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# The property of smallness up to a complemented Banach subspace

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**Abstract.** This article investigates locally convex spaces which satisfy the property of smallness up to a complemented Banach subspace, the $SCBS$ property, which was introduced by Djakov, Terzioğlu, Yurdakul and Zahariuta. It is proved that a bounded perturbation of an automorphism on a complete barrelled locally convex space with the $SCBS$ is stable up to a Banach subspace. New examples are given, and the relation of the $SCBS$ with the quasinormability is analyzed. It is proved that the Fréchet space $l_{p+}$ does not satisfy the $SCBS$, therefore this property is not inherited by subspaces or separated quotients.

## 1. Introduction

Our terminology for locally convex spaces is standard and we refer the reader to [11] or [8]. For a locally convex space $X$, $U(X)$ denotes a basis of absolutely convex neighborhoods of the origin in $X$ and for $U \in U(X)$, $p_U$ is the gauge of $U$. A linear operator $T$ from a locally convex space $X$ into another $Y$ is bounded if $T(U)$ is a bounded subset of $Y$ for some $U \in U(X)$.

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We say that a pair $(X, Y)$ has *the bounded factorization property* and write $(X, Y) \in \mathcal{BF}$ if each continuous linear operator from $X$ into $X$ that factors over $Y$ is bounded.

By Grothendieck ([7], p. 107), a locally convex space $X$ is said to be *quasi-normable* if for each $U \in U(X)$ there is $V \in U(X)$ so that for every $\epsilon > 0$ there exists a bounded subset $A$ of $X$ such that $V \subset A + \epsilon U$.

Following [5], we say that a locally convex space $X$ satisfies *the property of smallness up to a complemented Banach subspace*, abbreviated from now on as the $SCBS$ property, if for each bounded subset $A$ of $X$, for each $U \in U(X)$, and for every $\epsilon > 0$, there are complementary subspaces $B$ and $E$ of $X$ such that $B$ is a Banach space and $A \subset B + \epsilon U \cap E$.

In [5], it was proved that all Banach-valued $l_p$-Köthe spaces have the $SCBS$ property and the bounded perturbation of an automorphism on an $l_p$-Köthe space is stable up to some Banach basic subspace. This was essential there to get a modification of the generalized Douady lemma [16].

We still have no complete characterization of Fréchet or $DF$-spaces with the $SCBS$ property. In this work we analize the $SCBS$ property, and show that more general form of Köthe spaces, say, $l$-Köthe spaces, some quasi-normable Fréchet spaces and the strong duals of some asymptotically normable Fréchet spaces have this property. We also get a characterization of quasi-normable $l$-Köthe spaces in terms of their basic Banach subspaces (see [2], [4]). Modifying Theorem 1 in [5] we obtain, as an extension of Proposition 3 there, that any infinite dimensional complemented Banach subspace of a $c_0$-Köthe space is basic. This result may be considered as a partial answer to the well-known Pelczynski problem: Does a complemented subspace of a space with basis have a basis? Moreover, in this case we confirm the conjecture of Bessaga [1] that each complemented subspace of a Köthe space is isomorphic to a basic subspace.

## 2. Results

Note that a locally convex direct sum of Banach spaces has the $SCBS$ property trivially, since every bounded subset of it is contained in a finite sum of its components.

We follow Dragilev [6] to define $l$-Köthe spaces. First, we need:

*Definition.* Let $(X, \|\,.\,\|)$ be a Banach space with a basis $(x_n)$ and let $(x'_n)$ denote the sequence of coefficient functionals. The norm $\|\,.\,\|$ of $X$ is called *monotonous* if the following implication holds: for any $x, y \in X$, $|x'_n(x)| \leq |x'_n(y)|$ $(\forall n \in \mathbb{N})$ implies $\|x\| \leq \|y\|$ (see also [9]).

It is known that every Banach space with an unconditional basis has a monotonous norm which is equivalent to its original norm. Indeed, it is enough to put

$$\|x\| = \sup_{|\alpha_k| \leq 1} \left| \sum_k x'_k(x) \alpha_k x_k \right|$$

where $|\,.\,|$ denotes the original norm.

Throughout this work we denote by $l$ a Banach sequence space in which the canonical system $(e_n)$ is an unconditional basis, with a monotonous norm $\|\,.\,\|$ satisfying $\|e_n\| = 1$ for each $n \in \mathbb{N}$. Let $\Lambda$ be the class of all such spaces; in particular, $l_p$ and $c_0$ are in this class.

