Some examples of factors.

To the memory of my teacher and friend, T. Szele.

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The purpose of the present paper is to construct some examples of factors[1]) of type III[2]), which will illustrate that certain phenomena known for factors of type II_1 subsist also in this case. First we establish the existence of non * isomorphic factors of type III by giving the corresponding examples (Theorem 1). This result has been obtained by F. J. MURRAY and J. v. NEUMANN for the case II_1 (R. O. IV, Theorem XVI'). Secondly we give examples of factors of type III, containing particular maximal abelian subrings. Let **N** be a subring of an operator ring **M**, and denote by **T** the subring $\subseteqq \mathbf{M}$ generated by the unitary operators of **M** satisfying $U\mathbf{N}U^* \subseteqq \mathbf{N}$. We say that **N** is *singular* if $\mathbf{T}=\mathbf{N}$, and *semi-regular* if **T** is a factor $\neq \mathbf{M}$. As DIXMIER has recently shown, there exist singular and semi-regular maximal abelian subrings in an approximately finite factor of type II_1 (cf. [1], Théorème 1). In the second part of our paper we intend to give examples of factors of type III, which contain such abelian subrings (cf. Theorem 2).

In the course of our proofs we make large use of the ideas and methods developed in the papers R. O. III and IV, and [1]. Nevertheless, the adaptation of these reasonings to the present case requires often essential modifications.

§ 1. Non * isomorphic factors of type III.

We introduce first the following definition, which is inspired by the Definition 6. 1. 1 in R. O. IV.

Definition. *We say that the ring of operators* **M** *possesses the property L, if there exists a sequence* U_k $(k=1,2,\ldots)$ *of unitary operators in* **M**, *such that* weak $\lim U_k=0$, *and* strong $\lim U_k A U_k^* = A$ *for every* $A \in \mathbf{M}$.

As it is shown in [5], Theorem II and III, the notions strong and weak convergence of an operator sequence are purely algebraic, i. e. they are invariant

[1]) For a theory of the factors in a Hilbert space cf. [3].

[2]) Cf. in particular [5]. We quote the papers [4] and [5] in the sequel as R. O. IV and R. O. III, respectively.

under $*$ isomorphism of operator rings. Therefore every operator ring $*$ isomorphic with another possessing the property L, has itself this property.

Our objective in this section is to give an example of a factor of type III with the property L and then another without this property.

Factors of type III can be obtained in the following way (cf. R. O. III, Chapters III and IV). Consider a totally σ-finite, complete measure space $(X, \mathbf{S}, \mu)$[3]) consisting of a separable σ-ring $\mathbf{S}$[4]) formed by the subsets of a set X, and a σ-finite complete measure defined on $\mathbf{S}$. Let $\mathfrak{G}$ be a countable group acting as a group of one-to-one mappings of X into itself and at the same time as a group of automorphisms of $\mathbf{S}$. For $x \in X$ and $a \in \mathfrak{G}$ we denote the effect of the mapping corresponding to a on x by xa. The measure μ is *quasi-invariant* under $\mathfrak{G}$ if $\mu(E)=0$ for $E \in \mathbf{S}$ implies $\mu(Ea)=0$[5]) for every $a \in \mathfrak{G}$. In this case the "translated measure" μ_a defined for $F \in \mathbf{S}$ by $\mu_a(F)=\mu(Fa)$ is absolutely continuous with respect to μ, thus we can form the Radon—Nikodym derivative $\frac{d\mu_a}{d\mu}(x)$. The group $\mathfrak{G}$ is said to be (i) *free*, if for $a \in \mathfrak{G}, a \neq e$ the set of points satisfying the condition $x = xa$ $(x \in X)$ is of a μ-measure 0, (ii) *ergodic*, if $Ea \subseteq E$ for $E \in \mathbf{S}$ and every $a \in \mathfrak{G}$ implies either $\mu(E)=0$, or $\mu(X-E)=0$, (iii) *non-measurable*, if there exists no σ-finite measure ν, which is equivalent to μ and invariant under $\mathfrak{G}$ (i. e. which satisfies $\nu(Ea)=\nu(E)$ for $E \in \mathbf{S}$ and $a \in \mathfrak{G}$).

Suppose we are given a measure μ and a group $\mathfrak{G}$, such that μ is quasi-invariant, and $\mathfrak{G}$ possesses the properties (i)—(iii). We form the Hilbert space H of the complex-valued functions $F(a, x)$ $(a \in \mathfrak{G}, x \in X)$ satisfying $\sum_{a\in\mathfrak{G}} \int_X |F(a,x)|^2\, d\mu < +\infty$ with the inner product

$$(F, G) = \sum_{a\in\mathfrak{G}} \int_X F(a,x)\overline{G(a,x)}\, d\mu \qquad (F, G \in H).$$

For $a_0 \in \mathfrak{G}$, and an arbitrary complex-valued, bounded and measurable function $\varphi(x)$ we consider the operators $\bar{U}_{a_0}$ and $\bar{L}_{\varphi(x)}$ defined for $F \in H$ by

$$(\bar{U}_{a_0}F)(a,x) \equiv \left(\frac{d\mu_{a_0}}{d\mu}(x)\right)^{\frac{1}{2}} F(aa_0, xa_0),$$

$$(\bar{L}_{\varphi(x)}F)(a,x) \equiv \varphi(x)\, F(a,x) \qquad (a \in \mathfrak{G}, x \in X).$$

Then under the above conditions on μ and $\mathfrak{G}$ the ring of operators $\mathbf{M}$ generated by these operators in the space H is a factor of type III.

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[3]) In general we employ the terminology of [2].

[4]) We say, that $\mathbf{S}$ is *separable*, if there exists a countable subsystem $\mathbf{O}$ of sets in $\mathbf{S}$ with the following properties: (i) $\mathbf{O}$ generates $\mathbf{S}$, (ii) If $x \in E$ is equivalent with $y \in E$ $(x, y \in X)$ for every $E \in \mathbf{O}$, then $x = y$ (cf. R. O. III, Definition 3.2.3).

[5]) We put $Ea = \{xa; x \in E\}$.

The space H can be considered as the direct sum of an infinite number of replicas of the space $L_\mu^2(X)$ of quadratically integrable functions over X with respect to μ (indexed by the elements of $\mathfrak{G}$). So every bounded operator A in H can be represented by an operator matrix, in symbols $A \sim (A_{a,b})$ $(a, b \in \mathfrak{G})$, whose coefficients are operators in $L_\mu^2(X)$. If $A \in \mathbf{M}$, then $A_{a,b} = L_{\varphi_{b^{-1}a}(x)} U_{b^{-1}a}$[6]), where the operators U_a and $L_{\varphi_a(x)}$ $(a \in \mathfrak{G})$ are defined for $f(x) \in L_\mu^2(X)$ by

$$(U_a f)(x) \equiv \left(\frac{d\mu_a}{d\mu}(x) \right)^{\frac{1}{2}} f(xa)$$

$$(L_{\varphi_a(x)} f)(x) \equiv \varphi_a(x) f(x).$$

Here the functions $\varphi_a(x)$ are complex-valued, bounded and measurable. Conversely, to a sequence $\varphi_a(x)$ $(a \in \mathfrak{G})$ of such functions corresponds an operator of $\mathbf{M}$, described by the operator-matrix $(L_{\varphi_{b^{-1}a}(x)} U_{b^{-1}a})$ provided that the latter defines a bounded operator in H. For this operator $A \in \mathbf{M}$ we define $(A)_a \equiv \varphi_a(x)$.

Lemma 1. *Supposing $(A)_a \equiv \varphi_a(x)$ and $(B)_a \equiv \psi_a(x)$ $(a \in \mathfrak{G})$ for $A, B \in \mathbf{M}$, we have the following rules of computation:*

(i) $(\lambda A)_a \equiv \lambda \varphi_a(x)$, ($\lambda$ *arbitrary complex constant*),

(ii) $(A^*)_a \equiv \overline{\varphi_{a^{-1}}(xa)}$,

(iii) $(A+B)_a \equiv \varphi_a(x) + \psi_a(x)$,

(iv) $(AB)_a \equiv \Sigma_{c \in \mathfrak{G}} \varphi_{c^{-1}}(x) \psi_{ca}(xc^{-1})$, *where the last series converges almost everywhere with respect to μ, irrespective of the order of the summation.*[7])

Proof. The statements (i)—(iii) are identical with those of lemma 3. 7. 1 of R. O. III. To prove (iv) we observe that by the statement (iv) loc. cit., the series $\Sigma_{c \in \mathfrak{G}} \varphi_{c^{-1}}(x) \psi_{ca}(xc^{-1})$ converges in measure to $(AB)_a$, irrespective of the order of summation. Since the convergence in measure of a monotonous sequence of functions implies its convergence almost everywhere, we see, that the series $\Sigma_{c \in \mathfrak{G}} |\varphi_c(x)|^2$, and $\Sigma_{c \in \mathfrak{G}} |\varphi_c(xc^{-1})|^2$ converge almost everywhere to $(AA^*)_e$ and $(A^*A)_e$, respectively. Hence by an obvious application of the Cauchy—Schwarz inequality we get that the series $\Sigma_{c \in \mathfrak{G}} \varphi_{c^{-1}}(x) \psi_{ca}(xc^{-1})$ converges also almost everywhere, irrespective of the order of the summation.

The commutant of $\mathbf{M}$, i. e. the ring $\mathbf{M}'$ consisting of the operators permutable with every element of $\mathbf{M}$, is generated by the operators

$$(\bar{V}_{a_0} F)(a, x) \equiv F(a_0^{-1} a, x)$$

$$(\bar{M}_{\varphi(x)} F)(a, x) \equiv \varphi(xa^{-1}) F(a, x) \qquad (F \in H, a_0 \in \mathfrak{G}),$$

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[6]) Cf. lemma 3. 6. 3 (ii) and lemma 3. 6. 4 in R. O. III. For later purposes we write $\varphi_{a^{-1}}(x)$ for $\varphi_a(x)$ loc. cit.

