], PURDUE [14], SENGUPTA [15], STERN and ELWALID [17]). Necessary and sufficient conditions for the stability of a single server exponential queue with random fluctuations in the intensity of the arrival processes have also been derived (c.f., GELENBE and ROSENBERG [8]).

This paper is concerned with a queueing model to analyse the machine interference problem with N heterogeneous machines and one operative. The machines and the repair facility are assumed to operate in independent random environments governed by ergodic Markov chains. The running and repair times of a machine are supposed to be exponentially distributed random variables with parameter depending on the index of the machine and state of the corresponding random environment. Assuming that the repair rates are many times greater than the corresponding failure rates ("fast" service), it is shown that the time until the number of stopped machines reaches a certain level converges weakly, under appropriate norming, to an exponentially distributed random variable. Furthermore, some numerical examples illustrate the problem in question in the field of textile winding.

2. Preliminary results

This section presents a brief survey of results (c.f., ANISIMOV *et al.* [2]) to be applied in the next section.

Let $(X_\varepsilon(k), k \geq 0)$ be a Markov chain depending on a small parameter

$\varepsilon > 0$, and let its state space be

$$\bigcup_{q=0}^{m+1} X_q, \quad X_i \cap X_j = 0, \quad i \neq j,$$

with $m+2$ levels of states, $i, j = 0, 1, \ldots, m+1$. Assume that the transition matrix $\left(p_\varepsilon(i^{(q)}, j^{(z)})\right)$, $i^{(q)} \in X_q$, $j^{(z)} \in X_z$, $q, z = 0, 1, \ldots, m+1$ satisfies the following conditions:

1. $p_\varepsilon(i^{(0)}, j^{(0)}) \to p_0(i^{(0)}, j^{(0)})$, as $\varepsilon \to 0$, $i^{(0)}, j^{(0)} \in X_0$, and matrix $P_0 = \left(p_0(i^{(0)}, j^{(0)})\right)$ is irreducible;
2. $p_\varepsilon(i^{(q)}, j^{(q+1)}) = \varepsilon \alpha^{(q)}(i^{(q)}, j^{(q+1)}) + o(\varepsilon)$, $i^{(q)} \in X_q, j^{(q+1)} \in X_{q+1}$, where $\alpha^{(q)}(i^{(q)}, j^{(q+1)})$ is an appropriate transition matrix;
3. $p_\varepsilon(i^{(q)}, f^{(q)}) \to 0$, as $\varepsilon \to 0$, $i^{(q)}, f^{(q)} \in X_q$, $q \geq 1$;
4. $p_\varepsilon(i^{(q)}, f^{(z)}) \equiv 0$, $i^{(q)} \in X_q$, $f^{(z)} \in X_z$, $z - q \geq 2$.

In the sequel the set of states X_q is called the q-th level of the chain, $q = 0, \ldots, m+1$. Let us single out the subset of states

$$\langle \alpha_m \rangle = \bigcup_{q=0}^{m} X_q.$$

Denote by $\{\pi_\varepsilon(i^{(q)}),\ i^{(q)} \in X_q\}$, $q = 1, \ldots, m$ the stationary distribution of a chain with transition matrix

$$\left(\frac{p_\varepsilon(i^{(q)}, j^{(z)})}{1 - \sum\limits_{k^{(m+1)} \in X_{m+1}} p_\varepsilon(i^{(q)}, k^{(m+1)})}\right), \quad i^{(q)} \in X_q,\ j^{(z)} \in X_z,\ q, z \leq m.$$

Furthermore, denote by $g_\varepsilon(\langle \alpha_m \rangle)$ the steady state probability of exit from $\langle \alpha_m \rangle$, that is

$$g_\varepsilon(\langle \alpha_m \rangle) = \sum_{i^{(m)} \in X_m} \pi_\varepsilon(i^{(m)}) \sum_{j^{(m+1)} \in X_{m+1}} p_\varepsilon(i^{(m)}, j^{(m+1)}).$$