*Definition.* Let $l \in \Lambda$ and $\|\,.\,\|$ be a norm (monotonous in this work) in $l$. If $(a_{n,p})$ is a Köthe matrix, then *the $l$-Köthe space* $\lambda^l(a_{n,p})$ is the locally convex space of all sequences $(t_n)$, such that $(t_n a_{n,p}) \in l$ for any $p \in \mathbb{N}$, with the topology generated by the seminorms

$$|(t_n)|_p = \|(t_n a_{n,p})\|, \quad p \in \mathbb{N}.$$

Note that $|e_n|_p = a_{n,p}\|e_n\| = a_{n,p}$.

**Proposition 1.** *Every $l$-Köthe space has the SCBS property. In particular, every $c_0$-Köthe space has the SCBS property.*

Proof. Let $X = \lambda^l(a_{n,k})$ be a $l$-Köthe space. Let $A$ be a bounded subset of $X$. Without loss of generality we may assume that

$$A = \left\{ x \in X : |x|_k = \|(x_i a_{i,k})\| = \sup_{|\alpha_i| \leq 1} \left| \sum_i \alpha_i x_i a_{i,k} e_i \right| \leq c_k \forall k \right\}$$

where $(c_k)$ is a sequence of real numbers increasing to $\infty$. Choose $(c_k)$ so that $(\frac{a_{i,k}}{c_k})_k$ tends to zero for all i. For this purpose, it is enough to replace those $c_k$'s with $k \geq i$ by bigger ones, for example by $k \max_{1 \leq i \leq k} a_{i,k}$. Here $\|\,.\,\|$, $|\,.\,|$ respectively denote the monotonous and the original norm in $l$.

We set $\gamma_i = \sum_k \frac{a_{i,k}}{2^k c_k}$. Then, for any $x \in A$

$$\left|\sum_i \alpha_i \gamma_i x_i e_i\right| = \left|\sum_i \alpha_i x_i \left(\sum_k \frac{a_{i,k}}{2^k c_k}\right) e_i\right|$$
$$= \left|\sum_k \frac{1}{2^k}\left(\sum_i \alpha_i \frac{a_{i,k}}{c_k} x_i e_i\right)\right| \leq 1$$

or $\|(x_i \gamma_i)\| \leq 1$ holds.

Fix $\epsilon > 0$, $k_0 \in \mathbb{N}$ and set $B = [e_i : \epsilon\gamma_i \leq a_{i,k_0}]$, $E = [e_i : \epsilon\gamma_i > a_{i,k_0}]$ where the square brackets denote the closed linear span of the corresponding vectors. For $x \in B$ and $k \in \mathbb{N}$, since

$$|x_i a_{i,k_0}| \geq |x_i \gamma_i \epsilon| \geq \left|x_i \epsilon \frac{a_{i,k}}{2^k c_k}\right|$$

we have

$$\frac{1}{\epsilon} 2^k c_k \|(x_i a_{i,k_0})\| \geq \frac{1}{\epsilon} 2^k c_k \|(x_i \gamma_i \epsilon)\|$$
$$\geq \frac{1}{\epsilon} 2^k c_k \left\|\left(x_i \epsilon \frac{a_{i,k}}{2^k c_k}\right)\right\| = |x|_k,$$

that is $B$ is a basic Banach subspace.

If $x \in A \cap E$, since $|x_i a_{i,k_0}| \leq |x_i \gamma_i \epsilon|$ we have

$$|x|_{k_0} = \|(x_i a_{i,k_0})\| < \|(\epsilon x_i \gamma_i)\| \leq \epsilon$$

which means that $X$ has the *SCBS* property. □

The following condition should be compared with the weak quasi-normability condition of Grothendieck (see for example [4]).

*Definition.* A Fréchet space $X$ satisfies *the condition* $(QN)$ if there is a map $\pi \in \mathbb{N}^{\mathbb{N}}$ such that for every $p \in \mathbb{N}$ and for every $\epsilon > 0$ there is a complemented Banach subspace $B$ of $X$ on which the topology given by $p$-th norm and the original topology coincide and

$$U_{\pi(p)} \subset B + \epsilon U_p \qquad (1)$$

holds.

**Proposition 2.** *Every quasi-normable $l$-Köthe space $X$ satisfies the condition $(QN)$.*

Proof. $|\,.\,|_p^*$ denotes the gauge functional of $U_p^\circ$ i.e.

$$|x'|_p^* := \sup\left\{|x'(x)| : x \in X, |x|_p \le 1\right\},\ x' \in X'.$$

Assume $a_{j,p} \neq 0$. Since $\|\,.\,\|$ is monotonous, for any $x \in U_p$ one has

$$|e_j'(x)| = |x_j| = \|x_j e_j\| \le \left\|\left(\frac{1}{a_{j,p}} x_i a_{i,p}\right)_i\right\| = \frac{1}{a_{j,p}}|x|_p \le \frac{1}{a_{j,p}}.$$

Since the coefficient functionals $e_i'$ are continuous, we have

$$1 = |e_i'(e_i)| \le |e_i'|_p^* a_{i,p}$$

from which $|e_j'|_p^* = \frac{1}{a_{j,p}}$ follows.