[7]) We use the sign $\equiv$ to denote an identity, which is fulfilled disregarding a set of μ-measure 0.

where $\varphi(x)$ is an arbitrary complex-valued, bounded and measurable function on X. This gives immediately the following

Lemma 2. *Suppose $\mu(X) < +\infty$. Denoting by F_0 the vector of H, for which $F_0(e, x) \equiv 1$, $F_0(a, x) \equiv 0$ $(a \neq e)$, the elements $A' F_0$ $(A' \in M')$ span the space H.*

We proceed now to discuss special examples [8]). We consider the measure space $(X_0, \mathbf{S}_0, \mu_0)$, where X_0 contains the points 0 and 1 only, $\mathbf{S}_0$ consists of all subsets of X_0, $\mu_0(\{0\}) = p$, $\mu_0(\{1\}) = q$, and $q > p > 0, p + q = 1$. For $n = 1, 2, \ldots$ we put $(X_n, \mathbf{S}_n, \mu_n) = (X_0, \mathbf{S}_0, \mu_0)$, form the Cartesian product $(X, \mathbf{S}', \mu') = (\mathbf{X}_{n=1}^{\infty} X_n, \mathbf{X}_{n=1}^{\infty} \mathbf{S}_n, \mathbf{X}_{n=1}^{\infty} \mu_n)$ of these spaces, and take the completion μ of μ'. We denote the measure space obtained in this way by $(X, \mathbf{S}, \mu)$. A point $x \in X$ may be identified with a sequence (x_n) $(n = 1, 2, \ldots)$, where $x_n = 0$ or $= 1$. Defining for $x = (x_n), y = (y_n) \in X$ the sum $x + y$ by the sequence $(x_n + y_n)$ reduced mod 2, X becomes a group. The set $\Delta = \{(x_n);\, x_n \neq 0$ for a finite number of n only$\}$ forms a countable subgroup of X, generated by the elements $\gamma_k = ((\gamma_k)_n)$ $(k = 1, 2, \ldots)$, where $(\gamma_k)_n \neq 0$ only for $n = k$. For $\gamma \in \Delta$ we define a one to one mapping of X onto itself by $x\gamma = x + \gamma$ $(x \in X)$; it is clear, that these mappings represent Δ as a group of automorphism of $\mathbf{S}'$. We show next that μ' is quasi-invariant, which gives immediately that $\mathbf{S}$ is mapped onto itself along with $\mathbf{S}'$ and that the measure μ is quasi-invariant too.

Lemma 3. *The measure μ' is quasi-invariant under Δ.*

PROOF. We have to show that the measure μ'_γ defined by $\mu'_\gamma(E) = \mu'(E\gamma)$ for $\gamma \in \Delta$ and $E \in \mathbf{S}'$, is absolutely continuous with respect to μ. Since every $\gamma \in \Delta$ is the product of some of the generators γ_k, we need to do this for the measures μ'_{γ_k} only. Consider now for a fixed k the function

$$f_k(x) = \begin{cases} \dfrac{p}{q} & \text{if } x_k = 1, \\[2mm] \dfrac{q}{p} & \text{if } x_k = 0 \qquad (x \in X). \end{cases}$$

It suffices plainly to show that for $E \in \mathbf{S}'$ $\mu'_{\gamma_k}(E) = \int_E f_k(x)\, d\mu'$, but taking in view the definition of the product measure μ' we need to consider only the „cylindrical" sets of the form $E = \{x; x_{m_i} = 0,\ x_{n_j} = 1,\ i = 1, 2, \ldots, u, j = 1, 2, \ldots, v\}$. Suppose first that k occurs among the numbers m_i and n_j, $k = m_i$ say. Then

$$\int_E f_k(x)\, d\mu' = \frac{q}{p}\, \mu'(E) = \frac{q}{p}\, p^u q^v = p^{u-1} q^{v+1} = \mu'(E\gamma_k) = \mu'_{\gamma_k}(E).$$

[8]) For the following cf. [2] § 38, in particular the top of page 159 and 4 in the introduction of R. O. III.

The case if $k=n_j$ is quite similar. If k does not occur among m_i and n_j, then

$$\int_E f_k(x)\,d\mu' = \frac{p}{q}\,q\mu'(E) + \frac{q}{p}\,p\mu'(E) = \mu'(E) = \mu'(E\gamma_k) = \mu'_{\gamma_k}(E)$$

So lemma 3 is proved.

Corollary. *The measure μ is quasi-invariant under Δ.*

For this it suffices to observe that every null-set in $\mathbf{S}$ is contained in a null-set of $\mathbf{S}'$.

The proof of lemma 1 gives immediately that $\frac{d\mu_{\gamma_k}}{d\mu}(x) \equiv f_k(x)$. For later use we observe that for $\gamma \in \Delta$ and $\gamma = \gamma_{k_1} + \gamma_{k_2} + \cdots + \gamma_{k_n}$.

$$\frac{d\mu_\gamma}{d\mu}(x) \equiv f_{k_1}(x) f_{k_2}(x) \ldots f_{k_n}(x) \equiv \Pi_{n=1}^{\infty} \left(\frac{p}{q}\right)^{(2x_n-1)\gamma_n},$$

as it is easily seen.

The elements of Δ are in one to one correspondence with the finite subsets of the positive integers. From now on we denote by $\alpha \vee \beta$ and $\alpha \wedge \beta$ the elements $\in \Delta$ corresponding to the union and intersection of the sets belonging to α and β, respectively ($\alpha, \beta \in \Delta$).

Lemma 4. *The system of functions*

$$\omega_\alpha(x) = (-1)^{\sum_{n=1}^{\infty} \alpha_n x_n} \prod_{n=1}^{\infty} \left(\frac{p}{q}\right)^{\left(x_n - \frac{1}{2}\right)\alpha_n} \qquad (\alpha \in \Delta)$$

forms a complete orthonormal system in the space $L^2_\mu(X)$.

PROOF. For $\alpha, \beta \in \Delta$

$$\int_X \omega_\alpha(x)\omega_\beta(x)\,d\mu = \int_X \omega_{\alpha+\beta}(x)\omega^2_{\alpha\wedge\beta}(x)\,d\mu = \int_X \omega_{\alpha+\beta}(x)\,d\mu \int_X \omega^2_{\alpha\wedge\beta}(x)\,d\mu,$$

hence to establish the orthonormality we need only to show that for $\alpha \in \Delta$, $\alpha \neq 0$,

$$\int_X \omega_\alpha(x)\,d\mu = 0, \int_X \omega^2_\alpha(x)\,d\mu = 1.$$

Supposing $\alpha = \gamma_{k_1} + \gamma_{k_2} + \cdots + \gamma_{k_n}$, we have

$$\int_X \omega_\alpha(x)\,d\mu = \Pi_{i=1}^{n} \int_X \omega_{\gamma_{k_i}}(x)\,d\mu,$$

$$\int_X \omega^2_\alpha(x)\,d\mu = \Pi_{i=1}^{n} \int_X \omega^2_{\gamma_{k_i}}(x)\,d\mu.$$

But for every k $\int_X \omega_{\gamma_k}(x)\,d\mu = -q\sqrt{\frac{p}{q}} + p\sqrt{\frac{q}{p}} = \sqrt{pq} - \sqrt{pq} = 0$, and

$$\int_X \omega^2_{\gamma_k}(x)\,d\mu = q\,\frac{p}{q} + p\,\frac{q}{p} = 1.$$

Thus the system $\{\omega_\alpha(x)\}$ is orthonormal.

To prove the completeness it suffices to show that the characteristic functions of the cylindrical sets are linear combinations of members of the system $\{\omega_\alpha(x)\}$. Every cylindrical set is the intersection of a finite number of sets of the form $E_j=\{x; x_j=0\}$ and $F_k=\{x; x_k=1\}$ $(j, k=1, 2, \ldots)$. Denoting the characteristic functions of E_j and F_k by $g_j(x)$ and $h_k(x)$, respectively, we have $g_j(x)\equiv\sqrt{pq}\,\omega_{\gamma_j}(x)+p\,\omega_0(x)$, $h_k(x)\equiv-\sqrt{pq}\,\omega_{\gamma_k}(x)+q\,\omega_0(x)$. Observing that for $\alpha, \beta\in\Delta$ and $\alpha\wedge\beta=0$, $\omega_{\alpha+\beta}(x)\equiv\omega_\alpha(x)\,\omega_\beta(x)$, our result follows immediately.

Lemma 5. *For $f(x)\in L^2_\mu(X)$ we have*

$$\lim_{k\to\infty}\int_X |f(x\gamma_k)-f(x)|^2\,d\mu=0.$$

PROOF. For $g(x)\in L^2_\mu(X)$ we put $\|g(x)\|_2=\left(\int_X |g(x)|^2\,d\mu\right)^{\frac{1}{2}}$. By lemma 4 for every $\varepsilon>0$ we can determine a finite linear combination $\omega(x)\equiv\Sigma_{\alpha\in\Delta'}c_\alpha\omega_\alpha(x)$ (Δ' = a finite subset of Δ), such that $\|f(x)-\omega(x)\|_2<\varepsilon$. If k is sufficiently large, then $\omega_\alpha(x\gamma_k)\equiv\omega_\alpha(x)$ for $\alpha\in\Delta'$, therefore $\omega(x\gamma_k)\equiv\omega(x)$ too. We have further for every $g(x)\in L^2_\mu(X)$

$$\|g(x\gamma_k)\|_2=\left(\int_X |g(x\gamma_k)|^2\,d\mu\right)^{\frac{1}{2}}=\left(\int_X |g(x)|^2\frac{d\mu_{\gamma_k}}{d\mu}(x)\,d\mu\right)^{\frac{1}{2}}\leqq\frac{q}{p}\|g(x)\|_2,$$

since (cf. lemma 3) $\left|\dfrac{d\mu_{\gamma_k}}{d\mu}\right|\leqq\dfrac{q}{p}$, So finally $\|f(x\gamma_k)-f(x)\|_2\leqq\|f(x\gamma_k)-$ $-\omega(x\gamma_k)\|_2+\|\omega(x\gamma_k)-\omega(x)\|_2+\|\omega(x)-f(x)\|_2\leqq\left(1+\dfrac{q}{p}\right)\|f(x)-\omega(x)\|_2\leqq$ $\leqq\left(1+\dfrac{q}{p}\right)\varepsilon$, provided that $k\geqq k_0$, say. This clearly proves our lemma.