Denote by $\{\pi_0(i^{(0)}), i^{(0)} \in X_0\}$ the stationary distribution corresponding to P_0 and let

$$\bar{\pi}_0 = \{\pi_0(i^{(0)}, i^{(0)} \in X_0\}, \quad \bar{\pi}_\varepsilon^{(q)} = \{\pi_\varepsilon(i^{(q)}), i^{(q)} \in X_q\},$$

be row vectors. Finally, let the matrix

$$A^{(q)} = \left(\alpha^{(q)}(i^{(q)}, j^{(q+1)})\right), \quad i^{(q)} \in X_q, \quad j^{(q+1)} \in X_{q+1}, q = 0, \ldots, m$$

defined by condition 2. Conditions 1–4 enable us to compute the main terms of the asymptotic expression for $\bar{\pi}_\varepsilon^{(q)}$ and $g_\varepsilon(\langle \alpha_m \rangle)$. Namely, we obtain

$$(1)\qquad \begin{aligned} \bar{\pi}_{\varepsilon}^{(q)} &= \varepsilon^{q}\bar{\pi}_0 A^{(0)}A^{(1)}\dots A^{(q-1)} + o(\varepsilon^{q}), \quad q=1,\dots,m,\\ g_{\varepsilon}(\langle\alpha_m\rangle) &= \varepsilon^{m+1}\bar{\pi}_0 A^{(0)}A^{(1)}\dots A^{(m)}\underline{1} + o(\varepsilon^{m+1}), \end{aligned}$$

where $\underline{1} = (1,\dots,1)^*$ is a column vector, (c.f., ANISIMOV *et al.* [2], pp. 141–153).

Let $(\eta_\varepsilon(t),\ t \ge 0)$ be a Semi Markov Process (SMP) given by the embedded Markov chain $(X_\varepsilon(k),\ k \ge 0)$ satisfying conditions 1–4. Let the times $\tau_\varepsilon(j^{(s)}, k^{(z)})$– transition times from state $j^{(s)}$ to state $k^{(z)}$– fulfil the condition

$$E\exp\{i\theta\beta_\varepsilon\tau_\varepsilon(j^{(s)},k^{(z)})\} = 1 + a_{j^{(s)}k^{(z)}}(\theta)\varepsilon^{m+1} + o(\varepsilon^{m+1}), \quad (i^2=-1),$$

where β_ε is some normalizing factor. Denote by $\Omega_\varepsilon(m)$ the instant at which the SMP reaches the $(m+1)$-th level for the first time, exit time from $\langle\alpha_m\rangle$, provided $\eta_\varepsilon(0)\in\langle\alpha_m\rangle$. Then we have:

Theorem 1. *(c.f., ANISIMOV et al. [2], pp. 153) If the above 1–4 conditions are satisfied then*

$$\lim_{\varepsilon\to 0} E\exp\{i\theta\beta_\varepsilon\Omega_\varepsilon(m)\} = (1-A(\theta))^{-1},$$

where

$$A(\theta) = \frac{\sum_{j^{(0)},k^{(0)}\in X_0} \pi_0(j^{(0)})p_0(j^{(0)},k^{(0)})a_{j^{(0)}k^{(0)}}(\theta)}{\bar{\pi}_0 A^{(0)}A^{(1)}\dots A^{(m)}\underline{1}}.$$

Corollary 1. *In particular, if $a_{j^{(s)}k^{(z)}}(\theta) = i\theta m_{j^{(s)}k^{(z)}}$ then the limit is an exponentially distributed random variable with mean*

$$\frac{\sum_{j^{(0)},k^{(0)}\in X_0} \pi_0(j^{(0)})p_0(j^{(0)},k^{(0)})m_{j^{(0)}k^{(0)}}}{\bar{\pi}_0 A^{(0)}A^{(1)}\dots A^{(m)}\underline{1}}.$$

