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Transcendence measures for the values of generalized Mahler functions in arbitrary characteristic

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I. Introduction and statement of results

Let $f(z) \in \boldsymbol{C}[[z]]$ be a power series at $z = 0$ having algebraic coefficients. Suppose that $f(z)$ converges in a neighborhood U of the origin and that it is transcendental over $\boldsymbol{C}(z)$. The following can be seen as the basic problem of transcendence theory:

(*) Let $\alpha \in U$ be algebraic. Determine whether $f(\alpha)$ is transcendental or algebraic.

Thus one wants to know to what extent the function-theoretic transcendence of $f(z)$ already implies the number-theoretic transcendence of $f(\alpha)$. Since there exist transcendental entire functions with rational coefficients and $f(\alpha)$ algebraic for all algebraic $\alpha \neq 0$ (cf. [Ma], p. 48), it is not possible to give a general solution to this problem, at least not in the complex case.

Nevertheless, in many cases a complete answer can be given if further restrictions are imposed on the function $f(z)$. Let us mention only Lindemann's result on the transcendence of e^{α} for algebraic $\alpha \neq 0$ or Mahler's result that the Fredholm series

$$f_d(z) = \sum_{h \geq 0} z^{d^h} \quad (d \in \boldsymbol{Z},\ d \geq 2)$$

takes on transcendental values at all algebraic points α with $0 < |\alpha| < 1$.

It is also possible to study problem (*) in the function field case. Let $f(z) \in F[[z]]$, where $F = \boldsymbol{F}_q(x)$ is the function field over the field of q

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elements. Suppose that $f(z)$ converges in a neighborhood U of $z = 0$ and that it is transcendental over $F(z)$. Here (as in the classical case $\boldsymbol{F} = \boldsymbol{C}$) a general solution to the problem $(*)$ is impossible, but, as Allouche observed, an answer can be given if the power series coefficients of f belong to the field $\boldsymbol{F}_q$.

Proposition A (ALLOUCHE [A]). *Let $f(z) \in \boldsymbol{F}_q[[z]]$ be transcendental over $\boldsymbol{F}_q(z)$. Let $\alpha \in U_1 \setminus \{0\}$ be algebraic over $\boldsymbol{F}_q(x)$. Then $f(\alpha)$ is transcendental over $\boldsymbol{F}_q(x)$.*

By U_1 we denote the set of all Laurent series g in the variable x^{-1} with the property $xg(x^{-1}) \in \boldsymbol{F}_q[[x^{-1}]]$.

ALLOUCHE's original result (cf. [A], Ch. IV), which is slightly more general than the one stated here, leads to a generalization of an earlier result of Wade. In [Wa] WADE showed a transcendence result for the values of the above introduced Fredholm series $f_d(z)$ (now interpreted as elements of $\boldsymbol{F}_q[[z]]$.)

Once $(*)$ is solved for some specific function $f(z)$ one might ask whether there is also a relation between "transcendence measures" for the function $f(z)$, which are usually called zero estimates, and transcendence measures for the transcendental values $f(\alpha)$. First let us consider the classical situation $f(z) \in \boldsymbol{C}[[z]]$. Similarly as for the purely qualitative problem $(*)$ it is impossible to make any general assertion. Nevertheless, one should note that almost all "sharp" transcendence measures for a function value $f(\alpha)$ rely on a suitable zero estimate for the corresponding function $f(z)$. Later we will give examples for this connection.

Again, the situation is different if we restrict our attention to the special setting studied by Allouche. Before we can state a quantitative version of Proposition A we have to introduce some notation.

For $a, b \in \boldsymbol{F}_q[x] \setminus \{0\}$, let $|a/b| = q^{\deg a - \deg b}$ and let $|0| = 0$. Let $\mathcal{R}$ be the (unique) completion of $\boldsymbol{F}_q(x)$ with respect to the valuation $|\,.\,|$. We use the same symbol for the extension of $|\,.\,|$ to $\mathcal{R}$. We have $x^{-1}\boldsymbol{F}_q[[x^{-1}]] = \{\omega \in \mathcal{R} \mid |\omega| < 1\} = U_1$, that is, U_1 is the analog of the unit intervall.

Furthermore, we write $\mathcal{Z} = \boldsymbol{F}_q[x]$ and $\mathcal{Q} = \boldsymbol{F}_q(x)$. For $P = a_0 + a_1 y + \cdots + a_d y^d \in \mathcal{Z}[y]$ we denote by $d(P)$ the degree of P with respect to y and by $H(P)$ the height of P, that is, $\max_\nu |a_\nu|$. Similarly we define the degree and the height of polynomials with complex coefficients but, of course, based on the usual absolute value. Let Q be a rational function. We define its degree $d(Q)$ by $d(Q) = \max\{d(P_1), d(P_2)\}$ if P_1 and P_2 are coprime polynomials with $Q = P_1/P_2$.

Let F be a field. For a formal power series $g(z) = \sum_{\nu \geq 0} \gamma_\nu z^\nu \in F[[z]]$, we denote by ord g the smallest ν with $\gamma_\nu \neq 0$. We have the following quantitative version of Proposition A.