Lemma 6. *The group Δ is* (i) *free,* (ii) *ergodic and* (iii) *non-measurable.*

PROOF. Ad (i): For $\gamma\in\Delta$, $\gamma\neq0$ the equality $x\gamma=x$ or $x+\gamma=x$ $(x\in X)$ is impossible.

Ad (ii). It suffices to show, that if $\varphi(x\gamma_k)\equiv\varphi(x)$ $(k=1, 2, \ldots)$ for a bounded measurable function $\varphi(x)$ on X, then $\varphi(x)$ is constant almost everywhere. For this observe first that $\dfrac{d\mu_{\gamma_k}}{d\mu}(x)\equiv\dfrac{q-p}{\sqrt{pq}}\,\omega_{\gamma_k}(x)+1$. Therefore we have for $\alpha\in\Delta$ and $\gamma_k\wedge\alpha=0$: $c_\alpha=\int_X\varphi(x)\,\omega_\alpha(x)\,d\mu=\int_X\varphi(x\gamma_k)\omega_\alpha(x)\,d\mu=$

$$=\int_X\varphi(x)\,\omega_\alpha(x\gamma_k)\frac{d\mu_{\gamma_k}}{d\mu}(x)\,d\mu=\frac{q-p}{\sqrt{pq}}\int_X\varphi(x)\,\omega_{\alpha+\gamma_k}(x)\,d\mu+\int_X\varphi(x)\,\omega_\alpha(x)\,d\mu=$$

$=\frac{q-p}{\sqrt{qp}}c_{\alpha+\gamma_k}+c_\alpha$, that is $c_{\alpha+\gamma_k}=0$. From this it follows at once that $c_\alpha=0$ for $\alpha\neq 0$. By lemma 4 this proves our statement.

Ad (iii). Suppose, there exists a σ-finite measure ν on $\mathbf{S}$, which is equivalent to μ and invariant under the group Δ. Then we have for $E\in\mathbf{S}$: $\nu(E)=\int\limits_E f(x)\,d\mu$ ($\leqq+\infty$) where $f(x)$ is a suitable measurable and positive function on X. We have further $\int\limits_E f(x)d\mu=\nu(E)=\nu(E\gamma_k)=\int\limits_{E\gamma_k} f(x)\,d\mu=$ $=\int\limits_E f(x\gamma_k)\frac{d\mu_{\gamma_k}}{d\mu}(x)$ for every $E\in\mathbf{S}$, which gives $f(x\gamma_k)\frac{d\mu_{\gamma_k}}{d\mu}(x)\equiv f(x)$ almost everywhere with respect to μ ($k=1,2,\ldots$). Suppose now that $f(x)$ is bounded on $F\in\mathbf{S}$, and $\mu(F)>0$. Denoting by $f'(x)$ the function which equals $f(x)$ on the set $F\cup(\bigcup_{k=1}^{\infty}F\gamma_k)$ and vanishes otherwise, $f'(x)$ is bounded on X. We have for every $k=1,2,\ldots$ observing that $\frac{d\mu_{\gamma_k}}{d\mu}(x)$ takes the values $\frac{p}{q}$ and $\frac{q}{p}$ only:

$$\int\limits_X |f'(x\gamma_k)-f'(x)|^2\,d\mu\geqq\int\limits_F|f(x\gamma_k)-f(x)|^2d\mu=\int\limits_F\left|\left(\frac{d\mu_{\gamma_k}}{d\mu}(x)\right)^{-1}-1\right|^2|f(x)|^2d\mu\geqq$$
$$\geqq\left(1-\frac{p}{q}\right)^2\int\limits_F|f(x)|^2\,d\mu.$$

Since $p\neq q$, by lemma 5 this implies $\int\limits_F|f(x)|^2\,d\mu=0$. But then $\mu(F)=0$ and $\nu(F)=0$, which contradicts to our assumption $\mu(F)>0$.

The proof of lemma 6 is completed.

We form the Hilbert space H of the functions $F(\gamma,x)$ $(\gamma\in\Delta, x\in X)$ for which $\sum_{\gamma\in\Delta}\int\limits_X|F(\gamma,x)|^2\,d\mu<+\infty$, with the inner product $(F,G)=$ $=\sum_{\gamma\in\Delta}\int\limits_X F(\gamma,x)\overline{G(\gamma,x)}\,d\mu$ $(F,G\in H)$. By lemma 3 and 6 the operator ring $\mathbf{M}_1$ in H generated by the operators

$$(\bar{U}_\alpha F)(\gamma,x)\equiv\left(\frac{d\mu_\alpha}{d\mu}(x)\right)^{\frac{1}{2}}F(\gamma+\alpha,x\alpha)$$
$$(\bar{L}_{\varphi(x)}F)(\gamma,x)\equiv\varphi(x)\,F(\gamma,x)$$

$(\alpha,\gamma\in\Delta,\ F\in H,\varphi(x)$ arbitrary complex-valued, bounded and measurable function on $X)$ is a factor of type III.

Lemma 7. *The ring* $\mathbf{M}_1$ *possesses the property L.*

PROOF. We have to establish the existence of a sequence U_k $(k=1,2,\ldots)$ of unitary operators in $\mathbf{M}_1$, such that weak lim $U_k=0$ and strong lim $U_k A U_k^*=A$ for every $A\in\mathbf{M}_1$. In the following we show that putting $U_k=\bar{U}_{\gamma_k}$ we obtain a sequence with the required properties. We divide the proof in two parts.

(i) We assert first that weak lim $\bar{U}_{\gamma_k}=0$. To see this we observe that the set $H'=\{F; F(\gamma,x)\equiv 0$, except for $\gamma=\gamma_0$, γ_0 arbitrary in $\Delta\}$ is fundamental in H. But it is evident that, for $F,G\in H'$ and every sufficiently large k $(\bar{U}_{\gamma_k}F,G)=0$, which proves our statement.

(ii) Let A be an arbitrary operator in $\mathbf{M}_1$. We consider the element $F_0\in H$, which satisfies $F_0(0,x)\equiv 1$, $F_0(\gamma,x)\equiv 0$, if $\gamma\neq 0$. By lemma 2 the elements $A'F_0$ $(A'\in\mathbf{M}_1)$ are dense in H. Therefore to prove that strong $\lim \bar{U}_{\gamma_k}A\bar{U}_{\gamma_k}^*=A$, it suffices to show that $\lim\|(\bar{U}_{\gamma_k}A\bar{U}_{\gamma_k}^*)F_0-AF_0\|=0$, since the operators $\bar{U}_{\gamma_k}A\bar{U}_{\gamma_k}^*$ $(k=1,2,\ldots)$ are uniformly bounded in norm. Using the fact that $\bar{U}_{\gamma_k}^*=\bar{U}_{-\gamma_k}=\bar{U}_{\gamma_k}$ (R. O. III, lemma 3.6.2), this is equivalent to $\lim\|(A\bar{U}_{\gamma_k}-\bar{U}_{\gamma_k}A)F_0\|=0$. Observe now that obviously $(\bar{U}_{\gamma_k})_\alpha\equiv\delta_{\gamma_k,\alpha}$, where $\delta_{\alpha,\beta}$ denotes a function on X, which is $\equiv 1$ for $\alpha=\beta$, and $\equiv 0$ otherwise. Supposing $(A)_\alpha\equiv\varphi_\alpha(x)$, we have by (iv) of lemma 1 $(A\bar{U}_{\gamma_k})_\alpha\equiv\varphi_{\alpha+\gamma_k}(x)$, $(\bar{U}_{\gamma_k}A)\equiv\varphi_{\alpha+\gamma_k}(x\gamma_k)$, and by (i) and (iii) of the same lemma $(A\bar{U}_{\gamma_k}-\bar{U}_{\gamma_k}A)_\alpha\equiv$ $\equiv\varphi_{\alpha+\gamma_k}(x)-\varphi_{\alpha+\gamma_k}(x\gamma_k)$ $(\alpha\in\Delta, k=1,2,\ldots)$. If C is arbitrary in $\mathbf{M}_1$ and $(C)_\alpha\equiv\psi_\alpha(x)$, then $C\sim(L_{\psi_{\alpha+\beta}(x)}U_{\alpha+\beta})$. Hence

$$(CF_0)(\alpha,x)\equiv\psi_\alpha(x)\left(\frac{d\mu_\alpha}{d\mu}(x)\right)^{\frac{1}{2}},$$

and so

$$\|CF_0\|^2=\sum_{\alpha\in\Delta}\int_X|\psi_\alpha(x)|^2\frac{d\mu_\alpha}{d\mu}(x)d\mu=\sum_{\alpha\in\Delta}\int_X|\psi_\alpha^{-}(x\alpha)|^2d\mu.$$