By [10], Theorem 7, since $X$ is quasi-normable, there exists a strictly increasing function $\phi$ such that for all $p \in \mathbb{N}$ there is $q \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ there exists $c > 0$ such that

$$|e_i'|_q^* \le c\phi\left(\frac{|e_i'|_p^*}{|e_i'|_q^*}\right)|e_i'|_k^*$$

or $\frac{a_{i,k}}{a_{i,q}} \le c\phi(\frac{a_{i,q}}{a_{i,p}})$ holds for each $i$.

Fix any $p \in \mathbb{N}$. Then there exists $q = \pi(p) \in \mathbb{N}$ such that the above property holds. For the given $\epsilon > 0$, we set

$$I_1 = \{i : \epsilon a_{i,q} \le a_{i,p}\}, \quad B = [e_i : i \in I_1],$$

where the square bracket denotes the closed linear span of the corresponding vectors. Then obviously $\epsilon|x|_q = \epsilon\|(x_i a_{i,q})\| \le \|(x_i a_{i,p})\| = |x|_p$ for any $x \in B$, since $|\epsilon x_i a_{i,q}| \le |x_i a_{i,p}|$ and the norm is monotonous. On the other hand for any $k > q$

$$\frac{a_{i,k}}{a_{i,q}} \le c\phi\left(\frac{a_{i,q}}{a_{i,p}}\right) \le c\phi\left(\frac{1}{\epsilon}\right)$$

for every $i \in I_1$, so we have $|x|_k \le c\phi(\frac{1}{\epsilon})|x|_q$ for any $x \in B$. Hence $B$ is a Banach space whose topology is given by the $p$-th norm.

Let $E = [e_i : i \notin I_1]$; then $X = B \oplus E$ and $\epsilon|x|_q > |x|_p$ for $x \in E$, therefore the condition $(QN)$ holds. □

Further, the condition $(QN)$ characterizes quasinormability in terms of basic Banach subspaces in $l$-Köthe spaces (see [2], Proposition 3.3 and [4], Corollary 8). Indeed:

**Proposition 3.** *If $X$ satisfies the condition $(QN)$ then $X$ is a quasi-normable Fréchet space.*

Proof. Since $\pi(p) > p$, $U_{\pi(p)} \subset U_p$ and condition 1 implies that $U_{\pi(p)} \subset A + \epsilon U_p$ where $A$ is a ball in $B$ with radius $c(1+\epsilon)$. In fact, when $x \in U_{\pi(p)}$ we can write $x = b + \epsilon u$ for some $b \in B$ and $u \in U_p$. Let $|\,.\,|_B$ denote the norm of $B$. Then $|b|_B \leq c|b|_p \leq c(|x|_p + |\epsilon u|_p) \leq c(1+\epsilon)$ since $p$-th norm and the space $X$ induce the same topology on the Banach space $B$. Hence $X$ is a quasi-normable Fréchet space. □

**Proposition 4.** *Every Fréchet space $X$ with the condition $(QN)$ has the $SCBS$ property.*

Proof. Let $A$ be a bounded subset in $X$ and $\pi(p)$ be as in condition 1. Find $\rho > 0$ such that $A \subset \rho U_{\pi(p)}$. Then by assumption it follows that $A \subset \rho U_{\pi(p)} \subset B + \rho\epsilon U_p \cap E$. □

*Definition.* A Fréchet space $X$ is said to satisfy *the condition* $(AN)$ if for every $q \in \mathbb{N}$ there is $r \in \mathbb{N}$ so that for each $\epsilon > 0$ there exist two subspaces $B$ and $E$ of $X$ ($B$ is Banach and $X = B \oplus E$) such that

$$U_q^\circ \subset B' + \epsilon\, U_r^\circ \cap E'. \tag{2}$$

Here $B'$ denotes the dual of $B$.