Theorem 1. *Let $f(z) \in \boldsymbol{F}_q[[z]]$ and suppose that there exists a function $\mu : [0,\infty)^2 \to [0,\infty)$ which is not decreasing with respect to each of its two variables and for which the following property holds:*

For all polynomials $R(z,y) \in \boldsymbol{F}_q[z,y] \setminus \{0\}$ with $\deg_z R \le M$ and $\deg_y R \le N$ we have

$$\operatorname{ord} R(z, f(z)) \le \mu(N, M). \tag{1}$$

Let $\alpha \in U_1 \setminus \{0\}$ be algebraic over $\mathcal{Q}$. Then there exist constants C_1, C_2, $C_3 > 0$ depending only on α and f such that for all polynomials $P(y) \in \mathcal{Z}[y] \setminus \{0\}$ with $H(P) \le H$ and $d(P) \le n$ the following inequality holds

$$\log |P(f(\alpha))| \ge -C_1 \mu(C_2 n, C_3 \log H). \tag{2}$$

This result enables us to deduce transcendence measures (2) for function field transcendental numbers of the type $f(\alpha)$ provided we have zero estimates (1) for the function f. The following theorem yields such zero estimates for functions satisfying functional equations of the type

$$f(Tz) = Q(z, f(z)), \tag{3}$$

where $Tz = T(z)$ is a rational function of z and where $Q(z,y)$ is a rational function of z and y. If we take $Tz = z^d$ with some $d \ge 2$ we have the functional equation usually considered in the one variable case of Mahler's method for transcendence. Here zero estimates were given by GALOCHKIN [Ga], MILLER [Mi], and by NISHIOKA and TÖPFER [NT]. These results are all included in

Theorem 2. *Let $\boldsymbol{F}$ be a field (of arbitrary characteristic). Let $f(z) \in \boldsymbol{F}[[z]]$ be transcendental over $\boldsymbol{F}(z)$. Let $Tz = T(z) \in \boldsymbol{F}(z)$ have a zero of order $d \ge 2$ at $z = 0$. Suppose that $Q(z,y) \in \boldsymbol{F}(z,y)$ satisfies (3) and at least one of the conditions*

$$\text{(i) } \deg_y Q < d \quad \text{or} \quad \text{(ii) } \operatorname{char} \boldsymbol{F} = 0.$$

Then there exists a constant C depending only on f such that for all $P(z,y) \in \boldsymbol{F}[z,y]$ with $\deg_z P \le M$ and $\deg_y P \le N$ the following inequality holds

$$\operatorname{ord} P(z, f(z)) \le CN(N+M).$$

A combination of Theorems 1 and 2 yields the following

Corollary 1. *Let $p = \operatorname{char} \boldsymbol{F}_q$ and let $d \ge 2$ be a positive integer which is not a power of p. Let $\alpha \in U_1 \setminus \{0\}$ be algebraic over $\mathcal{Q}$. Suppose that ω is one of the following numbers*

$$\text{(a) } \sum_{h \ge 0} \left(\frac{\alpha^{d^h}}{1 - \alpha^{d^h}} \right)^m, \quad \text{(b) } \sum_{h \ge 0} \alpha^{d^h}, \quad \text{(c) } \prod_{h \ge 0} (1 - \alpha^{d^h}).$$

Then there exists a constant $C > 0$ depending only on α, d, (and in case (a) also on m) such that for all polynomials $P \in \mathcal{Z}[y] \setminus \{0\}$ with $H(P) \leq H$ and $d(P) \leq n$ the following estimate holds

$$\log |P(\omega)| \leq -Cn(n + \log H)\,. \tag{4}$$

Remarks. 1) The transcendence of the numbers of type (a) and type (c) was shown in [Be3], Cor. 3. The transcendence of the values of the Fredholm series (b) was shown by WADE [Wa].

2) Mahler's classification of transcendental numbers according to their respective transcendence measures can be carried over to the function field case (cf. [Bu], p. 412 or [Sp], Chapt. 3). It was shown by SPRINDZUK that almost all numbers (in the sense of a suitable Haar measure) belong to the class of S–numbers. From (4) it is immediately clear that these numbers ω are S–numbers. As far as we know this is the first example of S–numbers for the function field case.

Another way to compare transcendence measures is based on the so-called transcendence type (cf. [Wl1], p. 100). A transcendental number is said to have a transcendence type not exceeding $\tau > 0$, if there is a constant $C > 0$ such that for all $P(y) \in \mathcal{Z}[y] \setminus \{0\}$

$$\log |P(\omega)| > -Ct(P)^{\tau}\,,$$

where $t(P) = d(P) + \log H(P)$. Since it can be shown that one always has $\tau \geq 2$ (cf. [Ge], Th. 10.6), Corollary 1 yields the best bound for the transcendence type of the numbers in consideration. With respect to the classical situation one should note that there are no explicitly given complex numbers whose transcendence type is known to be 2. The result for π shown by WALDSCHMIDT [Wl2], which says that π has a transcendence type not exceeding $2 + \varepsilon$, is almost best possible.