In particular for $C=A\bar{U}_{\gamma_k}-\bar{U}_{\gamma_k}A$,

$$\|(A\bar{U}_{\gamma_k}-\bar{U}_{\gamma_k}A)F_0\|^2=\sum_{\alpha\in\Delta}\int_X|\varphi_{\alpha+\gamma_k}(x\alpha)-\varphi_{\alpha+\gamma_k}(x[\alpha+\gamma_k])|^2d\mu=$$

$$=\sum_{\alpha\in\Delta}\int_X|\varphi'_\alpha(x)-\varphi'_\alpha(x\gamma_k)|^2d\mu,$$

provided that $\varphi_\alpha(x\alpha)\equiv\varphi'_\alpha(x)$, Observe in addition that $\|AF_0\|^2=$ $=\sum_{\alpha\in\Delta}\int_X|\varphi_\alpha(x\alpha)|^2d\mu=\sum_{\alpha\in\Delta}\int_X|\varphi'_\alpha(x)^2|d\mu=<+\infty$. Putting again $\|f(x)\|_2=$ $=\left(\int_X|f(x)|^2d\mu\right)^{\frac{1}{2}}$ for $f(x)\in L^2_\mu(X)$, we have

$$\|\varphi'_\alpha(x\gamma_k)-\varphi'_\alpha(x)\|_2\leqq\|\varphi'_\alpha(x\gamma_k)\|_2+\|\varphi'_\alpha(x)\|_2,$$

and

$$||\varphi'_\alpha(x\gamma_k)||_2^2 = \int_X |\varphi'_\alpha(x)|^2 \frac{d\mu_{\gamma_k}}{d\mu}(x)\, d\mu \leqq \frac{q}{p} ||\varphi'_\alpha(x)||_2^2,$$

which gives that $\int_X |\varphi'_\alpha(x\gamma_k) - \varphi'_\alpha(x)|^2 d\mu \leqq \left(1+\frac{q}{p}\right)^2 \int_X |\varphi'_\alpha(x)|^2 d\mu$ $(\alpha \in \Delta, k = 1, 2, \ldots)$. So applying lemma 5 we get finally

$$\lim_{k\to\infty} ||(A\bar{U}_{\gamma_k} - \bar{U}_{\gamma_k}A)F_0||^2 = \sum_{\alpha\in\Delta} \lim_{k\to\infty} \int_X |\varphi'_\alpha(x\gamma_k) - \varphi'_\alpha(x)|^2 d\mu = 0.$$

Thus the proof of lemma 7 is completed.

Our next objective is to give an example of a factor of type III, which does not possess the property L. Some of the constructions which are necessary for this purpose, will be used also later, therefore we describe them under more general assumptions than immadiately needed.[9])

We take a countable infinite group G. Let $(X, \mathbf{S}, \mu)$ be again the measure space used in the previous considerations, but from now on the components X_i $(i = 1, 2, \ldots)$ of the product space X will be indexed by the elements of the group G. Thus every $x \in X$ may be identified with a function (x_g) $(g \in G)$ defined on G taking the values 0 and 1 only.

We denote the set of pairs (α, a) $(\alpha \in \Delta, a \in G)$ by $\mathfrak{G}$. To an element $(\alpha, a) = \mathfrak{a}$ of $\mathfrak{G}$ we associate the mapping $x \to x\mathfrak{a}$ of X onto itself defined by $x\mathfrak{a} = (x_{ag} + \alpha_g)$ [10]) $(x \in X)$.

These mappings are obviously one to one. Introducing the notation $\alpha^a = (\alpha_{ag})$ $(\alpha \in \Delta, a \in G)$, for $\mathfrak{b} = (\beta, b)$ we get $(x\mathfrak{a})\mathfrak{b} = (x_{ag} + \alpha_g)\mathfrak{b} = (x_{abg} + \alpha_{bg} + \beta_g) = x\mathfrak{c}$, where $\mathfrak{c} = (\alpha^b + \beta, ab)$. If $x\mathfrak{a} = x\mathfrak{a}'$ for $\mathfrak{a}, \mathfrak{a}' \in \mathfrak{G}$ and every $x \in X$, then $\mathfrak{a} = \mathfrak{a}'$. Therefore with the law of composition $\mathfrak{a}\mathfrak{b} = (\alpha, a)(\beta, b) = (\alpha^b + \beta, ab) = \mathfrak{c}$ $\mathfrak{G}$ is a semigroup. Observing that for $(\alpha, a) \in \mathfrak{G}$

$$(\alpha, a)(0, e) = (\alpha^e + 0, ae) = (\alpha, a)\ ^{11})$$
$$(0, e)(\alpha, a) = (0^a + \alpha, e \cdot a) = (\alpha, a)$$

and

$$(\alpha, a)(\alpha^{a^{-1}}, a^{-1}) = (\alpha^{a^{-1}} + \alpha^{a^{-1}}, a \cdot a^{-1})$$
$$(\alpha^{a^{-1}}, a^{-1})(\alpha, a) = (\alpha^{a^{-1}a} + \alpha, a^{-1}a) = (0, e)$$

we see that with the unit $(0, e)$ and the inverse $(\alpha, a)^{-1} = (\alpha^{a^{-1}}, a^{-1})$ $\mathfrak{G}$ is a group.

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[9]) For the following cf. the construction in R. O. IV, § 5.5.

[10]) We recall that, for $x = (x_g)$ and $y = (y_g)$, $(x_g + y_g)$ denotes the element of X defined by the function (z_g), which is $= 1$ if $x_g + y_g = 1$, and $= 0$ otherwise.

[11]) We denote by 0 and e the unit elements in Δ and G, respectively.

It is easily verified that the correspondences $\alpha \to (\alpha, e)$ $(\alpha \in \Delta)$ and $a \to (0, a)$ $(a \in G)$ define isomorphisms of the groups Δ and G with subgroups of $\mathfrak{G}$. We denote these subgroups in the sequel again by Δ and G, and their elements by $\alpha, \beta, \ldots$ and $a, b, \ldots$, respectively. (Thus e. g. (α, e) is identified with α). For $\mathfrak{a} = (\alpha, a) \in \mathfrak{G}$ we have $\mathfrak{a} = (0, a)(\alpha, e) = a\alpha$, and the "components" a and α are uniquely determined by $\mathfrak{a}$; so the group $\mathfrak{G}$ may be considered as the semi-direct product of the groups Δ and G. The mappings of X belonging to the elements of Δ are identical with those used in the construction of the ring $\mathbf{M}_1$ (cf. lemma 7).

The mapping $x \to x\mathfrak{a}$ carries a cylindrical subset of X into a similar one, therefore it determines an automorphism of the σ-ring $\mathbf{S}'$ generated by these sets. We prove next that it defines an automorphism of the σ-ring $\mathbf{S}$ too. As in the case of the group Δ (cf. lemma 3) this follows, along with the quasi-invariancy of μ, from the following

Lemma 8. *The measure μ' is quasi-invariant under $\mathfrak{G}$.*

Proof. We have to show that $\mu'(E) = 0$ for $E \in \mathbf{S}'$ implies $\mu'(E\mathfrak{a}) = 0$ $(\mathfrak{a} \in \mathfrak{G})$. Suppose $\mathfrak{a} = a\alpha$ $(a \in G, \alpha \in \Delta)$. The mapping $x \to xa$ for $a \in G$ leaves invariant the measure of a cylindrical set, hence also that of every set in $\mathbf{S}'$. So $\mu'(E) = 0$ implies $\mu'(Ea) = 0$. But applying lemma 3 we get $\mu'(E\mathfrak{a}) = \mu'([Ea]\alpha) = 0$, which proves our lemma.

Corollary. *The measure μ is quasi-invariant under $\mathfrak{G}$.*

Lemma 9. *The group $\mathfrak{G}$ is* (i) *free,* (ii) *ergodic and* (iii) *non-measurable.*

Proof. Ad (i)[12]). We have to prove, that if $\mathfrak{a}$ is not the unit element in $\mathfrak{G}$, then the set E of the elements $x \in X$ satisfying $x = x\mathfrak{a}$, is of μ-measure 0. Suppose that $\mathfrak{a} = a\alpha$ $(a \in G, \alpha \in \Delta)$. If $a = e$, then $\mathfrak{a} \in \Delta$, and this case is already settled in (i) of lemma 6. We assume therefore $a \neq e$, and choose an infinite sequence g_i of elements in G such that the g_i and ag_j are all different, and $\alpha_{g_i} = 0$ $(i, j = 1, 2, \ldots)$. If $x = x\mathfrak{a} = (x_{ag} + \alpha_g)$ for $x = (x_g) \in X$, then we have in particular $x_{ag_i} = x_{g_i}$ $(i = 1, 2, \ldots, m)$, where m is an arbitrarily fixed positive integer. But the measure of the set E_m of elements $\in X$ satisfying this condition is $(p^2 + q^2)^m$. By virtue of our assumptions $q > p > 0$, $p + q = 1$, hence $p^2 + q^2 < 1$. Since m is arbitrary, and clearly $E \subset E_m$, we have necessarily $\mu(E) = 0$.

Ad (ii) and (iii). Since by lemma 6 the subgroup Δ of $\mathfrak{G}$ prossesses these properties, these subsist a fortiori for the group $\mathfrak{G}$.

The following lemma is an adaptation of the reasoning of lemma 6.2.1 in R. O. IV for our purposes.

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[12]) For this cf. the reasoning at the top of p. 795 in [4].

Lemma 10. *Let $\mathfrak{G}$ be a group and $\mathfrak{B}$ a subset of $\mathfrak{G}$. Suppose there exists a subset $\mathfrak{F} \subset \mathfrak{B}$ and two elements $g_1, g_2 \in \mathfrak{G}$ such that* (i) $\mathfrak{F} \cup g_1 \mathfrak{F} g_1^{-1} = \mathfrak{B}$, (ii) *the sets $\mathfrak{F}, g_2^{-1}\mathfrak{F}g_2$ and $g_2\mathfrak{F}g_2^{-1} \subset \mathfrak{B}$ are pairwise disjoint. Let $f(a)$ be a complex-valued function on $\mathfrak{G}$ such that $\sum_{a\in\mathfrak{G}}|f(a)|^2 < +\infty$, and*

$$\left(\sum_{a\in\mathfrak{G}}|f(g_i a g_i^{-1}) - f(g)|^2\right)^{\frac{1}{2}} < \varepsilon \qquad (i = 1, 2).$$

Then $\left(\sum_{a\in\mathfrak{B}}|f(a)|^2\right)^{\frac{1}{2}} < k_1\varepsilon$, where k_1 does not depend on ε.