**Proposition 5.** *If $X$ is a Fréchet space satisfying the condition $(AN)$, then $X$ is an asymptotically normable Fréchet space.*

Proof. Suppose $X$ has the condition $(AN)$. Since $U_q^\circ \subset U_r^\circ$ for $r > q$, condition 2 implies that $U_q^\circ \subset B' \cap (1+\epsilon)U_r^\circ + \epsilon U_r^\circ$. Since $U_r^\circ$ is weakly relatively compact, $U_r^\circ$ is strongly bounded. Hence $B' \cap (1+\epsilon)U_r^\circ$ is strongly bounded or bounded in the norm topology of $B'$. So $U_q^\circ \subset MB_{B'} + \epsilon\, U_r^\circ$ holds for some $M > 0$, where $B_{B'}$ denotes the ball of $B'$. Of course this implies that $X$ has $DN_\phi$ (in polarized form) i.e. $X$ is asymptotically normable [13]. □

**Proposition 6.** *The strong dual of a Fréchet space with the condition $(AN)$ has the $SCBS$ property.*

PROOF. Let $X$ be a Fréchet space with the condition $(AN)$ and let $A'$ be a bounded subset of $X'$. Then there exists a $q > 0$ such that $A' \subset U_q^\circ$. By assumption, for this $q$, we find an $r > 0$ such that condition 2 holds. Also for an arbitrary zero-neighborhood $C_\alpha^\circ$ of $X'$ (where $C_\alpha$ is a bounded subset of $X$) there exists $s_\alpha > 0$ such that $C_\alpha \subset s_\alpha U_r$ and thus $U_r^\circ \subset s_\alpha C_\alpha^\circ$. Now apply condition 2 for $\frac{\epsilon}{s_\alpha}$ to obtain $B'$ and $E'$ depending on $\epsilon$ and $\alpha$ such that $U_q^\circ \subset B' + \frac{\epsilon}{s_\alpha} U_r^\circ \cap E'$ or $A' \subset U_q^\circ \subset B' + \epsilon\, C_\alpha^\circ \cap E'$. Hence $X'$ has the $SCBS$ property. □

The following example shows that the $SCBS$ property is not a necessary condition for the stability of a bounded perturbation of an automorphism up to some Banach subspace. Moreover it is not inherited by subspaces or quotients:

*Example* [12]. For $1 \leq p < \infty$, let $l_{p^+} = \cap_{q>p}\, l_q = \cap_k l_{p_k}$, where $p_k \downarrow p$ (projective limit of $l_p$ spaces). The topology of the Fréchet space $l_{p^+}$ may be then represented by means of the $p_k$-norms:

$$\|(x_n)\|_k = \left( \sum_n |x_n|^{p_k}| \right)^{\frac{1}{p_k}}.$$

This space has the following properties:

(i) It is a reflexive quasi-normable Fréchet space.

(ii) Since $l_p \subset l_{p^+} \subset l_q$ for all $q > p$, with continuous inclusions that are not compact, $l_{p^+}$ is not a Montel space.

(iii) Since the canonical inclusions from $l_{p_{k+1}}$ into $l_{p_k}$ are strictly singular and strictly cosingular, $l_{p^+}$ can have no infinite dimensional Banach subspace or quotient.

Since the canonical linking maps are strictly singular, we observe that any linear bounded operator $T$ from $l_{p^+}$ to $l_{p^+}$ is also super strictly singular and hence $I + T$ is Frédholm ([15], [14]).

If $l_{p^+}$ had the $SCBS$ property, then for a given bounded subset $A$, $k > 0$ and $\epsilon > 0$, by assumption and (iii), there would exist a finite dimensional subspace $B$ of $l_{p^+}$ such that:

$$A \subset B + \epsilon U_k,$$

that is $A$ is precompact. Hence $l_{p^+}$ would be Montel, which contradicts (ii). Therefore, $l_{p^+}$ does not have the $SCBS$ property and consequently it can not satisfy the condition $(QN)$.

**Proposition 7.** *The $SCBS$ property does not pass to quotients.*

PROOF. $l_{p^+}$ is a quasi-normable Fréchet space, and thus by MEISE–VOGT's characterization [10] of quasinormability in Fréchet spaces, it is isomorphic to a quotient of an $l_1$-valued generalized Köthe space which, by [5], has the $SCBS$ property, but $l_{p^+}$ does not. $\square$

**Proposition 8.** *The $SCBS$ property does not pass to subspaces.*

PROOF. $l_{p^+}$ is a countable projective limit of $l_p$-spaces and thus by Remark 24.5 in [11], it is isomorphic to a subspace of some countable product of Banach spaces. This product can be understood as a Banach-valued Köthe space which, by [5], has the $SCBS$ property. However, $l_{p^+}$ does not have. $\square$

One can follow the steps of the proof of Theorem 1 in [5] to get the following:

**Theorem 9.** *If $X$ is a complete barrelled locally convex space with the $SCBS$ property and $T$ is a linear bounded (respectively, compact) operator on $X$ into $X$, then there exist complementary subspaces $B$ and $E$ of $X$ such that:*

(i) *$B$ is a Banach (respectively, finite dimensional) space; and*

(ii) *if $\pi_E$ and $i_E$ are the canonical projection onto $E$ and embedding into $X$, respectively, then $1_E + \pi_E T i_E$ is an automorphism on $E$.*

PROOF. Since $T$ is a bounded operator there exists $U_0 \in U(X)$ such that $T(U_0)$ is a bounded set in $X$, therefore

$$\forall U \in U(X)\ \exists C_U > 0\ :\ p_U(Tx) \leq C_U p_{U_0}(x).$$

Since $X$ has the $SCBS$ property there exist complementary subspaces $B$ and $E$ of $X$ such that $B$ is a Banach (respectively finite dimensional) space and $T(U_0) \subset B + \frac{1}{2}U_0 \cap E$. Therefore, setting $T_1 = \pi_E T i_E : E \to E$, we obtain that

$$p_{U_0}(T_1 x) \leq \frac{1}{2} p_{U_0}(x) \qquad \forall\, x \in E.$$

Now it is easy to see that the operator $1_E - T_1$ is an automorphism of $E$. Indeed, consider for any $x \in E$ the series

$$Sx = x + T_1 x + T_1^2 x + \ldots + T_1^m x + \ldots . \tag{3}$$

It is convergent in $E$ because, for any $U \in U(X)$, we have

$$p_U(T_1^m x) \le C_U p_{U_0}(T_1^{m-1} x) \le C_U \left(\frac{1}{2}\right)^{m-1} p_{U_0}(x), \quad m = 1, 2, \ldots ,$$

so by Banach–Steinhaus theorem, since $X$ is barrelled and complete, the formula (3) defines a linear continuous operator $S : E \to E$.
Since $(1_E - T_1)Sx = S(1_E - T_1)x = x$, the operator $S$ is inverse to the operator $1_E - T_1$. □

Exactly in the same way as in ([5], Theorem 2), Theorem 9 enables us to have the following modification of the generalized Douady lemma in ([16], Section 6):

**Theorem 10.** *Suppose $X_1$ is a complete barrelled locally convex space with the $SCBS$ property and $X_2$, $Y_1$, $Y_2$ are topological vector spaces. If $X_1 \times X_2 \simeq Y_1 \times Y_2$ and $(X_1, Y_2) \in \mathcal{BF}$, then there exists complementary subspaces $E$ and $B$ in $X_1$ and complementary subspaces of $G$ and $F$ in $Y_1$ such that $B$ is a Banach space, $F \simeq E$ and $B \times X_2 \simeq G \times Y_2$.*

*If, in addition, $(Y_1, X_2) \in \mathcal{BF}$, then $G$ is a Banach space.*

**Proposition 11.** *Since the space $\phi$ of finite sequences with the direct sum topology is a complete barrelled $DF$ space, then in the above theorem $X_1$ can be taken to be $\phi$ and $Y_2$ to be either a Fréchet space or a locally convex space with the individual countable boundedness condition (see [3]).*

Since, by Proposition 1, $c_0$-Köthe spaces have the $SCBS$ property, we get:

**Proposition 12.** *Let $X$ be a $c_0$-Köthe space, and $F, G$ be complementary subspaces in $X$, i.e. $X = F \oplus G$. If $G$ is an infinite dimensional Banach space then $G \simeq c_0$, and moreover, $F$ and $G$ are isomorphic to some basic subspaces of $X$.*

Proof. We have $X \times \{0\} \simeq F \times G$ and by Theorem 10 there exist complementary basic subspaces $E$ and $B$ in $X$ and complementary subspaces

$F_1$ and $G_1$ in $F$ such that $B$ is a Banach space and

$$F_1 \simeq E, \qquad B \simeq G_1 \times G.$$

Since every infinite dimensional basic Banach subspace of a $c_0$-Köthe space is isomorphic to $c_0$ we obtain that $B \simeq c_0$. On the other hand, each infinite dimensional complemented subspace of $c_0$ is isomorphic to $c_0$ (see [9]), so $G$ is isomorphic to $c_0$. Finally, since $B \simeq c_0$, its complemented subspace $G_1$ is isomorphic to some basic subspace of $B$ and $F \simeq E \oplus G_1$ is isomorphic to some basic subspace of $X$. □

It would be interesting to have this result for $l$-Köthe spaces (see, for example [6], Proposition 2.2.1).

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