Our zero estimate, Theorem 2, which holds in arbitrary characteristic, can also be applied deduce transcendence measures for the values of generalized Mahler functions with complex coefficients. As an example of such an application we give a quantitative version of the transcendence result established as Theorem 2 in [BB].

Definition. Let $p(z) = a_0 + a_1 z + \cdots + a_d z^d$ and $q(z) = b_0 + b_1 z + \cdots + b_d z^d$ be polynomials of degree $d \geq 2$. Let $\lambda \in \boldsymbol{C}$ satisfy $\lambda^{d-1} = a_d / b_d$. The unique function f defined and analytic in a neighborhood of ∞ with $f(z) \sim \lambda z$ as $z \to \infty$ and

$$f(p(z)) = q(f(z)) \tag{5}$$

is called a *Böttcher function* with respect to p and q.

The polynomials p and q are said to be *linearly conjugate* if there exists a linear Böttcher function $f(z) = \alpha z + \beta, \;\; \alpha, \beta \in \boldsymbol{C}, \;\; \alpha \neq 0$ with respect to p and q.

Remarks. 1) A proof for the existence and uniqueness of Böttcher functions can be found in [Ba], Th. 6.10.1, or in [St], § 3.3.

2) It was shown in [BB], Theorem 1, that an algebraic Böttcher function with respect to p and q is already linear or both p and q are linearly conjugate to M_d, T_d, or $-T_d$, where $M_d(z) = z^d$ and where T_d is the d–th Chebychev polynomial.

Corollary 2. *Let p and q be polynomials of degree $d \geq 2$ having algebraic coefficients. Suppose that at least one of them is not linearly conjugate to M_d, T_d, or $-T_d$. Let f be a nonlinear Böttcher function with respect to p and q and suppose that f is defined and analytic in a punctured neighbourhood G of ∞ such that $p(G) \subset G$ and $p_m|_G \to \infty$ as $m \to \infty$, where p_m denotes the m–th iterate of p. Let $\alpha \in G$ be an algebraic number.*

There exists a constant $C > 0$ depending only on α and f such that for all polynomials $P \in \boldsymbol{Z}[y] \setminus \{0\}$

$$|P(f(\alpha))| \leq \exp(-Ct(P)^4)\,. \tag{6}$$

Remarks. 1) Our result shows that the numbers in consideration have a transcendence type not exceeding 4 and it is therefore comparable to the result of MILLER [Mi], who derived a transcendence measure for the values of functions satisfying linear Mahler type functional equations. Unfortunately, Corollary 2 is not sharp enough to imply that $f(\alpha)$ is an S–number according to Mahler's classification.

2) It is clear that the numbers studied in Corollaries 1 and 2 of [BB] have also the transcendence measures (6).

II. The zero estimate: Proof of Theorem 2

Let $P(z, y) = P^{(1)}(z, y) \ldots P^{(t)}(z, y)$ be the decomposition of the polynomial P with $P^{(i)} \in \boldsymbol{F}[z, y]$ irreducible over $\boldsymbol{F}(z)$. Let $d_i = \deg_y P^{(i)} \geq 1$, $\nu_i = \operatorname{ord} P^{(i)}(z, f(z))$, and $\nu = \nu_1 + \cdots + \nu_t$. Since $d_1 + \cdots + d_t \leq N$, we may assume without loss of generality that $\nu_1/\nu \geq d_1/N$. Suppose that $T(z) = T_1(z)/T_2(z)$ with relatively prime polynomials $T_1, T_2 \in \boldsymbol{F}[z]$ and $Q(z, y) = Q_1(z, y)/Q_2(z, y)$ also with relatively prime polynomials $Q_1, Q_2 \in \boldsymbol{F}[z, y]$. We define polynomials

$$R_0(z, y) = P^{(1)}(z, y)\,,$$

$$R_1(z, y) = T_2(z)^M Q_2(z, y)^{d_1} R_0(Tz, Q(z, y))\,. \tag{7}$$

The proof is rather easy if condition (i) of Theorem 2 is satisfied. We treat this case first.

(i) $\deg_y Q < d$.

Since $R_0(z,y)$ is irreducible, there is a number $n \in \boldsymbol{N}_0$ such that $R_0(z,y)$ and $\tilde{R}_1(z,y) = R_1(z,y)R_0(z,y)^{-n}$ are coprime polynomials from $\boldsymbol{F}(z)[y]$. Let $D = D(z) \in \boldsymbol{F}[z]$ be the leading coefficient of $R_0(z,y)^n$ with respect to the variable y. Then we have $R_2(z,y) = D\tilde{R}_1(z,y) \in \boldsymbol{F}[z,y]$. Clearly, $\deg_y R_1 \leq d_1 \deg_y Q$ and $\deg_z R_1 \leq M \deg_z T + d_1 \deg_z Q$. Thus the degrees of R_2 with respect to y and z are bounded as follows

$$(8) \qquad 0 \leq \deg_y R_2 \leq d_1 \deg_y Q - d_1 n < (d-n)d_1 \,,$$

and

$$\deg_z R_2 \leq nM + M \deg_z T + d_1 \deg_z Q \,.$$

Since R_0 and R_2 are relatively prime as polynomials in $\boldsymbol{F}(z)[y]$, they have a nonzero resultant $R(z) \in \boldsymbol{F}(z)$. It satisfies

$$(9) \quad \min\{\operatorname{ord} R_0(z,f(z)), \operatorname{ord} R_2(z,f(z))\} \leq \operatorname{ord} R(z) \leq \deg_z R(z) \,.$$