Proof. We put for a subset $\mathfrak{A} \subset \mathfrak{G}$ $\nu(\mathfrak{A}) = \sum_{a\in\mathfrak{A}}|f(a)|^2$. Then we have by an application of the triangle inequality

$$\varepsilon > \left(\sum_{a\in\mathfrak{G}}|f(g_1 a g_1^{-1}) - f(a)|^2\right)^{\frac{1}{2}} \geqq \left|\nu(g_1\mathfrak{F}g_1^{-1})^{\frac{1}{2}} - \nu(\mathfrak{F})^{\frac{1}{2}}\right|.$$

Putting $\nu(\mathfrak{B}) = s$, this gives that

$$|\nu(g_1\mathfrak{F}g_1^{-1}) - \nu(\mathfrak{F})| = \left|\nu(g_1\mathfrak{F}g_1^{-1})^{\frac{1}{2}} + \nu(\mathfrak{F})^{\frac{1}{2}}\right| \left|\nu(g_1\mathfrak{F}g_1^{-1})^{\frac{1}{2}} - \nu(\mathfrak{F})^{\frac{1}{2}}\right| < 2s\varepsilon,$$

and thus $\nu(g_1\mathfrak{F}g_1^{-1}) < \nu(\mathfrak{F}) + 2s\varepsilon$. Hence

$$s^2 \leqq \nu(g_1\mathfrak{F}g_1^{-1}) + \nu(\mathfrak{F}) < 2(\nu(\mathfrak{F}) + s\varepsilon), \quad \text{or} \quad \nu(\mathfrak{F}) > \frac{s^2}{2} - s\varepsilon.$$

Observing that $\left(\sum_{a\in\mathfrak{G}}|f(g_2 a g_2^{-1}) - f(a)|^2\right)^{\frac{1}{2}} < \varepsilon$ gives by aid of the substitution $a \to g_2^{-1} a g_2$ $\left(\sum_{a\in\mathfrak{G}}|f(g_2^{-1} a g_2) - f(a)|^2\right)^{\frac{1}{2}} < \varepsilon$, we have similarly as before

$$|\nu(g_2\mathfrak{F}g_2^{-1}) - \nu(\mathfrak{F})| < 2s\varepsilon, \quad |\nu(g_2^{-1}\mathfrak{F}g_2) - \nu(\mathfrak{F})| < 2s\varepsilon$$

or

$$\nu(g_2\mathfrak{F}g_2^{-1}) > \nu(\mathfrak{F}) - 2s\varepsilon > \frac{s^2}{2} - 3s\varepsilon, \qquad \nu(g_2^{-1}\mathfrak{F}g_2) > \nu(\mathfrak{F}) - 2s\varepsilon > \frac{s^2}{2} - 3s\varepsilon.$$

Hence finally

$$s^2 = \nu(\mathfrak{B}) \geqq \nu(\mathfrak{F}) + \nu(g_2^{-1}\mathfrak{F}g_2) + \nu(g_2\mathfrak{F}g_2^{-1}) > \frac{3}{2}s^2 - 7s\varepsilon,$$

that is $s < 14\varepsilon$, which proves our lemma.

Lemma 11. *Let G be the free group with the two generators g_1 and g_2. Suppose that $f(g)$ is a complex-valued function on G such that $\sum_{g\in G}|f(g)|^2 < +\infty$ and $\left(\sum_{g\in G}|f(gg_i) - f(g)|^2\right)^{\frac{1}{2}} < \varepsilon$, $(i = 1, 2)$. Then $\left(\sum_{g\in G}|f(g)|^2\right)^{\frac{1}{2}} < k_2\varepsilon$, where k_2 does not depend on ε.*

Proof. We denote by F_1 the set of elements $\in G$ ending with a non-zero power of g_1 in their reduced form. Let F_2' be the set corresponding in the same way to g_2, and $F_2 = F_2' \cup \{e\}$ (e denotes the unit in G). For a subset $A \subset G$ we write again $\nu(A) = \sum_{g \in A} |f(g)|^2$. Then we have

$$\varepsilon > \left(\sum_{g \in A} |f(gg_i) - f(g)|^2\right)^{\frac{1}{2}} \geqq \left| \nu(Ag_i)^{\frac{1}{2}} - \nu(A)^{\frac{1}{2}} \right| \qquad (i = 1, 2)$$

for every subset A of G. Putting $A = F_1, i = 2$, and $A = F_1 g_2, i = 1$, we get

$$\left| \nu(F_1 g_2)^{\frac{1}{2}} - \nu(F_1)^{\frac{1}{2}} \right| < \varepsilon \text{ and } \left| \nu(F_1 g_2 g_1)^{\frac{1}{2}} - \nu(F_1 g_2)^{\frac{1}{2}} \right| < \varepsilon$$

and so $\left| \nu(F_1 g_2 g_1)^{\frac{1}{2}} - \nu(F_1)^{\frac{1}{2}} \right| < 2\varepsilon$. Writing $\nu(G)^{\frac{1}{2}} = s$ and observing that $F_1 g_2 g_1 \subset F_1$, we have

$$\nu(F_1 - F_1 g_2 g_1) = \left| \nu(F_1)^{\frac{1}{2}} - \nu(F_1 g_2 g_1)^{\frac{1}{2}} \right| \left| \nu(F_1)^{\frac{1}{2}} + \nu(F_1 g_2 g_1)^{\frac{1}{2}} \right| < 4s\varepsilon.$$

Similarly we get $\nu(F_2 - F_2 g_1 g_2) < 4s\varepsilon$.

Applying again the same reasonings, it follows that

$$|\nu(F_1 g_2 g_1^2) - \nu(F_1 g_2 g_1)| < 2s\varepsilon.$$

Since $F_1 g_2 g_1^2 \subset F_1 - F_1 g_2 g_1$, we have $\nu(F_1 g_2 g_1) < \nu(F_1 g_2 g_1^2) + 2s\varepsilon < 6s\varepsilon$. Similarly $\nu(F_2 g_1 g_2) < 6s\varepsilon$. So we get finally $s^2 = \nu(G) = \nu(F_1 - F_1 g_2 g_1) + \nu(F_1 g_2 g_1) + \nu(F_2 - F_2 g_1 g_2) + \nu(F_2 g_1 g_2) < 4s\varepsilon + 6s\varepsilon + 4s\varepsilon + 6s\varepsilon = 20s\varepsilon$. Thus $s < 20\varepsilon$, which proves our lemma.

From now on throughout the present section we suppose that the group G in the construction described before the lemma 8 is the free group with the two generators g_1 and g_2.

Lemma 12. *Suppose, that the function $f(x)$ $(x \in X)$ is quadratically integrable with respect to μ. Suppose further*

$$\left(\int_X |f(xg_i) - f(x)|^2 \, d\mu\right)^{\frac{1}{2}} < \varepsilon \qquad (i = 1, 2).$$

Then

$$\left| \left(\int_X |f(x)|^2 \, d\mu\right)^{\frac{1}{2}} - \left| \int_X f(x) \, d\mu \right| \right| < k_3 \varepsilon$$

where k_3 does not depend on ε.

Proof. Let $f(x) \sim \sum_{\alpha \in \Delta} c_\alpha \omega_\alpha(x)$ be the expansion of $f(x)$ in terms of the system $\{\omega_\alpha(x)\}$ (cf. lemma 4). Since the mappings $x \to xg$ $(x \in X)$ leave invariant the measure μ (cf. lemma 8) and $\omega_\alpha(xg) = \omega_{\alpha^{g^{-1}}}(x)$[13]) for $\alpha \in \Delta$ and $g \in G \subset \mathfrak{G}$, we have

$$\int_X f(xg_i) \, \omega_\alpha(x) \, d\mu = \int_X f(x) \, \omega_\alpha(xg_i^{-1}) \, d\mu = \int_X f(x) \, \omega_{\alpha^{g_i}}(x) \, d\mu = c_{\alpha^{g_i}},$$

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[13]) We recall, that for $\alpha = \{\alpha_g\}$ and $a \in G$ $\alpha^a = (\alpha_{ag})$.

hence $f(xg_i) \sim \sum_{\alpha \in \Delta} c_{\alpha^{g_i}} \omega_\alpha(x)$ and so

$$\Big(\int_X |f(xg_i) - f(x)|^2 \, d\mu\Big)^{\frac{1}{2}} = \Big(\sum_{\alpha \in \Delta} |c_{\alpha^{g_i}} - c_\alpha|^2\Big)^{\frac{1}{2}} \qquad (i = 1, 2).$$

For $\alpha, \beta \in \Delta$ we write $\alpha \sim \beta$ if there exists a $g \in G$ such that $\alpha^g = \beta$. It is easily verified, that in this way we obtain an equivalence relation on the set of the elements of Δ. We denote by Λ the totality of the equivalence classes not containing the null element 0 of Δ. If α_λ is an element of the class $\lambda \in \Lambda$, then every element of λ can be written uniquely in the form α_λ^g $(g \in G)$.[14] We introduce now the function $f^{(\lambda)}(g) = c_{\alpha_\lambda^g}$ $(\lambda \in \Lambda,\ g \in G)$. Putting

$$a_\lambda = \Big(\sum_{g \in G} |f^{(\lambda)}(g)|^2\Big)^{\frac{1}{2}}, \quad b_\lambda = \sup_{i=1,2} \Big(\sum_{g \in G} |f^{(\lambda)}(gg_i) - f^{(\lambda)}(g)|^2\Big)^{\frac{1}{2}},$$

we have $\sum_{\lambda \in \Lambda} b_\lambda^2 < 2\varepsilon^2$, and by lemma 11 $a_\lambda \leqq k_2 b_\lambda$. Hence

$$\Big| \int_X |f(x)|^2 \, d\mu - \Big| \int_X f(x) \, d\mu \Big|^2 \Big| = \sum_{\substack{\alpha \in \Delta \\ \alpha \neq 0}} |c_\alpha|^2 = \sum_{\lambda \in \Lambda} \Big(\sum_{g \in G} |c_{\alpha_\lambda^g}|^2\Big) =$$
$$= \sum_{\lambda \in \Lambda} a_\lambda^2 \leqq k_2^2 \sum_{\lambda \in \Lambda} b_\lambda^2 < 2 \cdot k_2^2 \varepsilon^2,$$

and so finally

$$\Big| \Big(\int_X |f(x)|^2 \, d\mu\Big)^{\frac{1}{2}} - \Big| \int_X f(x) \, d\mu \Big| \Big| < \sqrt{2 \cdot k_2}\, \varepsilon$$

which proves our lemma.

