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On the structure of optimal measures and some of its applications

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Abstract. By an optimal measure space we mean a triple $(\Omega, \mathcal{F}, p)$, where $(\Omega, \mathcal{F})$ is a measurable space and p is an optimal measure, i.e. $p : \mathcal{F} \to [0,1]$ is a set function satisfying the following properties:

P1. $p(\emptyset) = 0$ and $p(\Omega) = 1$.

P2. p is F-additive, i.e. $p(B \cup E) = p(B) \vee p(E)$ for all $B, E \in \mathcal{F}$, where $\vee$ denotes the maximum.

P3. p is continuous from above.

We give a structure theorem describing optimal measure spaces. As a consequence, the corresponding Radon-Nikodym derivative is explicitly given.

0. Introduction

In [1] we have defined the so-called optimal average, in analogy to the mathematical expectation. We shall here recall some notation, preliminary definitions and results.

Notation. *i*) The symbol $\vee$ (resp. $\wedge$) stands for the maximum (resp. minimum).

ii) $\chi(B)$ denotes the characteristic function of the set B.

Convention: $0 \cdot \infty = 0$.

Let $(\Omega, \mathcal{F})$ be any measurable space. In the sequel, measurable sets (resp. functions) will be referred to as events (resp. random variables, abbreviated r.v.'s), measurable simple functions as discrete r.v.'s. The complement of any event B will be denoted by B', and we write $\mathbf{1} := \chi(\Omega)$.

By an optimal measure we mean a set function $p : \mathcal{F} \to [0,1]$ satisfying the following properties:

P1. $p(\emptyset) = 0$ and $p(\Omega) = 1$.

P2. p is F-additive. (i.e. $p(B \cup E) = p(B) \vee p(E)$ for all events B and E.)

P3. p is continuous from above. (That is, if $B_n \in \mathcal{F}$ and $B_1 \supset B_2 \supset \cdots$, then $p\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} p(B_n)$.)

The triple $(\Omega, \mathcal{F}, p)$ is called an optimal measure space.

Let $s = \sum_{i=1}^{n} b_i \chi(B_i)$ be a nonnegative discrete r.v. We have shown that the functional $I(s) = \bigvee_{i=1}^{n} b_i \cdot p(B_i)$ (called optimal average of s), does not depend on the decompositions of s. The optimal average of s on an event B is defined by $I_B(s) = I(s \cdot \chi(B))$.

Let $f \geq 0$ be a r.v. The quantity $Af := \sup I(s)$ (where the supremum is taken over all discrete r.v.'s $s \geq 0$ such that $s \leq f$) is called the optimal average of f. The optimal average of f on an event B is defined by

$$A_B f := A(f\chi(B)),$$

Let f be any r.v. We shall say that f belongs to:

i) $\mathcal{A}^\infty$, if $p(|f| \leq b) = 1$ for some $b \in \mathbb{R}_+$.

ii) $\mathcal{A}^\alpha$, if $A|f|^\alpha < \infty$, $\alpha \in [1, \infty)$.

For $\alpha \in [1, \infty]$, the functional

$$\|f\|_\alpha := \begin{cases} \inf(b \in \mathbb{R}_+ : p(|f| \leq b) = 1), & \text{if } f \in \mathcal{A}^\infty,\ \alpha = \infty \\ \{A|f|^\alpha\}^{1/\alpha}, & \text{if } f \in \mathcal{A}^\alpha,\ \alpha \in [1, \infty) \end{cases}$$

is a norm. For $\alpha \in [1, \infty]$, the space $\mathcal{A}^\alpha$ endowed with this norm is a Banach space.

From now on let us fix an optimal measure space $(\Omega, \mathcal{F}, p)$ and in analogy to the symbol "$\int$" of the Lebesgue integral we shall adopt the symbol "$\setminus$" to designate the optimal average, i.e. $Af = \setminus_{\Omega} f\,dp$, $A_B f = \setminus_{B} f\,dp$ where $f \geq 0$ is a r.v. and $B \in \mathcal{F}$.