We have $\deg_z R(z) \leq \deg_z R_0 \deg_y R_2 + \deg_z R_2 \deg_y R_0 \leq \gamma_1 d_1 (d_1 + M)$. Here and in the sequel $\gamma_1, \gamma_2, \ldots$ represent positive constants depending only on f, but not on M or N. From the definition of R_2 and from the assumption that $T(z)$ has a zero of order d at $z = 0$ we get

$$\operatorname{ord} R_2(z,f(z)) \geq \operatorname{ord} R_1(z,f(z)) - n \operatorname{ord} R_0(z,f(z)) \geq (d-n)\nu_1 \,.$$

Equation (8) yields $d - n \geq 1$. Thus, by (9) and $\nu \leq \nu_1 N d_1^{-1}$, we have the asserted inequality

$$\nu \leq (d-n)\nu_1 N d_1^{-1} \leq \gamma_1 N (d_1 + M) \leq \gamma_1 N (N + M) \,.$$

Now we assume

(ii) $\operatorname{char} \boldsymbol{F} = 0$ and distinguish two cases. The first one is

$(\mathrm{ii})_1$ $R_0(z,y)$ and $R_1(z,y)$ are coprime as polynomials in $\boldsymbol{F}(z)[y]$.

Here one can repeat the argument exposed in part (i). Since $n = 0$ in this case, no further restriction with respect to $\deg_y Q$ and d is necessary.

$(\mathrm{ii})_2$ R_0 and R_1 are not relatively prime.

In this case we apply Lemma 1 below to see that $R_0(z,y)$ and

$$\tilde{R}_1(z,y) = \frac{R_1(z,y)}{R_0(z,y)}$$

are relatively prime or $R_0(z,y)$ belongs to a certain set of exceptional polynomials (described in Lemma 1). If R_0 and $\tilde{R}_1$ are relatively prime, then we can proceed as in part (i) to derive the zero estimate. (Note: $n = 1$ and $d \geq 2$!)

Otherwise, that is, if R_0 is one of the exceptional polynomials we have $R_0(z,y) = a(z)S(z,y)$, where $a(z) \in \boldsymbol{F}[z]$ and $S(z,y)$ belongs to a finite set $\mathcal{S}$ of irreducible polynomials with leading coefficient 1. Thus

$$\operatorname{ord} R_0(z, f(z)) \leq \deg_z a(z) + \max_{S \in \mathcal{S}} \operatorname{ord} S(z,(f(z)) \leq M + \gamma_2 .$$

Lemma 1. *There is a finite set $\mathcal{S} \subset \boldsymbol{F}(z)[y]$ of irreducible polynomials with leading coefficient 1 such that the following holds:*

If $R_0 \in \boldsymbol{F}[z,y]$ is irreducible and $R_1 \in \boldsymbol{F}[z,y]$ is constructed according to (7), then R_1 has no multiple roots in $\overline{\boldsymbol{F}(z)}$ (the algebraic closure of $\boldsymbol{F}(z)$) or R_0 is of the form $a(z)S(z,y)$ with $a(z) \in \boldsymbol{F}[z]$ and $S \in \mathcal{S}$.

PROOF. We assume that $g(z) \in \overline{\boldsymbol{F}(z)}$ is a multiple root of $R_1(z,y)$, i.e. $R_1(z,g(z)) = \frac{d}{dy}R_1(z,y)|_{y=g(z)} = 0$. Since $R_1(z,g(z)) = 0$, it is clear that one of the factors in (7) has to vanish after substituting $y = g(z)$. Of course, $T_2(z) \neq 0$, and $Q_2(z,g(z)) = 0$ would imply

$$0 = R_1(z,g(z)) = T_2(z)^M r_{d_1}(Tz) Q_1(z,g(z))^{d_1} ,$$

where $r_{d_1}(z) \neq 0$ is the leading coefficient of $R_0(z,y)$. This would yield a contradiction to the coprimality of Q_1 and Q_2. Thus

$$R_0(Tz, Q(z,g(z))) = 0 . \tag{10}$$

We have

$$\begin{aligned} &\frac{d}{dy}R_1(z,y) = \\ &= T_2(z)^M d_1 \left(\frac{d}{dy}Q_2(z,y)\right) Q_2(z,y)^{d_1-1} R_0(Tz,Q(z,y)) + \\ &+ T_2(z)^M Q_2(z,y)^{d_1} \left(\frac{d}{dy}Q(z,y)\right)\left(\frac{d}{dy}R_0(Tz,y)|_{y=Q(z,y)}\right) . \end{aligned} \tag{11}$$