We arrive now to the final step. We form the Hilbert space H of the functions $F(a, x)$ $(a \in \mathfrak{G}, x \in X)$ satisfying $\|F\|^2 = \sum_{a \in \mathfrak{G}} \int_X |F(a, x)|^2 \, d\mu < +\infty$. By virtue of lemma 8 and 9 the ring of operators $\mathbf{M}_2$ generated in H by the operators

$$(\bar{U}_{a_0} F)(a, x) \equiv \Big(\frac{d\mu_{a_0}(x)}{d\mu}\Big)^{\frac{1}{2}} F(aa_0, xa_0)$$
$$(L_{\varphi(x)} F)(a, x) \equiv \varphi(x) F(a, x)$$

$(F \in H, a_0 \in \mathfrak{G}, \varphi(x)$ arbitrary complex-valued, bounded and measurable function on $X)$ is a factor of type III.

Lemma 13. *The ring $\mathbf{M}_2$ does not possess the property L.*

PROOF. We consider the element F_0 of the space H for which $F_0(e, x) \equiv 1$ and $F_0(a, x) \equiv 0$, if $a \neq e$ $(a \in \mathfrak{G})$. We denote again by g_1 and g_2

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[14]) Indeed, suppose that $\alpha^g = \alpha^{g'}$ and $g \neq g'$. Then $\alpha^{gg'^{-1}} = \alpha$; putting $B = \{h;\ \alpha_h = 1,\ h \in G\}$, and $gg'^{-1} = a$ we have $aB = B$. Since B is finite, this is possible only if a of finite order, which occurs in our case only for $a = e$.

the generators of the group G, which is now a subgroup of $\mathfrak{G}$. To prove our lemma it suffices obviously to show that $\|(\bar{U}_{g_i} - U\bar{U}_{g_i}U^*)F_0\| < \varepsilon$ $(i=1, 2)$ for a unitary U in $\mathbf{M}_2$ and a sufficiently small ε implies $|(UF_0, F_0)| \geqq \frac{1}{2}$. Supposing $(U)_a \equiv \varphi_a(x)$, we have by (ii) of lemma 1 $(U^*)_a \equiv \overline{\varphi_{a^{-1}}(xa)}$. Since plainly $(U_{g_i})_a \equiv \delta_{a, g_i}$, where δ_{a, g_i} denotes a function on X, which is $\equiv 1$ for $a = g_i$, and $\equiv 0$ otherwise, the (iv) of lemma 1 gives

$$(U^*\bar{U}_{g_i})_a \equiv \overline{\varphi_{g_i a^{-1}}(xag_i^{-1})},\ (\bar{U}_{g_i}U^*)_a \equiv \overline{\varphi_{a^{-1}g_i}(xa)},$$

and by (i) and (iii) of the same lemma

$$(U^*\bar{U}_{g_i} - \bar{U}_{g_i}U^*)_a \equiv \overline{\varphi_{g_i a^{-1}}(xag_i)} - \overline{\varphi_{a^{-1}g_i}(xa)}\ \ (a \in \mathfrak{G}, i=1, 2).$$

Hence, similarly as in lemma 7,

$$\|(\bar{U}_{g_i} - U\bar{U}_{g_i}U^*)F_0\|^2 = \|(U^*\bar{U}_{g_i} - \bar{U}_{g_i}U^*)F_0\|^2 =$$
$$= \sum_{a\in\mathfrak{G}} \int_X |\varphi_{g_i a}(xg_i^{-1}) - \varphi_{ag_i}(x)|^2 d\mu = \sum_{a\in\mathfrak{G}} \int_X |\varphi_{g_i a g_i^{-1}}(xg_i^{-1}) - \varphi_a(x)|^2 d\mu.$$

Putting $f(a) = \Big(\int_X |\varphi_a(x)|^2 d\mu\Big)^{\frac{1}{2}}$, and using the fact, that the measure μ is invariant under the mappings of the subgroup $G \subset \mathfrak{G}$, we get from the last equation by an application of the triangle inequality

$$\Big(\sum_{a\in\mathfrak{G}} |f(g_i a g_i^{-1}) - f(a)|^2\Big)^{\frac{1}{2}} \leqq \|(U^*\bar{U}_{g_i} - \bar{U}_{g_i}U^*)F_0\| < \varepsilon.$$

We put now $\mathfrak{B} = \{(\alpha, g); g \neq e\} \subset \mathfrak{G}$ and $\mathfrak{F} = \{(\alpha, g); g \in F_1\} \subset \mathfrak{B}$ (for the definition of F_1 cf. lemma 11). For $(\alpha, a) \in \mathfrak{G}$ and $g \in G$ we have $(0, g)(\alpha, a)(0, g)^{-1} = (\alpha, ga)(0, g^{-1}) = (\alpha_{g^{-1}}, gag^{-1})$. Since $g_1F_1g_1^{-1} \cup F_1 = G - \{e\}$, and the sets $F_1, g_2F_1g_2^{-1}, g_2^{-1}F_1g_2$ are pairwise disjoint, it can be seen easily, that $\mathfrak{F} \cup g_1\mathfrak{F}g_1^{-1} = \mathfrak{B}$, and the sets $\mathfrak{F}, g_2\mathfrak{F}g_2^{-1}, g_2^{-1}\mathfrak{F}g_2 \subset \mathfrak{B}$ are also pairwise disjoint. We have in addition $\Big(\sum_{a\in\mathfrak{B}} |f(g_i a g_i^{-1}) - f(a)|^2\Big)^{\frac{1}{2}} < \varepsilon$, $(i=1, 2)$. Thus an application of lemma 10 gives

$$\Big(\sum_{a\in\mathfrak{B}} |f(a)|^2\Big)^{\frac{1}{2}} < k_1\varepsilon.$$

Since $g_i\alpha g_i^{-1} = \alpha^{g_i^{-1}}$, we have further $\Big(\sum_{\alpha\in\Delta} |f(\alpha^{g_i^{-1}}) - f(\alpha)|^2\Big)^{\frac{1}{2}} < \varepsilon$ $(i=1, 2)$. Applying the reasoning detailed in the proof of lemma 12, we get

$$\Big(\sum_{\substack{\alpha\in\Delta \\ \alpha\neq 0}} |f(\alpha)|^2\Big)^{\frac{1}{2}} < k_3\varepsilon.$$

Observing that

$$1 = \|U^*F_0\|^2 = \sum_{a\in\mathfrak{G}} \int_X |\varphi_a(x)|^2 d\mu = \sum_{a\in\mathfrak{G}} |f(a)|^2$$

we have from the previous considerations

$$\Big(\int_X |\varphi_e(x)|\, d\mu\Big)^{\frac{1}{2}} > 1 - k\varepsilon,$$

where k does not depend on ε. We have also

$$\Big(\int_X |\varphi_e(xg_i) - \varphi_e(x)|^2\, d\mu\Big)^{\frac{1}{2}} < \varepsilon$$

$(i = 1, 2)$. Applying lemma 12 we obtain

$$|(\int_X |\varphi_e(x)|^2\, d\mu)^{\frac{1}{2}} - |\int_X \varphi_e(x)\, d\mu|| < k_3\varepsilon$$

and so

$$|(UF_0, F_0)| = |\int_X \varphi_e(x)\, d\mu| > 1 - K\varepsilon$$

(K independent of ε). Therefore, if $\varepsilon < \frac{1}{2K}$ then $|(UF_0, F_0)| \geqq \frac{1}{2}$ which proves our lemma.

Taking in view that the property L of an operator-ring in a Hilbert space is purely algebraic (cf. the remarks at the begin of the present section) we get from lemma 7 and 11 the following

Theorem 1. *There exist factors of type* III, *wich are not* $*$ *isomorphic.*

REMARK. If $p = q = \frac{1}{2}$ in our construction, then $\mathbf{M}_1$ and $\mathbf{M}_2$ are non $*$ isomorphic factors of type II_1.

§ 2. Semi-regular and singular maximal abelian rings in factors of type III.

In this section we use again the construction described before lemma 8; the group G will be specified later. We denote by $\mathbf{M}$ the factor of type III obtained in this way.

Lemma 14. *Suppose that* G_0 *is an abelian subgroup of* G *such that for each* $a \in G, a \bar{\in} G_0$ *the set* $\{g^{-1}ag; g \in G_0\}$ *contains infinitely many different elements of* G. *Then the operator ring* $\mathbf{P} \subset \mathbf{M}$ *generated by the operators* $\bar{U}_g$ $(g \in G_0)$ *is maximal abelian in* $\mathbf{M}$.[15])

PROOF. We have to prove, that if $A \in \mathbf{M}$ commutes with the operators $\bar{U}_g$ $(g \in G_0)$ then $(A)_a$ is constant for each $a \in G_0$ and vanishes identically if $a \in G - G_0$. Supposing $(A)_a \equiv \varphi_a(x)$ we have as in lemma 13 $(A\bar{U}_g)_a \equiv$

[15]) Cf. lemma 1 in [1].