1. The fundamental theorem

By a ($p-$)atom we mean an event H, $p(H) > 0$ such that whenever $B \in \mathcal{F}$, $B \subset H$, then $p(H) = p(B)$ or $p(B) = 0$.

Definition 1.1. We shall say that a $(p-)$atom H is decomposable, if there exists a subatom $B \subset H$ such that $p(B) = p(H) = p(H \setminus B)$. If no such subatom exists, we shall say that H is indecomposable.

Lemma 1.1. *Any atom H can be expressed as the union of finitely many disjoint indecomposable subatoms of the same optimal measure as H.*

PROOF. We say that an event E is good, if it can be expressed as the union of finitely many disjoint indecomposable subatoms. Let H be an atom and suppose that H is not good. Then H is decomposable. Set $H = B_1 \cup C_1$, where $B_1 \cap C_1 = \emptyset$, $p(B_1) = p(C_1) = p(H)$. Since H is not good, at least one of the two events B_1 and C_1 is not good; suppose, e.g. B_1 is not good. Then B_1 is decomposable. Let $B_1 = B_2 \cup C_2$, where B_2 and C_2 are disjoint events with $p(B_2) = p(C_2) = p(H)$. Continuing this process for every $n \geq 1$ we obtain two events B_n and C_n such that the C_n's are pairwise disjoint with $p(C_n) = p(H)$. This however is impossible since $E_n = \bigcup\limits_{k=n}^{\infty} C_k$ tends decreasingly to the empty set and hence, by property P3, $p(E_n) \to p(\emptyset) = 0$ as $n \to \infty$ which contradicts $p(E_n) \geq p(C_n) = p(H) > 0$, $n \geq 1$. □

Remark 1.1. Let H be any indecomposable atom and E any event with $p(E) > 0$. Then, either $p(H) = p(H \setminus E)$ and $p(E \cap H) = 0$, or $p(H) = p(H \cap E)$ and $p(H \setminus E) = 0$.

Theorem 1.2 (Fundamental Optimal Measure). *Let $(\Omega, \mathcal{F}, p)$ be an optimal measure space. Then there exists a collection $\mathcal{H} := \{H_n : n \in \mathcal{J}\}$ of disjoint indecomposable $(p-)$atoms, where $\mathcal{J}$ is some countable (i.e. finite or countably infinite) index set, such that for every event B with $p(B) > 0$ we have*

$$p(B) = \max\{p(B \cap H_n) : n \in \mathcal{J}\}. \tag{1.1}$$

Moreover if $\mathcal{J}$ is countably infinite, then the only limit point of the set $\{p(H_n) : n \in \mathcal{J}\}$ is 0.

Before the proof, let us state the following results.

Lemma 1.3. *Let E be an event with $p(E) > 0$ and $B_k \in \mathcal{F}$, $B_k \subset E$ $(k \in \mathcal{J})$ where $\mathcal{J}$ is any countable index set. Then,*

$$p\left(\bigcup_{k \in \mathcal{J}} B_k\right) < p(E) \quad \textit{if and only if for all } k \in \mathcal{J}, \tag{1.2}$$

$$p(B_k) < p(E). \tag{1.2'}$$

Proof. The lemma is obvious if the index set $\mathcal{J}$ is finite. We may thus assume that $\mathcal{J} = \{1, 2, 3, \dots\}$. Suppose that (1.2)$'$ holds for all $k \geq 1$. Put $C_k = \bigcup_{i=1}^{k} B_i$, $k \geq 1$. Clearly, (C_k), $k \geq 1$, is an increasing sequence of events and the inequality

$$p(C_k) < p(E) \tag{1.3}$$

holds for all $k \geq 1$. Assume that $p(E) = p\left(\bigcup_{k=1}^{\infty} C_k\right)$. Then via (1.3) we obtain that $p(E) = p\left(\left(\bigcup_{i=1}^{\infty} C_i\right) \setminus C_k\right)$ for all $k \geq 1$. This is impossible, since $E_k = \left(\left(\bigcup_{i=1}^{\infty} C_i\right) \setminus C_k\right)$, $k \geq 1$, tends decreasingly to the empty set and thus, by P1 and P3, $p(E_k) \to 0$ as $k \to \infty$. Hence (1.2) holds.