From our assumption $\operatorname{char} \boldsymbol{F} = 0$ we conclude that the irreducible polynomial $R_0(z,y)$ has no multiple roots. The monomorphism $z \mapsto Tz$ of $\boldsymbol{F}(z)$ into $\boldsymbol{F}(z)$ can be extended to a monomorphism of the splitting field L of $R_0(z,y)$ into $\overline{\boldsymbol{F}(z)}$ (cf. [L], p. 369). Therefore $R_0(Tz,y)$ has no multiple zeros and, by (10), $\frac{d}{dy}R_0(Tz,y)|_{y=Q(z,g(z))} \neq 0$. Hence as a consequence of $\frac{d}{dy}R_1(z,y)|_{y=g(z)} = 0$, (10), and (11) we get

$$\frac{d}{dy}Q(z,y)|_{y=g(z)} = 0 . \tag{12}$$

Since char $\boldsymbol{F} = 0$ and $f(z)$ is transcendental, it is clear that $\frac{d}{dy}Q(z,y) \neq 0$. Thus (12) is satisfied only for $g(z) \in \mathcal{G}$, where $\mathcal{G} \subset \overline{\boldsymbol{F}(z)}$ is a finite set. Let $u \in \overline{\boldsymbol{F}(z)}$ be the unique solution of $u(Tz) = z$ and let $\mathcal{H}$ be the finite set

$$\{h(z) \in \overline{\boldsymbol{F}(z)} \mid h(z) = Q(u(z), g(u(z))) \text{ for some } g \in \mathcal{G}\}.$$

Thus $R_0(Tz, Q(z, g(z))) = 0$ with $g \in \mathcal{G}$ implies $R_0(z, h(z)) = 0$ for some $h \in \mathcal{H}$, i.e. R_0 is associated to one of the finitely many minimal polynomials of the elements of $\mathcal{H}$. □

III. Measures in positive characteristic: Proofs of Theorem 1 and Corollary 1

We prove the following result, which is more general than Theorem 1 and which shows the connection between zero estimates for algebraic independent functions $f_1, \ldots, f_m$ and measures of algebraic independence of the corresponding values $f_1(\alpha), \ldots, f_m(\alpha)$. Theorem 1 is just the special case $m = 1$ of

Proposition B. *Let $f_1(z), \ldots, f_m(z) \in \boldsymbol{F}_q[[z]]$ and suppose that there exists a function $\mu : [0,\infty)^2 \to [0,\infty)$ which is not decreasing with respect to each of its two variables and for which the following property holds:*

For all polynomials $R(z, y_1, \ldots, y_m) \in \boldsymbol{F}_q[z, y_1, \ldots, y_m] \setminus \{0\}$ with $\deg_z R \leq M$ and $\text{deg.tot}_{y_1,\ldots,y_m} R \leq N$ we have

$$\text{ord}\, R(z, f_1(z), \ldots, f_m(z)) \leq \mu(N, M). \tag{13}$$

Let $\alpha \in U_1 \setminus \{0\}$ be algebraic over $\mathcal{Q}$. Then there exist constants $C_1, C_2, C_3 > 0$ depending only on α and $f_1, \ldots, f_m$ such that for all polynomials $P(y_1, \ldots, y_m) \in \mathcal{Z}[y_1, \ldots, y_m] \setminus \{0\}$ with $H(P) \leq H$ and $\text{deg.tot}_{y_1,\ldots,y_m} P \leq n$ the following inequality holds

$$\log |P(f_1(\alpha), \ldots, f_m(\alpha))| \geq -C_1 \mu(C_2 n, C_3 \log H).$$

Remarks. 1) Since $\mu(N, M) < \infty$, the functions $f_1(z), \ldots, f_m(z)$ have to be algebraically independent over $\boldsymbol{F}_q(z)$ and the function values $f_1(\alpha), \ldots, f_m(\alpha)$ have to be algebraically independent over the field $\mathcal{Q}$. It is clear that for any set of algebraically independent functions $f_1, \ldots, f_m$ there exists a function μ having property (13). Hence Proposition B includes a generalization of Allouche's result, Proposition A, to the case of algebraic independent functions.

2) It will become clear from the proof that the proposition holds true also if $\boldsymbol{F}_q$ is substituted by an arbitrary field, not necessarily of positive characteristic.