$\equiv \varphi_{ag^{-1}}(x), (\bar{U}_g A)_a \equiv \varphi_{g^{-1}a}(xg)$ $(a \in \mathfrak{G})$. If $g \in G_0$, then $A\bar{U}_g = \bar{U}_g A$, hence in particular $\varphi_{ag^{-1}}(x) \equiv (A\bar{U}_g)_a = (\bar{U}_g A)_a \equiv \varphi_{g^{-1}a}(xg)$, or $\varphi_a(x) \equiv \varphi_{g^{-1}ag}(xg)$ for every $a \in \mathfrak{G}$. Suppose, that $a \bar{\in} G_0$, and $a = (\alpha, \bar{a})$, then $g^{-1}ag = (0, g^{-1})(\alpha, \bar{a})(0, g) = (\alpha^g, g^{-1}\bar{a}g)$. Since $a \bar{\in} G_0$, we have either $\alpha \neq 0$, or $\bar{a} \bar{\in} G_0$, in both cases the set $\{g^{-1}ag;\ g \in G_0\}$ contains infinitely many different elements. Observing again, that the mapping $x \to xg$ $(g \in G)$ leaves invariant the measure μ, we have

$$\int_X |\varphi_{g^{-1}ag}(x)|^2\, d\mu = \int_X |\varphi_{g^{-1}ag}(xg)|^2\, d\mu = \int_X |\varphi_a(x)|^2\, d\mu.$$

Since the series $\sum_{c \in \mathfrak{G}} \int_X |\varphi_c(x)|^2\, d\mu$ converges, it follows, that $\int_X |\varphi_a(x)|^2 d\mu = 0$, and so $\varphi_a(x) \equiv 0$, for $a \bar{\in} G_0$.

Suppose[16]) now, that $a \in G_0$. We have in this case $\varphi_a(xg) \equiv \varphi_a(x)$ for every $g \in G_0$. Let $\varphi_a(x) \sim \Sigma_{\alpha \in \Delta} c_\alpha \omega_\alpha(x)$ be the expansion of $\varphi_a(x)$ in terms of the system $\{\omega_\alpha(x)\}$. Then $c_\alpha = \int_X \varphi_a(xg)\omega_\alpha(x)\, d\mu = \int_X \varphi_a(x)\, \omega_\alpha(xg^{-1}) d\mu = \int_X \varphi_a(x)\, \omega_{\alpha^g}(x)\, d\mu = c_{\alpha^g}$ $(g \in G)$. If $\alpha \neq 0$, then the set $\{\alpha^g;\ g \in G_0\}$ is infinite, hence $c_\alpha = 0$, and so $\varphi_a(x) \equiv 0$. Thus lemma 14 is proved.

Lemma 15. *Suppose that U is a unitary operator in* $\mathbf{M}$, *and F a subset of $\mathfrak{G}$ such that the Radon—Nikodym derivatives $\frac{d\mu_\alpha}{d\mu}(x)$ $(a \in F)$ are uniformly bounded. Then for $\varepsilon > 0$ there exists a finite subset $B_0 \subset \mathfrak{G}$ such that for every finite $B \supseteq B_0$ and $\delta > 0$ one can determine a set of functions $\varphi'_c(x)$ $(c \in B)$ with the following properties:* 1) *each $\varphi'_c(x)$ is a finite linear combination of functions of the system $\{\omega_\alpha(x)\}$,* 2) $\|(U)_c - \varphi'_c(x)\|_2 < \delta$[17]) $(c \in B)$, 3) *putting*

$$U' = \sum_{c \in B} \bar{L}_{\varphi'_c(x)} \bar{U}_c,$$

we have

$$\|(U\bar{U}_g U^*)_a - (U'\bar{U}_g U'^*)_a\|_2 < \varepsilon$$

for every $g \in \mathfrak{G}$, and $a \in F$.

Proof. This will be given in two steps.

(i) Suppose that $(U_a) \equiv \varphi_a(x)$ $(a \in \mathfrak{G})$. We have by lemma 1 $(U\bar{U}_g)_a \equiv \varphi_{ag^{-1}}(x)$, $(U^*)_a \equiv \overline{\varphi_{a^{-1}}(xa)}$, and so $(U\bar{U}_g U^*)_a \equiv \Sigma_{c \in \mathfrak{G}} \varphi_{cg^{-1}}(x)\overline{\varphi_{a^{-1}c}(xa)} \equiv \Sigma_{c \in \mathfrak{G}} \varphi_c(x) \overline{\varphi_{a^{-1}cg}(xa)}$; these series converge almost everywhere. Let B be a finite subset of $\mathfrak{G}$. Putting $V = \Sigma_{c \in B} \bar{L}_{\varphi_c(x)} \bar{U}_c$, we have $(V\bar{U}_g V^*)_a \equiv \Sigma_{c \in B,\ a^{-1}cg \in B} \varphi_c(x)\overline{\varphi_{a^{-1}cg}(xa)}$ if $B \cap aBg^{-1}$ is not void, and $(V\bar{U}_g V^*)_a \equiv 0$ otherwise.

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16) For this reasoning cf. R. O. IV p. 796, (III).

17) $\|\ \|_2$ denotes the norm in the space $L^2_\mu(X)$.

Hence $(U\bar{U}_g U)_a - (V\bar{U}_g V^*)_a \equiv \Sigma_{\substack{c \in B \\ a^{-1}cg \in B}} \varphi_c(x)\overline{\varphi_{a^{-1}cg}(xa)} + \Sigma_{c \overline{\in} B}\varphi_c(x)\overline{\varphi_{a^{-1}cg}(xa)} \equiv$ $\equiv \Sigma_I + \Sigma_{II}$ $(a, g \in \mathfrak{G})$. Observe that

$$\sum\nolimits_{c \in \mathfrak{G}} |\varphi_c(x)|^2 \equiv (UU^*)_e \equiv (I)_e \equiv 1.$$

Therefore

$$|\sum\nolimits_I|^2 \leqq (\sum\nolimits_{c \in B} |\varphi(x)|^2)(\sum\nolimits_{c' \overline{\in} B} |\varphi_{c'}(xa)|^2) \leqq$$
$$\leqq \sum\nolimits_{c \overline{\in} B} |\varphi_c(xa)|^2$$

almost everywhere. Suppose, that $\frac{d\mu_a}{d\mu}(x) \leqq J^2$ $(a \in F)$ and choose B_0 such that $\sum_{c \overline{\in} B_0} \int\limits_X |\varphi_c(x)|^2 d\mu < \eta^2$. Then we have for a fixed finite $B \supseteqq B_0$

$$\|\sum\nolimits_I\|_2 \leqq \left(\sum\nolimits_{c \overline{\in} B} \int\limits_X |\varphi_c(xa)|^2 d\mu\right)^{\frac{1}{2}} = \left(\sum\nolimits_{c \overline{\in} B} \int\limits_X |\varphi_c(x)|^2 \frac{d\mu_{a^{-1}}}{d\mu}(x)\, d\mu\right)^{\frac{1}{2}} < J\eta,$$

for every $a \in F$ and $g \in \mathfrak{G}$.[18]) Similarly

$$|\sum\nolimits_{II}|^2 \leqq (\sum\nolimits_{c \overline{\in} B} |\varphi_c(x)|^2)(\sum\nolimits_{c \overline{\in} B} |\varphi_{a^{-1}cg}(xa)|^2) \leqq \sum\nolimits_{c \overline{\in} B} |\varphi_c(x)|^2$$

almost everywhere. Hence $\|\sum_{II}\|_2 \leqq \left(\sum_{c \overline{\in} B} \int\limits_X |\varphi_c(x)|^2 d\mu\right)^{\frac{1}{2}} < \eta$. Finally, if $\eta = \frac{\varepsilon}{2(J+1)}$: $\|(U\bar{U}_g U^*)_a - (V\bar{U}_g V^*)_a\|_2 \leqq \|\sum_I\|_2 + \|\sum_{II}\|_2 < (1+J)\eta = \frac{\varepsilon}{2}$ for every $a \in F$ and $g \in \mathfrak{G}$.

(ii) We consider again the set $B \supseteqq B_0$ and the corresponding operator V determined above. For each $\varphi_c(x)$ $(c \in B)$ we can find a finite linear combination $\varphi_c'(x)$ of the functions $\{\omega_a(x)\}$ such that $|\varphi_c'(x)| \leqq 1$ and $\|\varphi_c(x) - \varphi_c'(x)\|_2 < \vartheta$, where ϑ is a given positive number. Indeed, since $|\varphi_c(x)| \leqq 1$ almost everywhere, there exists a finite linear combination $\varphi_c'(x)$ of characteristic functions of cylindrical sets, such that $|\varphi_c'(x)| \leqq 1$ and $|\varphi_c(x) - \varphi_c'(x)| < \frac{\vartheta}{\sqrt{2}}$, except on a set E with a measure $\mu(E) < \frac{\vartheta}{\sqrt{8}}$. We have in this case $\|\varphi_c(x) - \varphi_c'(x)\|_2^2 = \int\limits_{X-E} |\varphi_c(x) - \varphi_c'(x)|^2 d\mu + \int\limits_X |\varphi_c(x) - \varphi_c'(x)|^2 d\mu <$ $< \left(\frac{\vartheta}{\sqrt{2}}\right)^2 + 4\left(\frac{\vartheta}{\sqrt{8}}\right)^2 = \vartheta^2$, or $\|\varphi_c(x) - \varphi_c'(x)\|_2 < \vartheta$. But the characteristic function of a cylindrical set is a finite linear combination of the functions $\omega_a(x)$ (cf.