3. The Queueing Model

Let us consider the machine interference problem with N heterogeneous machines which are looked after by one operative. The machines are assumed to operate in a random environment governed by an ergodic Markov chain $(\xi_1(t),\ t\ge 0)$ with state space $(1,\dots,r_1)$ and with transition density matrix $(a_{ij},\ i,j=1,\dots,r_1,\ a_{ii} = \sum_{j\ne i} a_{ij})$. Whenever the environmental process is in state i the probability that machine p breaks down in the time interval $(t,t+h)$ is $\lambda_p(i)h + o(h)$. Each machine is immediately repaired if the operative is idle, otherwise a queueing line is formed. The service discipline is First Come-First Served (FCFS). The repair facility is also supposed to operate in a random environment governed by an ergodic Markov chain $(\xi_2(t),\ t \geq 0)$ with state space $(1, \dots, r_2)$ and with transition density matrix $(b_{kq},\ k, q = 1, \dots, r_2,\ b_{kk} = \sum_{q \neq k} b_{kq})$. Whenever the environmental process is in state k and there are s machines stopped, $s = 1, \dots, N$, the probability that the repair of machine p is completed in time interval $(t, t+h)$ is $\mu_p(k, s; \varepsilon)h + o(h)$. After being repaired each machine immediately starts operating. All random variables involved here and the random environments are supposed to be independent of each other.

Let us consider the system under the assumption of "fast" repair, i.e., $\mu_p(k, s; \varepsilon) \to \infty$ as $\varepsilon \to 0$. For simplicity let $\mu_p(k, s; \varepsilon) = \mu_p(k, s)/\varepsilon$, $p = 1, \dots, N$. Denote by $Y_\varepsilon(t)$ the number of stopped machines at time t, and let

$$\Omega_\varepsilon(m) = \inf\{t : t > 0, Y_\varepsilon(t) = m + 1 / Y_\varepsilon(0) \leq m\},$$

that is, the instant at which the number of stopped machines reaches the $(m+1)$–th level for the first time, provided that at the beginning their number is not greater than $m; m = 1, \dots, N-1$.

Denote by $(\pi_i^{(1)},\ i = 1, \dots, r_1)$, $(\pi_k^{(2)},\ k = 1, \dots, r_2)$ the steady-state distributions of the governing Markov chains $(\xi_1(t),\ t \geq 0)$, $(\xi_2(t),\ t \geq 0)$, respectively and let V_N^s be the set of all variations of order s of integers $1, \dots, N$. Now we have:

Theorem 2. *For the system in question under the above assumptions, independently of the initial state, the distribution of the normalized random variable $\varepsilon^m \Omega_\varepsilon(m)$ converges weakly to an exponentially distributed random variable with parameter*

$$\Lambda = \sum_{i=1}^{r_1} \sum_{k=1}^{r_2} \pi_i^{(1)} \pi_k^{(2)} \sum_{(p_1, \dots, p_{m+1}) \in V_N^{m+1}} \lambda_{p_1}^{(i)} \prod_{s=1}^{m} \frac{\lambda_{p_{s+1}}^{(i)}}{\mu_{p_1}(k, s)}.$$

Proof. It is easy to see that the process

$$Z_\varepsilon(t) = (\xi_1(t), \xi_2(t), Y_\varepsilon(t); \gamma_1(t), \dots, \gamma_{Y_\varepsilon(t)}(t))$$

is a multi-dimensional Markov chain with state space

$$\begin{aligned} E = &((i, k, s; p_1, \dots, p_s),\ i = 1, \dots, r_1,\ k = 1, \dots, r_2, \\ &(p_1, \dots, p_s) \in V_N^s,\ s = 0, \dots, N), \end{aligned}$$

where $\gamma_1(t), \dots, \gamma_{Y_\varepsilon(t)}(t)$ denote the indices of failed machines at time t in the order of their breakdowns, and by definition $p_0 = \{0\}$. Furthermore, let

$$\begin{aligned}\langle\alpha_m\rangle =&((i,k,s;p_1,\ldots,p_m),\ i=1,\ldots,r_1,\ k=1,\ldots,r_2,\\ &(p_1,\ldots,p_m)\in V_N^m).\end{aligned}$$