PROOF. Let $\underline{y}$ be the vector notation for $(y_1, \ldots, y_m)$. Similarly we use the symbols $\underline{f}, \underline{f}(z), \ldots$. Let $R \in \boldsymbol{F}_q[z, \underline{y}]$ be a nonzero polynomial with $\deg_z R \leq M$ and $\deg.\mathrm{tot}_{\underline{y}} R \leq N$. By our assumption we have

$$R(z, \underline{f}(z)) = \sum_{\nu \geq \nu_0} a_\nu z^\nu$$

with $a_\nu \in \boldsymbol{F}_q$, $a_{\nu_0} \neq 0$, and $\nu_0 \leq \mu(N, M)$. This yields for $0 < |\alpha| < 1$ the estimate

$$\log |R(\alpha, \underline{f}(\alpha))| \geq -\mu(N, M) \log |\alpha| . \tag{14}$$

Our hypotheses $\alpha \in \bar{\mathcal{Q}}$ (the algebraic closure of $\mathcal{Q}$) and $0 < |\alpha| < 1$ imply that α is transcendental over $\boldsymbol{F}_q$ and algebraic over $\boldsymbol{F}_q(x)$. Since x is also transcendental over $\boldsymbol{F}_q$, it has to be algebraic over $\boldsymbol{F}_q(\alpha)$. There is an integer $e \geq 0$ such that $x_1 = x^{p^e}$ is separable over $\boldsymbol{F}_q(\alpha)$ and it is no restriction to assume that e is a multiple of $\log_p q$. Let e be minimal with these properties.

Let $P \in \mathcal{Z}[\underline{y}] \setminus \{0\}$. We have $P(\underline{y}) = P_1(x, \underline{y})$ with some $P_1 \in \boldsymbol{F}_q[x, \underline{y}]$ and consequently

$$P(\underline{f}(\alpha))^{p^e} = P_1(x, \underline{f}(\alpha))^{p^e} = P_1(x_1, \underline{f}(\alpha)^{p^e}) .$$

It is, therefore, sufficient to deduce lower bounds for the absolute value of polynomials in the variables x_1 and $\underline{f}(\alpha)$. Let L be the splitting field of the minimal polynomial of x_1 over $K = \boldsymbol{F}_q(\alpha)$. The extension L/K is Galois.

Let $D \in \boldsymbol{F}_q[\alpha] \setminus \{0\}$ be a denominator for x_1, i.e. such that the monic minimal polynomial of Dx_1 over $\boldsymbol{F}_q(\alpha)$ has coefficients from $\boldsymbol{F}_q[\alpha]$.

Let $S \in \mathcal{Z} \setminus \{0\}$ with $S(\underline{y}) = S_1(x, \underline{y})$ for some $S_1 \in \boldsymbol{F}_q[x, \underline{y}] \setminus \{0\}$ and $\kappa = \deg_x S_1 = \log_q H(S)$. There exists a polynomial $R(z, \underline{y}) \in \boldsymbol{F}_q[z, \underline{y}] \setminus \{0\}$ with

$$R(\alpha, \underline{y}) = \prod_\sigma D^\kappa S_1(\sigma(x_1), \underline{y}) , \tag{15}$$

where the product runs over the Galois group of L over K. The degrees of R are bounded as follows

$$\deg_z R \leq \kappa\, [L : K](\log_q |D| + \log_q \overline{|x_1|}) ,$$
$$\deg.\mathrm{tot}_{\underline{y}} R \leq [L : K] d(S) .$$

Here $d(S) = \deg.\mathrm{tot}_{\underline{y}} S$ and $\overline{|x_1|} = \max_\sigma |\sigma(x_1)|$ is the maximum of the absolute values of the conjugates of x_1. Let $\gamma_1, \gamma_2, \ldots$ represent positive constants depending only on α and $\underline{f}$. Thus we have $\deg.\mathrm{tot}_{\underline{y}} R \le \gamma_1 d(S)$ and $\deg_z R \le \gamma_2 \log H(S)$. It is a consequence of (14) and (15) that

$$\sum_{\sigma} \log |D^{\kappa} S_1(\sigma(x_1), \underline{f}(\alpha))| \ge -\gamma_3 \mu(\gamma_1 d(S), \gamma_2 \log H(S)). \tag{16}$$

The following estimate holds for all σ

$$\log |D^{\kappa} S_1(\sigma(x_1), \underline{f}(\alpha))| \le \log_q H(S) \left(\log |D| + \log \overline{|x_1|}\right) + \\ + d(S) \log \max_i |f_i(\alpha)| \le \gamma_4 (d(S) + \log H(S)).$$

We have $\mu(N, M) \ge N + M$, since it is always possible to construct a polynomial $\tilde{R} \in \boldsymbol{F}_q[z, \underline{y}]$ with $\deg_z \tilde{R} \le M$, $\deg.\mathrm{tot}_{\underline{y}} \tilde{R} \le N$, and $\mathrm{ord}\, \tilde{R}(z, \underline{f}(z)) \ge N + M$. Hence we may conclude from inequality (16)

$$\log |S_1(x_1, \underline{f}(\alpha))| \ge -\gamma_5 \mu(\gamma_1 d(S), \gamma_2 \log H(S)).$$

Now we choose S in such a way that $S_1(x_1, \underline{y}) = P_1(x_1, \underline{y}^{p^e})$ for the above defined e and $P_1 \in \boldsymbol{F}_q[x, \underline{y}]$. This yields

$$\log |P(\underline{f}(\alpha))| \ge -\gamma_6 \mu(\gamma_7 d(P), \gamma_8 \log H(P)). \quad \Box$$