The converse is obvious, proving the lemma. □

Lemma 1.4. *For every sequence* (B_k), $k \geq 1$, *of events we have* $p\left(\bigcup_{k=1}^{\infty} B_k\right) = \max\{p(B_k) : k \geq 1\}$.

(The proof is immediate from Lemma 1.3.)

Lemma 1.5. *Every event* E *with* $p(E) > 0$ *contains an atom* $H \subset E$ *such that* $p(H) = p(E)$.

Proof. If E is an atom, there is nothing to be proved. We may assume that E is not an atom. Let $\mathcal{U} \subset \mathcal{F}$ be such that:

i) if $B \in \mathcal{U}$ then, $B \subset E$ and $0 < p(B) < p(E)$,

ii) if $B, C \in \mathcal{U}$ and $B \neq C$, then $B \cap C = \emptyset$.

Clearly the collection of all such $\mathcal{U}$, denoted by $\mathcal{C}$, is partially ordered by set inclusion.

It is also obvious that every subset of $\mathcal{C}$ has an upper bound. Therefore, by the Zorn lemma, it follows that $\mathcal{C}$ contains a maximal element, which we shall denote by $\mathcal{U}^*$. For any fixed constant $\delta \in (0, 1)$, let us show that the set

$$\{B \in \mathcal{U}^* : p(B) > \delta\}$$

is finite. Suppose that the contrary holds. Then there exists a sequence $(B_n) \subset \mathcal{U}^*$ with $p(B_n) > \delta$ for all $n \geq 1$. But since $E_n = \bigcup_{i=n}^{\infty} B_i$ tends decreasingly to the empty set, we must have that $p(E_n) \to 0$ as $n \to \infty$. This however contradicts $p(E_n) = \bigvee_{i=n}^{\infty} p(B_i) > \delta$, $n \geq 1$. Hence $\mathcal{U}^* = \{B_k :$

$k \in \Delta\}$ with $p(B_k) < p(E)$ for all $k \in \Delta$, where Δ is a countable index set. By Lemma 1.3, it follows that

$$p\left(\bigcup_{k\in\Delta} B_k\right) < p(E).$$

Thus it is obvious that $H = E \setminus \bigcup_{k\in\Delta} B_k$ is an atom with $p(H) = p(E)$. This ends the proof of the lemma. □

PROOF of Theorem 1.2. Let $\mathcal{G}$ be a set of pairwise disjoint atoms. Clearly the collection of all such $\mathcal{G}$, denoted by Γ, is partially ordered by set inclusion and every subset of Γ has an upper bound. Then, the Zorn lemma entails that Γ contains a maximal element, which we shall denote by $\mathcal{G}^*$. As above, it can be easily seen that the set

$$\left\{K \in \mathcal{G}^* : p(K) > \frac{1}{n}\right\}$$

is finite. Hence $\mathcal{G}^* = \{K_i : i \in \nabla\}$ where ∇ is a countable index set. It is obvious that $p(K_i) \to 0$ as $i \to \infty$ whenever ∇ is countably infinite. Consequently it ensues, via Lemma 1.1, that each atom $K_i \in \mathcal{G}^*$ can be expressed as the union of finitely many disjoint indecomposable subatoms of the same optimal measure as K_i.

Let us list these indecomposable atoms occurring in the decompositions of the elements of $\mathcal{G}^*$ as follows: $\mathcal{H} := \{H_n : n \in \mathcal{J}\}$ where $\mathcal{J}$ is a countable index set. □

Lemma 1.6. *Let $\mathcal{H} = \{H_n : n \in \mathcal{J}\}$ be as above. Then for every event B with $p(B) > 0$, the identity*

$$p\left(B \setminus \bigcup_{n\in\mathcal{J}} (B \cap H_n)\right) = 0 \tag{1.4}$$

holds.

In fact, assume that the left side of (1.4) is positive. Then $B \setminus \bigcup_{n\in\mathcal{J}} (B \cap H_n)$ would contain an atom K such that $K \cap K_n = \emptyset$ for every $K_n \in \mathcal{G}^*$. This, however, would contradict the maximality of $\mathcal{G}^*$, proving Lemma 1.6.