Hence our aim is to determine the distribution of the first exit time of $Z_\varepsilon(t)$ from $\langle\alpha_m\rangle$, provided that $Z_\varepsilon(o)\in\langle\alpha_m\rangle$. It can easily be verified that the transition probabilities for the embedded Markov chain as $\varepsilon\to 0$ are

$$p_\varepsilon[(i,k,0;0),(j,k,0;0)] = \frac{a_{ij}}{a_{ii}+b_{kk}+\sum\limits_{p=1}^{N}\lambda_p(i)},$$

$$p_\varepsilon[(i,k,0;0),(i,q,0;0)] = \frac{b_{kq}}{a_{ii}+b_{kk}+\sum\limits_{p=1}^{N}\lambda_p(i)},$$

$$p_\varepsilon[(i,k,s;p_1,\ldots,p_s),(j,k,s;p_1,\ldots,p_s)] = o(1),$$

$$p_\varepsilon[(i,k,s;p_1,\ldots,p_s),(i,q,s;p_1,\ldots,p_s)] = o(1),\ \text{for } s=1,\ldots,N,$$

$$p_\varepsilon[(i,k,0;0),(i,k,1;p)] = \frac{\lambda_p(i)}{a_{ii}+b_{kk}+\sum\limits_{p=1}^{N}\lambda_p(i)},$$

$$p_\varepsilon[(i,k,s;p_1,\ldots,p_s),(i,k,s+1;p_1,\ldots,p_{s+1})] = \frac{\lambda_{p_{s+1}}(i)\varepsilon}{\mu_{p_1}(k,s)}(1+o(1)),$$
$$\text{for } s=0,\ldots,N-1,$$

$$p_\varepsilon[(i,k,s;p_1,\ldots,p_s),(i,k,s-1;p_2,\ldots,p_s)] = 1-o(1),\quad \text{for } s=1,\ldots,N.$$

This agrees with the conditions 1–4, but here the zero level is the set

$$((i,k,0;0),(i,k,1;p),\ i=1,\ldots,r_1,\ k=1,\ldots,r_2,\ p=1,\ldots,N),$$

while the q-th level is the set

$$\begin{aligned}((i,k,q+1;p_1,\ldots,p_{q+1}),\ i=1,\ldots,r_1,\ k=1,\ldots,r_2,\\ (p_1,\ldots,p_{q+1})\in V_N^{q+1}).\end{aligned}$$

Since the level 0 in the limit forms an essential class, the probabilities $\pi_0(i,k,0;0)$, $\pi_0(i,k,1;p)$, $i=1,\ldots,r_1$, $k=1,\ldots,r_2$, $p=1,\ldots,N$ satisfy the following system of equations

$$(2)\quad \pi_0(j,q,0;0) = \sum_{i\neq j} \pi_0(i,q,0;0)a_{ij}/(a_{ii}+b_{qq}+\sum_{p=1}^{N}\lambda_p(i))$$
$$+\sum_{k\neq q}\pi_0(j,k,0;0)b_{kq}/(a_{jj}+b_{kk}+\sum_{p=1}^{N}\lambda_p(j))+\sum_{p=1}^{N}\pi_0(j,q,1;p)$$

$$(3)\qquad \pi_0(j,q,1;p)=\pi_0(j,q,0;0)\lambda_p(j)/(a_{jj}+b_{qq}+\sum_{s=1}^{N}\lambda_s(j)).$$

It is clear that

$$(4)\qquad \pi_j^{(1)}a_{jj}=\sum_{i\neq j}\pi_i^{(1)}a_{ij},\quad \pi_q^{(2)}b_{qq}=\sum_{k\neq q}\pi_k^{(2)}b_{kq}.$$

It can easily be verified, that the solution of (2),(3) subject to (4) is

$$\pi_0(i,k,0;0)=B\pi_i^{(1)}\pi_k^{(2)}(a_{ii}+b_{kk}+\sum_{p=1}^{N}\lambda_p(i)),$$
$$\pi(i,k,1;p)=B\pi_i^{(1)}\pi_k^{(2)}\lambda_p(i),$$

where B is the normalizing constant, i.e.,

$$1/B=\sum_{i=1}^{r_1}\sum_{k=1}^{r_2}\pi_i^{(1)}\pi_k^{(2)}[a_{ii}+b_{kk}+2\sum_{p=1}^{N}\lambda_p(i)].$$