PROOF of Corollary 1. (a) Let

$$\sum_{h \ge 0} \left(\frac{z^{d^h}}{1 - z^{d^h}} \right)^m$$

for some $m \in \boldsymbol{N}$. f satisfies the functional equation

$$f(z) = f(z^d) + \left(\frac{z}{1 - z} \right)^m .$$

It was shown in Corollary 3 of [Be3] that $f(z)$ is a transcendental function. Thus we can apply Theorem 2 with $Tz = z^d$ and $Q(z, y) = y - (z/1 - z)^m$. The assertion of Theorem 2 guarantees that the assumption of Theorem 1 is satisfied if we take $\mu(N, M) = C_1 N(N + M)$ with a suitable constant $C_1 > 0$. Theorem 1 yields

$$\log |P(\omega)| \ge -C_2 n(n + \log H)$$

for all $P \in \mathcal{Z} \setminus \{0\}$ with $H(P) \le H$ and $d(P) \le n$, where the positive constant C_2 depends only on α, d, and m.

(b) and (c) follow by a similar reasoning applied to the functions $f_1(z) = \sum_{h \ge 0} z^{d^h}$ and $f_2(z) = \prod_{h \ge 0} (1 - z^{d^h})$. $\Box$

IV. Measures in characteristic 0: Proof of Corollary 2

For $j \geq 0$, let $\sum_{h\geq -j} f_{hj} z^{-h}$ be the Laurent series development of f^j at $z = \infty$. Let K be the algebraic number field containing $\alpha, f_{-1,1}$, and the coefficients of the polynomials p and q. O_K denotes the ring of algebraic integers in K. For algebraic numbers β we define $\overline{|\beta|}$, the house of β, as the maximum of the absolute values of the conjugates of β. $\gamma_1, \gamma_2, \ldots$ represent positive constants independent of the parameters M, N, and k which will be introduced later.

The corollary is shown in an equivalent form as a result on approximations $|f(\alpha) - \xi|$ for algebraic numbers ξ. To sketch its proof we start with the following lemma, which gives estimates for the houses and denominators of the Laurent series coefficients f_{hj}. Such estimates are needed for the construction of an auxiliary function, which is done in Lemma 3. Lemma 4 gives upper and lower bounds for suitable values of the auxiliary function and Lemma 5 shows how a good approximation of $f(\alpha)$ by an algebraic number ξ would lead to a good approximation of those function values. A Liouville estimate and a suitable choice of M, N, and k then yield a contradiction.

Lemma 2. *There is a constant $\gamma_1 > 0$ and a natural number D such that for $j \geq 0$ and $h \geq -j$*

$$\text{(i) } \log \overline{|f_{hj}|} \leq \gamma_1(h + 2j) \quad \text{and} \quad \text{(ii) } D^{h+2j} f_{hj} \in O_K .$$

PROOF. Part (ii) of the lemma is an immediate consequence of Proposition 1 and Lemma 6 in [Be2]. The coefficients f_{h1} satisfy the conditions

$$f_{-1,1}^{d-1} = \frac{a_d}{b_d} \tag{17}$$

and for $h \geq 0$

$$f_{h1} = R_h(a_0, \ldots, a_d, b_0, \ldots, b_d, f_{-1,1}, \ldots, f_{h-1,1}), \tag{18}$$

where $R_h \in \boldsymbol{Q}(x_0, \ldots, x_d, y_0, \ldots, y_d, w_{-1}, \ldots, w_{h-1})$ depends only on the common degree of the polynomials p and q but not on their coefficients. The recursion formula (18) is a consequence of the functional equation (5) satisfied by f.

Let σ be a field isomorphism of K. From (17) and (18) we get

$$\sigma(f_{-1,1})^{d-1} = \frac{\sigma(a_d)}{\sigma(b_d)}$$

and for $h \geq 0$

$$\sigma(f_{h1}) = R_h(\sigma(a_0), \ldots, \sigma(a_d), \sigma(b_0), \ldots, \sigma(b_d), \sigma(f_{-1,1}), \ldots, \sigma(f_{h-1,1})) .$$

Thus the Laurent series

$$(\sigma f)(z) = \sum_{h \geq -1} \sigma(f_{h1}) z^{-h}$$

is a Böttcher function with respect to the polynomials $\sum_i \sigma(a_i) z^i$ and $\sum_i \sigma(b_i) z^i$. Therefore it has to converge for all z of sufficiently large absolute value. Hence there exists a constant $\gamma_\sigma > 0$ with

$$|\sigma(f_{h1})| \leq {\gamma_\sigma}^{h+2} \quad (h \geq -1).$$

Since there are only finitely many isomorphisms to consider, we have

$$\overline{|f_{h1}|} = \max_\sigma |\sigma(f_{h1})| \leq (\max_\sigma \gamma_\sigma)^{h+2} \leq {\gamma_2}^{h+2}.$$

Now we apply the idea used in the proof of Lemma 6 in [Be2] to derive inequality (i). □

For a Laurent series $g(z) = \sum_{\nu \geq \nu_0} \gamma_{-\nu} z^{-\nu} \in \boldsymbol{C}((z^{-1}))$ at ∞ we denote by $\operatorname{ord}_\infty g$ the smallest ν with $\gamma_{-\nu} \neq 0$.