Now, via Lemma 1.4, the identity (1.4) and property P2, one can easily observe that (1.1) holds for every event B, $p(B) > 0$. It is also obvious that 0 is the only limit point of the set $\{p(H_n) : n \in \mathcal{J}\}$ whenever $\mathcal{J}$ is countably infinite. This completes the proof of the theorem. □

Definitions 1.2. The set $\mathcal{H} = \{H_n : n \in \mathcal{J}\}$ of disjoint indecomposable $(p-)$atoms (obtained in Theorem 1.2) will be called $(p-)$generating countable system:

i) it will be referred to as a $(p-)$ generating infinite system and will be denoted by $\mathcal{H}_\infty(p)$, if $\mathcal{J}$ is countably infinite;

ii) it will be called a $(p-)$ generating finite system and will be denoted by $\mathcal{H}^0(p)$, if $\mathcal{J}$ is finite.

2. Some applications

In the sequel we shall fix an optimal measure space $(\Omega, \mathcal{F}, p)$ with $\mathcal{H} = \{H_n : n \in \mathcal{J}\}$ as its generating countable system.

Remark 2.1. If a function $f : \Omega \to \mathbb{R}$ is measurable, then it is constant almost surely on every indecomposable p-atom.

Definition 2.1. By a quasi-optimal measure we mean a set function $q : \mathcal{F} \to \mathbb{R}_+$ satisfying properties P1–P3, with the hypothesis $q(\Omega) = 1$ in P1 replaced by the hypothesis $0 < q(\Omega) < \infty$.

Proposition 2.1. *If $f \geq 0$ is a bounded r.v., then the set function $q_f : \mathcal{F} \to \mathbb{R}_+$, defined by*

$$q_f(B) = \int_B f dp,$$

is a quasi-optimal measure.

(The proof is straightforward.)

Definition 2.2. We shall say that a quasi-optimal measure q is absolutely continuous relative to p (abbreviated $q \ll p$) if $q(B) = 0$ whenever $p(B) = 0$, $B \in \mathcal{F}$.

Proposition 2.2. *Let q be a quasi-optimal measure. Then $q \ll p$ if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $q(B) < \varepsilon$ whenever $p(B) < \delta$, $B \in \mathcal{F}$.*

(The proof is similarly done as in Measure Theory.)

Lemma 2.3. *Let q be a quasi-optimal measure, and let $\mathcal{H}$ be a p-generating countable system. If $q \ll p$ then,*

$$\mathcal{H}^* = \{H \in \mathcal{H} : q(H) > 0\}$$

is a q-generating countable system.

PROOF. Let H be an indecomposable p-atom. Suppose that there exists an event $E \subset H$ with $q(E) = q(H \setminus E) = q(H) > 0$. Since $q \ll p$, it is true that $p(E) > 0$ and $p(H \setminus E) > 0$ contradicting the fact that H is an indecomposable p-atom. Hence we can conclude that any indecomposable p-atom H is also an indecomposable q-atom, whenever $q(H) > 0$, and observe that

$$\mathcal{H}^* := \{H \in \mathcal{H} : q(H) > 0\} = \{H_k \in \mathcal{H} : k \in \mathcal{J}^*\}$$

where $J^* \subseteq \mathcal{J}$ is an index subset.

Let B be any event with $q(B) > 0$. Then, via Lemma 1.6 and the absolute continuity property it follows that

$$q\left(B \setminus \bigcup_{n \in \mathcal{J}^*} (B \cap H_n)\right) = 0.$$

Thus $q(B) = \max\{q(B \cap H_n) : n \in \mathcal{J}^*\}$.

If J^* is countably infinite, then Proposition 2.2 yields that $q(H_n)$ becomes arbitrarily small along with $p(H_n)$ as $n \to \infty$. This ends the proof. □

Theorem 2.4 (Optimal Radon–Nikodym). *Let q be a quasi-optimal measure such that $q \ll p$. Then there exists a unique* r.v. *$f \geq 0$ such that for every event B,*

$$q(B) = \int_B f dp. \tag{2.1.}$$

(This r.v., explicitly given in (2.2), will be called the Optimal Radon–Nikodym derivative of q relative to p and will be denoted by $\frac{dq}{dp}$.)