By using formula (1) it is easy to show that the probability of exit from $\langle\alpha_m\rangle$ is

$$g_\varepsilon(\langle\alpha_m\rangle)=\varepsilon^m B\sum_{i=1}^{r_1}\sum_{k=1}^{r_2}\pi_i^{(1)}\pi_k^{(2)}\sum_{(p_1,\ldots,p_{m+1})\in V_N^{m+1}}$$
$$\lambda_{p_1}(i)\prod_{s=1}^{m}\frac{\lambda_{p_{s+1}}(i)}{\mu_{p_1}(k,s)}(1+o(1)).$$

Taking into account the exponentiality of $\tau_\varepsilon(j,k,s;p_1,\ldots,p_s)$ for fixed θ we have

$$E\exp\{i\varepsilon^m\theta\tau_\varepsilon(j,k,0;0)\}=1+\varepsilon^m\frac{i\theta}{a_{jj}+b_{kk}+\sum_{p=1}^{N}\lambda_p(j)}(1+o(1)),$$

$$E\exp\{i\varepsilon^m\theta\tau_\varepsilon(j,k,s;p_1,\ldots,p_s)\}=1+o(\varepsilon^m),\quad s>0.$$

Notice that $\beta_\varepsilon=\varepsilon^m$ and therefore from Corollary 1 we immediately get the statement that $\varepsilon^m\Omega_\varepsilon(m)$ converges weakly to an exponentially distributed random variable with parameter

$$\Lambda=\sum_{i=1}^{r_1}\sum_{k=1}^{r_2}\pi_i^{(1)}\pi_k^{(2)}\sum_{(p_1,\ldots,p_{m+1})\in V_N^{m+1}}\lambda_{p_1}(i)\prod_{s=1}^{m}\frac{\lambda_{p_{s+1}}(i)}{\mu_{p_1}(k,s)},$$

which completes the proof.

Consequently, the distribution of the time until the number of stopped machines reaches the $(m+1)$-th level for the first time, can be approximated by

$$P(\Omega_\varepsilon(m)>t)=P(\varepsilon^m\Omega_\varepsilon(m)>\varepsilon^m t)\approx\exp(-\varepsilon^m\Lambda t),$$

i.e., $\Omega_\varepsilon(m)$ is asymptotically an exponentially distributed random variable with parameter $\varepsilon^m\Lambda$. In particular, for $m=N-1$, which means that there is no operating machine, we have

$$\Lambda^*=\varepsilon^{N-1}\Lambda=\varepsilon^{N-1}\sum_{i=1}^{r_1}\sum_{k=1}^{r_2}\pi_i^{(i)}\pi_k^{(2)}\sum_{(p_1,\ldots,p_{N-1})\in V_N^{N-1}}\lambda_{p_1}(i)\prod_{s=1}^{N-1}\frac{\lambda_{p_{s+1}}(i)}{\mu_{p_1}(k,s)}$$

(5)

$$=(N-1)!\sum_{i=1}^{r_1}\sum_{k=1}^{r_2}\pi_i^{(1)}\pi_k^{(2)}\sum_{p=1}^{N}\frac{\prod_{s=1}^{N}\lambda_s(i)}{\prod_{s=1}^{N-1}\mu_p(k,s)/\varepsilon}.$$

In homogeneous case, that is when $\lambda_p(i)=\lambda(i)$, $\mu_p(k,s)=\mu(k,s)$, $p=1,\ldots,N$ we get

$$\Lambda^*=N!\sum_{i=1}^{r_1}\sum_{k=1}^{r_2}\pi_i^{(1)}\pi_k^{(2)}\frac{\left(\lambda(i)\right)^N}{\prod_{s=1}^{N-1}\mu(k,s)/\varepsilon}. \tag{6}$$