Lemma 3. *Let $N \geq \gamma_3$ and $M \geq \gamma_4 N$ be integers. Then there exists a polynomial $P(z, y) \in \boldsymbol{Z}[z, y] \setminus \{0\}$ with the following properties:*

(a) $1 \leq \deg_y P \leq N, \quad \deg_z P \leq M$,

(b) $H(P) \leq \exp(\gamma_5 MN)$,

(c) $\operatorname{ord}_\infty P(z, f(z)) \geq \gamma_6 MN$,

(d) $\operatorname{ord}_\infty P(z, f(z)) \leq \gamma_7 MN$.

PROOF. Using Siegel's lemma (cf. [Wl1], p. 10) it is possible to construct a polynomial satisfying conditions (a), (b) and (c). (For more details of the construction the reader may wish to consult the proof of Lemma 1 in [Mi].) Since the Böttcher functions we have to consider here are all transcendental (cf. [BB], Theorem 1), we can apply Theorem 2 to the function $\tilde{f}(z) = z f(1/z)$. This function is analytic in a neighbourhood of $z = 0$ and satisfies a functional equation

$$\tilde{f}(Tz) = R(z, \tilde{f}(z)),$$

where $Tz = 1/p(1/z)$ and $R(z, w)$ is a suitable rational function of z and w. The estimate (d) is now a consequence of our zero estimate from Theorem 2. □

Now, for the parameters N and M, let $P(z, y)$ be a polynomial constructed according to Lemma 3.

Lemma 4. *Let M, N, and k be integers satisfying $N \geq \gamma_8$, $M \geq \gamma_9 N$, and $d^k \geq \gamma_{10} MN$. Then we have*

$$\exp(-\gamma_{11} MNd^k) < |P(p_k(\alpha), f(p_k(\alpha))| < \exp(-\gamma_{12} MNd^k).$$

PROOF. First we note that the Laurent series coefficients of f at ∞ satisfy the same estimates as the power series coefficients of the function f in [Mi]. Furthermore, we have for $k \geq 0$ the estimates $\gamma_{13} d^k \leq \log |p_k(\alpha)| \leq \gamma_{14} d^k$ (cf. Lemma 1 in [Be1]). Thus we can proceed as in the proof of Lemma 3 in [Mi]. □

Let ξ be an algebraic number of degree $d(\xi)$ and height $H(\xi)$. We define $t(\xi) = d(\xi) + \log H(\xi)$. If ξ is a good approximation to $f(\alpha)$, we may expect $q_k(\xi)$ to be a good approximation to $f(p_k(\alpha))$. An application of the mean-value theorem yields a corresponding result, which is stated as

Lemma 5. *Let M, N and $k \geq \gamma_{15}$ be integers. There is a positive number γ_{16} with the property: If*

$$|f(\alpha) - \xi| \leq \exp(-\gamma_{16} MNd^k), \tag{19}$$

then

$$|P(p_k(\alpha), f(p_k(\alpha))) - P(p_k(\alpha), q_k(\xi))| < \exp(-\gamma_{11} MNd^k),$$

with the constant γ_{11} of Lemma 4.

Now we can prove Corollary 2. Suppose that ξ satisfies (19). Then it is clear from Lemma 4 and Lemma 5 that $P(p_k(\alpha), q_k(\xi)) \neq 0$. A Liouville estimate (see [Ga], Lemma 5) yields

$$|P(p_k(\alpha), q_k(\xi))| \geq \exp(-\gamma_{17} Md^k(d(\xi) + \log H(\xi)))$$

if $M \geq \gamma_{18} N$. Combining this with Lemma 4 and Lemma 5 we get

$$\begin{aligned} \exp(-\gamma_{12} MNd^k) &> |P(p_k(\alpha), f(p_k(\alpha)))| \\ &\geq |P(p_k(\alpha), q_k(\xi))| - |P(p_k(\alpha, f(p_k(\alpha))) - P(p_k(\alpha), q_k(\xi))| \\ &> \exp(-\gamma_{17} Md^k t(\xi)) - \exp(-\gamma_{11} MNd^k). \end{aligned}$$

This yields a contradiction if we choose $N \geq \gamma_{19} t(\xi)$ with a sufficiently large constant γ_{19}. Hence the conditions of Lemma 4 and also

$$|f(\alpha) - \xi| > \exp(-\gamma_{16} MNd^k)$$

are satisfied if we take M and k with $M \geq \gamma_{20} t(\xi)$ and $d^k \geq \gamma_{21} t(\xi)^2$. We get

$$|f(\alpha) - \xi| > \exp(-\gamma_{22} t(\xi)^4),$$

from which the corollary can be deduced in the usual way (see [L], p. 61). □

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