PROOF. Define the following nonnegative r.v.:

$$f = \max\left\{\frac{q(H_n)}{p(H_n)} \cdot \chi(H_n) : n \in \mathcal{J}\right\}. \tag{2.2}$$

Fix an $n \in \mathcal{J}$ and let $B \in \mathcal{F}$, $p(B) > 0$. Then, Remark 1.1 and the absolute continuity property imply that

$$\frac{q(H_n)}{p(H_n)} \cdot p(B \cap H_n) = \begin{cases} 0, & \text{if } p(B \cap H_n) = 0 \\ q(B \cap H_n), & \text{otherwise.} \end{cases}$$

Hence, by a simple calculation, one can observe that

$$\mathop{\Big\backslash}\limits_{B} f dp = \max\{q(B \cap H_n) : n \in \mathcal{J}\}.$$

Consequently, Lemma 2.3 yields

$$\mathop{\Big\backslash}\limits_{B} f dp = \begin{cases} \max\{q(B \cap H_n) : q(H_n) > 0, n \in \mathcal{J}\}, & \text{if } q(B) > 0 \\ 0 & \text{otherwise,} \end{cases}$$

and thus (2.1) holds.

Let us show that the decomposition (2.1) is unique. Suppose that there exists two r.v.'s $f \geq 0$ and $g \geq 0$ satisfying (2.1). Then, for each event B, we have $\mathop{\Big\backslash}\limits_{B} f dp = \mathop{\Big\backslash}\limits_{B} g dp$. Put $E_1 = (f < g)$ and $E_2 = (g < f)$. Obviously, E_1 and $E_2 \in \mathcal{F}$. If the inequality $p(E_1) > 0$ hold, it would follow that

$$\mathop{\Big\backslash}\limits_{E_1} g dp = \mathop{\Big\backslash}\limits_{E_1} f dp < \mathop{\Big\backslash}\limits_{E_1} g dp,$$

which is impossible. This contradiction yields $p(E_1) = 0$. We can similarly show that $p(E_2) = 0$. These last two equalities imply that $p(f \neq q) = 0$, i.e. the decomposition (2.1) is unique. The theorem is thus proved. □

Let E be a fixed event with $p(E) > 0$. Consider the set function $p^* : \mathcal{F} \to [0, 1]$ defined by $p^*(B) = \dfrac{p(B \cap E)}{p(E)}$. Cleraly, p^* is an optimal measure and $p^* \ll p$. It is obvious that $\dfrac{dp^*}{dp} = \dfrac{\chi(E)}{p(E)} p$ a.s. (by the Optimal Radon–Nikodym theorem).

Definition 2.3. The above set function $p^*(B)$ will be called conditional optimal measure of B given E, and denoted by $p(B \mid E)$.

Definition 2.4. Let $f \in \mathcal{A}^1$ be a r.v. and $E \in \mathcal{F}$, with $p(E)>0$. The conditional optimal average of f given E, is defined by $A(|f| \mid E) := \mathop{\Big\backslash}\limits_{\Omega} |f| dp^*$.

The proofs of the following statements are straightforward:

Lemma 2.5. *If $f \in \mathcal{A}^1$ then, for every event E with $p(E) > 0$,*

$$A(|f| \mid E) = \frac{1}{p(E)} \int_E |f| dp.$$

Proposition 2.6. *If $f \in \mathcal{A}^1$, then*

$$A|f| = \sup_{n \in \mathcal{J}} \{b_n \cdot p(H_n)\}$$

where $b_n = |f(\omega)|$ for almost all $\omega \in H_n$ $(n \in \mathcal{J})$.

Proposition 2.7. *Let $f \in \mathcal{A}^1$ be a r.v. Then, f is bounded a.s. if and only if $b_n \leq \lambda$ $(n \in \mathcal{J})$, where $b_n = |f(\omega)|$ for almost all $\omega \in H_n$ $(n \in \mathcal{J})$ and λ is some positive constant.*

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