In particular, when there are no random environments (5) yields

$$\Lambda^*=(N-1)!\sum_{p=1}^{N}\frac{\prod_{s=1}^{N}\lambda_s}{\prod_{s=1}^{N-1}\mu_p(.,s)/\varepsilon}, \tag{7}$$

where $\lambda_s = \lambda_s(i)$, $i = 1, \ldots, r_1$, $\mu_p(.,s) = \mu_p(k,s)$, $k = 1, \ldots, r_2$, (see Sztrik [20]). Hence, by using (5) the steady-state probability Q_W that at least one machine works is

$$Q_W = \frac{\frac{1}{\varepsilon^{N-1}\Lambda}}{\frac{1}{\varepsilon^{N-1}\Lambda} + B_N}, \tag{8}$$

where B_N denotes the mean period of time during which all machines are stopped, that is

$$B_N = \sum_{p=1}^{N} \Big(\sum_{k=1}^{r_2} \frac{1}{\mu_p(k,N)/\varepsilon} \pi_k^{(2)} \Big) \Big(\sum_{i=1}^{r_1} \frac{\lambda_p(i)}{\sum_{s=1}^{N} \lambda_s(i)} \pi_i^{(1)} \Big).$$

In the case when there are no random environments from (8) we obtain

$$Q_W = 1 \Big/ \Big(1 + (N-1)! \sum_{p=1}^{N} \frac{\prod_{s=1}^{N} \lambda_s}{\prod_{s=1}^{N-1} \mu_p(.,s)/\varepsilon} \Big). \tag{9}$$

Finally, for the simplest case we have

$$Q_W = 1 \Big/ \Big(1 + N! \Big(\frac{\lambda}{\mu/\varepsilon} \Big)^N \Big). \tag{10}$$

4. Some Numerical Results and Applications in Textile Winding

In the context of the production department on the factory floor, most manufacturers will seek to establish a constant and optimal environment in which the various processes can be carried out. They will try to avoid the random environment. However, we do not live in the ideal world and variations in the repair rate and the breakdown rate will occur in spite of their best efforts. Machine operatives will feel "below par" with physical or mental problems from time to time and this in turn will affect their work rate. Their attitude to work at the start of a shift will be very different from their attitude just prior to the tea-break, just after the tea-break and again before the end of their shift. Of course, one could argue that the latter changes are more deterministic than random, although variations among workers will tend to make the overall effect more random than it might appear to be at first sight. The use of robots, and there is a marked trend in this direction in many industries, seeks to avoid these effects. The machinery used will suffer from minor faults due to wear and tear. These, although they may not in themselves constitute a breakdown, will have an adverse effect on the stoppage rate of the process. Another reason for variability in the stoppage rate arises from the quality of the raw materials used. This material may have been produced at an earlier stage in the production process, and unless very stringent quality control procedures have been used some variability is inevitable. In the particular case of the textile industry, especially where natural fibres such as wool or cotton are being used, variability between batches of raw yarns is difficult to avoid. Although it is not possible to generalise, because of the great variety of industrial production processes which exist, if the unit of time is taken to be the average repair time, then the average run time between successive stoppages due to yarn breaks of a single machine might be anything from about 20 time units to 100 time units. The idea of "fast" repair would therefore seem to be reasonable. However, the factors mentioned earlier could easily cause deviations of the order of 1050% of these times. We do not underestimate the practical difficulties of modelling these features of real manufacturing processes. The random environment idea would seem to be a first step in the right direction. In this section some numerical examples are given to illustrate the problem in question and the asymptotic results are compared to the classical exact formulae as well as the numerical ones obtained by Gaver *et al.* [7].

Case 1. In this section we illustrate how "good" the asymptotic results are by comparing them to the exact ones. Here $\rho = \frac{\lambda}{\mu/\varepsilon}$ and $P_W = 1 - N!\rho^N P_0$ (from Palm-formula). Using (10) we get the following results.

	N=5		N=10	
ρ	P_W	Q_W	P_W	Q_W
1	0.631901845	8.26446281 E-3	0.632120555	2.75573116 E-7
2^{-1}	0.862385321	0.210526316	0.864663592	2.82107342 E-4
2^{-2}	0.976671851	0.895104895	0.981632201	0.224180395
2^{-3}	0.998245819	0.996351253	0.999588836	0.996631800
2^{-4}	0.999918676	0.999885572	0.999998546	0.999996700
2^{-5}	0.999996963	0.999996424	0.999999998	0.999999997
2^{-6}	1	1	1	1

	N=15		N=20	
ρ	P_W	Q_W	P_W	Q_W
1	0.632120559	7.64716373 E-13	0.632120559	4.11031762 E-19
2^{-1}	0.864664717	2.50582255 E-8	0.864664717	4.30998041 E-13
2^{-2}	0.981684271	8.20434288 E-4	0.981684361	4.51933998 E-7
2^{-3}	0.999661753	0.9641165497	0.999664506	0.321522100
2^{-4}	0.999999759	0.9999988660	0.999999870	0.999997987
2^{-5}	1	1	1	1

	N=25		N=30	
ρ	P_W	Q_W	P_W	Q_W
1	0.632120559	6.44695028 E-26	0.632120559	3.76998763 E-33
2^{-1}	0.864664717	2.16323755 E-18	0.864664717	4.04799339 E-24
2^{-2}	0.981684361	7.25862072 E-11	0.981684361	4.34649981 E-15
2^{-3}	0.999664537	2.42967127 E-3	0.999664537	4.66699685 E-6
2^{-4}	0.999999886	0.9999987764	0.999999887	0.999800486
2^{-5}	1	1	1	1

Table 1

We can see how Q_W depends on N, ρ and how accurate it is. It should be noted that the greater the N the less the ρ for an acceptable approximation.

Case 2. In this section the machines operate in a random environment. We compare the asymptotic result to the numerical one obtained by GAVER *et al.* [7] and show how it depends on the intensities of the governing Markov chain. Here $\hat{Q}_W = 1 - g_\varepsilon(\langle\alpha\rangle)$ and P_W is the steady-state probability that at least one machine works obtained by GAVER *et al.* [7]. By using the notation of (6) we have the following parameters

$$N = 5 \quad m = 4 \quad r_1 = 2 \quad r_2 = 1$$
$$\lambda(1) = 0.12 \quad \mu(1;s)/\varepsilon = 1.00 \quad \pi_1^{(1)} = 2/3 \quad \pi_1^{(2)} = 1$$
$$\lambda(2) = 0.06 \quad \mu(2;s)/\varepsilon = 1.00 \quad \pi_2^{(1)} = 1/3 \quad s = 1,\dots,5$$

By the help of (8) we obtain

a_{11}	a_{22}	Q_W	$1/\Lambda^*$	P_W	$\hat{Q}_W$
50	100	0.99798	494.61	0.99932	0.99997
0.5	1	0.99798	494.61	0.99925	0.99879
0.05	0.1	0.99798	494.61	0.99908	0.99810

Table 2

We can observe that the corresponding probabilities are exact up to almost 3 digits while the mean failure-free operation time is very small compared to GAVER *et al.* [7] where $1/\Lambda \approx 1500$.

Case 3. In this section we approximate the results of GAVER *et al.* [7] by changing the service rate.

$$\mu(1;s)/\varepsilon = 1.30 \quad \mu(2;s)/\varepsilon = 1.30 \quad s = 1,\dots,5$$

the other parameters are the same as Case 2. We have

a_{11}	a_{22}	Q_W	$1/\Lambda^*$	P_W	$\hat{Q}_W$
50	100	0.99946	1412.68	0.99932	0.99999
0.5	1	0.99946	1412.68	0.99925	0.99958
0.05	0.1	0.99946	1412.68	0.99908	0.99934

Table 3

This example illustrates the situation when the repair rate is 1.3. We get almost the same results as GAVER *et al.* [7] but here the formulae are much simpler and we do not need numerical procedures.

Case 4. In this section we deal with the case when there are no random environments, the machines have different failure rate and the same repair rate. P_W denotes the steady-state probability that at least one machine works, found in SZTRIK [18]. By using the notation of (7) we have the following parameters

$$N = 4 \quad m = 3$$
$$\lambda_1 = 1 \quad \lambda_2 = 2 \quad \lambda_3 = 3 \quad \lambda_4 = 4$$
$$\mu_p(.,s) = \mu \qquad s = 1,\dots,4$$

With the aid of (7), (9) we get

μ/ε	$1/\Lambda^*$	Q_W	P_W
1	3.47 E -3	1.73310225 E -3	0.398119122
4	0.22	0.307692308	0.848101266
10	3.47	0.945537065	0.981161695
20	27.77	0.996412914	0.997902220
30	93.75	0.999289394	0.999500250

Table 4

We can see how the asymptotic value approaches the exat value.

Case 5. In this section we consider a general setup assuming that the operative works faster seeing a longer queue. We have the following parameters.

$$N = 4 \quad m = 3 \quad r_1 = 2 \quad r_2 = 1$$

$$\lambda_1(1) = 1 \quad \lambda_2(1) = 2 \quad \lambda_3(1) = 3 \quad \lambda_4(1) = 4$$

$$\lambda_1(2) = 1.5 \quad \lambda_2(2) = 2.5 \quad \lambda_3(2) = 3.5 \quad \lambda_4(2) = 4.5$$

$$\mu_1(1,1) = 30.0 \quad \mu_1(1,2) = 30.1 \quad \mu_1(1,3) = 30.2 \quad \mu_1(1,4) = 30.3$$
$$\mu_2(1,1) = 31.0 \quad \mu_2(1,2) = 31.1 \quad \mu_2(1,3) = 31.2 \quad \mu_2(1,4) = 31.3$$
$$\mu_3(1,1) = 32.0 \quad \mu_3(1,2) = 32.1 \quad \mu_3(1,3) = 32.2 \quad \mu_3(1,4) = 32.3$$
$$\mu_4(1,1) = 33.0 \quad \mu_4(1,2) = 33.1 \quad \mu_4(1,3) = 33.2 \quad \mu_4(1,4) = 33.3$$

$$\mu_1(2,1) = 35.0 \quad \mu_1(2,2) = 35.1 \quad \mu_1(2,3) = 35.2 \quad \mu_1(2,4) = 35.3$$
$$\mu_2(2,1) = 36.0 \quad \mu_2(2,2) = 36.1 \quad \mu_2(2,3) = 36.2 \quad \mu_2(2,4) = 36.3$$
$$\mu_3(2,1) = 37.0 \quad \mu_3(2,2) = 37.1 \quad \mu_3(2,3) = 37.2 \quad \mu_3(2,4) = 37.3$$
$$\mu_4(2,1) = 38.0 \quad \mu_4(2,2) = 38.1 \quad \mu_4(2,3) = 38.2 \quad \mu_4(2,4) = 38.3$$

We show how the system's behaviour depends on the stationary distributions of the corresponding Markov chains

$\pi_1^{(1)}$	$\pi_2^{(1)}$	$\pi_1^{(2)}$	$\pi_2^{(2)}$	$1/\Lambda^*$	Q_W
2/3	1/3	2/3	1/3	54.84	0.998920312
1/2	1/2	1/2	1/2	62.83	0.999079593

Table 5

Case 6. In this section we assume that the repair rates do not depend on the number of stopped machines. The repair rates vary but the remaining parameters are the same as in Case 5.

$$\mu_1(1,.) = 30 \quad \mu_2(1,.) = 31 \quad \mu_3(1,.) = 32 \quad \mu_4(1,.) = 33$$
$$\mu_1(2,.) = 35 \quad \mu_2(2,.) = 36 \quad \mu_3(2,.) = 37 \quad \mu_4(2,.) = 38$$

We show how the system's behaviour depends on the stationary distributions of the corresponding Markov chains

$\pi_1^{(1)}$	$\pi_2^{(1)}$	$\pi_1^{(2)}$	$\pi_2^{(2)}$	$1/\Lambda^*$	Q_W
2/3	1/3	2/3	1/3	54.32	0.998900145
1/2	1/2	1/2	1/2	62.24	0.999062589

Table 6

We can observe that this case $1/\Lambda^*$ and Q_W are slightly less than the corresponding ones in Table 4, as we have expected.

5. Concluding remarks

It should be noted that similar weak convergence arguments can be found in KEILSON [11] for Markov chains. It requires, however, the steady-state distribution of the underlying process, which in this case can be difficult to obtain due to the great number of states.

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