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[18]) If $\frac{d\mu_a}{d\mu}(x) \leqq J^2$, then we have $\frac{d\mu_{a^{-1}}}{d\mu}(x) \leqq J^2$ too. Supposing $a = (\varkappa, k)$, we have $a^{-1} = (\varkappa^{k^{-1}}, k^{-1})$. The upper bound for $\frac{d\mu_{a^{-1}}}{d\mu}(x)$ depends only on the number of elements in the set $\{g; \varkappa_g = 1,\ g \in G\}$, (cf. lemma 3 and 8). But this is the same for the set $\{g: \varkappa_g^{k^{-1}} = 1,\ g \in G\}$.

lemma 4), therefore the same is true for the functions $\varphi_c'(x)$ $(c \in B)$. Putting $U' = \sum_{c \in B} \bar{L}_{\varphi_c'(x)} \bar{U}_c$ and $B' = B \cap aBg^{-1}$, for $a \in F$ and $g \in \mathfrak{G}$, we have

$$(V\bar{U}_g V^*)_a - (U'\bar{U}_g U'^*)_a \equiv \sum_{c \in B'} (\varphi_c(x)\overline{\varphi_{a^{-1}cg}(xa)} - \varphi_c'(x)\overline{\varphi'_{a^{-1}cg}(xa)}) \equiv$$
$$\equiv \sum_{c \in B'} (\varphi_c(x) - \varphi_c'(x))\overline{\varphi_{a^{-1}cg}(xa)} + \sum_{c \in B'} \varphi_c(xa)\overline{(\varphi_{a^{-1}cg}(xa) - \varphi'_{a^{-1}cg}(xa))} \equiv$$
$$\equiv \sum_I + \sum_{II},$$

(if B' is void, then $(V\bar{U}_g V^*)_a - (U'\bar{U}_g U'^*)_a \equiv 0$). We have further

$$\| \sum_I \|_2 = \| \sum_{c \in B'} (\varphi_c(x) - \varphi_c'(x))\overline{\varphi_{a^{-1}cg}(xa)} \|_2 \leqq \sum_{c \in B'} \| \varphi_c(x) - \varphi_c'(x) \|_2 < n\vartheta$$

where n denotes the number of elements in B. Observe that

$$\|\varphi_c(xa) - \varphi_c'(xa)\|_2^2 = \int_X |\varphi_c(xa) - \varphi_c'(xa)|^2 \, d\mu = \int_X |\varphi_c(x) - \varphi_c'(x)|^2 \frac{d\mu_{a^{-1}}}{d\mu}(x) \, d\mu \leqq$$
$$\leqq J^2 \|\varphi_c(x) - \varphi_c'(x)\|_2^2.$$

Therefore using $|\varphi_c'(x)| \leqq 1$,

$$\| \sum_{II} \|_2 = \| \sum_{c \in B'} \varphi_c'(x)\overline{(\varphi_{a^{-1}cg}(xa) - \varphi'_{a^{-1}cg}(xa))} \|_2 \leqq$$
$$\leqq \sum_{c \in B} \| \varphi_c(xa) - \varphi_c'(xa) \|_2 < Jn\vartheta.$$

Hence $\|(V\bar{U}_g V^*)_a - (U'\bar{U}_g U'^*)_a\|_2 \leqq \| \sum_I \|_2 + \| \sum_{II} \|_2 < n(1+J)\vartheta$, for every $g \in \mathfrak{G}$ and $a \in F$. Choosing $\vartheta = \min\left(\delta, \frac{\varepsilon}{2(J+1)n}\right)$, (i) and (ii) give finally

$$\|(U\bar{U}_g U^*)_a - (U'\bar{U}_g U'^*)_a\|_2 \leqq \|(U\bar{U}_g U^*)_a - (V\bar{U}_g V^*)_a\|_2 +$$
$$+ \|(V\bar{U}_g V^*)_a - (U'\bar{U}_g U'^*)_a\|_2 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

for every $g \in \mathfrak{G}$ and $a \in F$, and at the same time we have $\| \varphi_c(x) - \varphi_c'(x)\|_2 < \delta$ $(c \in B)$.

Thus the proof of lemma 15 is completed.

We suppose in the sequel that the group G occuring in the construction of $\mathbf{M}$ has the following properties: 1) there exists an abelian normal subgroup G_0 of G such that for $a \bar{\in} G_0$ the set $\{x^{-1}ax;\ x \in G_0\}$ contains infinitely many different elements of G, 2) for a finite subset $B \subset G$ one can find an element $g \in G_0$ such that the equality $ugv^{-1} = g$ $(u, v \in B)$ is possible only for $u = v \in G_0$.[19])

In this case we have the

Lemma 16. *Suppose that B is a finite subset of $\mathfrak{G}$. Then there exists an infinity of elements $g \in G_0$ such that 1) the equality $ugv^{-1} = g$ $(u, v \in B)$ implies $u = v \in G_0$, 2) If $a_0 \bar{\in} G$ is a fixed element, then we have in addition $a_0 g a_0^{-1} \bar{\in} G$.*

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[19]) Cf. [1], lemma 3.

PROOF. Let $u_i=(\alpha_i, a_i)$ $(i=1,2,\ldots,n)$ be the elements of B and $a_0=(\delta, d)$, where $\delta \neq 0$. It can be easily seen that, by virtue of our conditions on G, there exists an infinity of elements $g \in G_0$, such that the equality $a_i g a_j^{-1}=g$ for any i,j, $1 \leqq i,j \leqq n$ implies $a_i=a_j \in G_0$. Therefore we may assume g to be chosen in such a way that $\delta^g+\delta \neq 0$, and $\alpha_i \wedge \alpha_j^g=0$ $(i,j=1,2,\ldots,n)$. But $u_i g u_j^{-1}=(\alpha_i, a_i)(0,g)(\alpha_j, a_j)^{-1}=(\alpha_i^g, a_i g)(\alpha_j^{a_j^{-1}}, a_j^{-1})=$ $=(\alpha_i^{ga_j^{-1}}+\alpha_j^{a_j^{-1}}, a_i g a_j^{-1})$, therefore $u_i g u_j^{-1}=g$ would imply $\alpha_i^g+\alpha_j=0$ and $a_i g a_j^{-1}=g$, which is possible only for $a_i=a_j$ and $\alpha_i=\alpha_j=0$, that is $u_i=u_j \in G_0$. We have in addition $a_0 g a_0^{-1}=(\delta, d)(0,g)(\delta, d)^{-1}=(\delta^{gd^{-1}}+\delta^{d^{-1}}, dgd^{-1})$ which shows immediately, that in this case $a_0 g a_0^{-1} \overline{\in} G$. Finally it is clear, that there exists an infinity of elements in G_0 possessing the properties of g. We arrive now to the final step. We suppose, that the group G satisfies the conditions described before lemma 16, and denote by $\mathbf{M}$ the corresponding factor of type III.

Lemma 17. *Denote by* $\mathbf{P}$ *and* $\mathbf{R}$ *the subrings of* $\mathbf{M}$ *generated by the sets of operators* $\{\bar{U}_g; g \in G_0\}$ *and* $\{\bar{U}_g; g \in G\}$, *respectively. Let* U *be a unitary operator in* $\mathbf{M}$, *such that* $U\mathbf{P}U^* \subseteqq \mathbf{P}$. *Then* $U \in \mathbf{R}$.

PROOF. We have to show, that under the above conditions $(U)_a$ is constant for each $a \in G$, and vanishes identically elsewhere. We suppose, that for $a_0 \in \mathfrak{G}$ $(U)_{a_0}$ does not possess this property, and show, that in this case we get a contradiction.

Let $\varphi_a(x) \sim \Sigma_{\alpha \in \Delta} f(a, \alpha)\, \omega_\alpha(x)$ be the expansion of $(U)_a \equiv \varphi_a(x)$ in terms of the system $\{\omega_\alpha(x)\}$ $(a \in \mathfrak{G})$. Since $\sum_{a \in \mathfrak{G}} \int_X |\varphi_a(x)|^2 d\mu=1$ (cf. e. g. lemma 15), we have $\sum_{\substack{c \in \mathfrak{G} \\ \alpha \in \Delta}} |f(c,\alpha)|^2=1$. We put $\varrho(H,\gamma)=(\Sigma_{c \in H}|f(c,\gamma)|^2)^{\frac{1}{2}}$ for a subset $H \subseteqq \mathfrak{G}$. Let γ_0 be an element $\in \Delta$ such that $\varrho(a_0 G_0, \gamma_0)>0$ and $\varrho(a_0 G_0, \gamma)<$ $<\varrho(a_0 G_0, \gamma_0)$ for $\gamma \geqq \gamma_0$.[20]) If $a_0 \overline{\in} G$ then we may suppose $\gamma_0 \neq 0$. We choose next a finite subset $B' \subset a_0 G_0$, for which $\varrho(B', \gamma_0)>\frac{2}{3}\varrho(a_0 G_0, \gamma_0)$. We consider now the set $F=\{a_0 g a_0^{-1}; g \in G_0\}$. If $a_0=(\delta, d)$, then by a calculation of the preceding lemma $a_0 g a_0^{-1}=(\delta^{gd^{-1}}+\delta^{d^{-1}}, dgd^{-1})$ $(g \in G_0)$, which shows, that the Radon—Nikodym derivatives $\frac{d\mu_a}{d\mu}(x)$ $(a \in F)$ are uniformly bounded (cf. footnote.[18])). We take the set B_0 of lemma 15, which corresponds to this F and an arbitrarily fixed $\varepsilon>0$. Putting $B=B' \cup B_0$ and choosing a $\delta>0$ we apply lemma 15 to the case of B and δ, and we denote by $\varphi_c'(x)$ the system obtained in this way, for which $\|\varphi_c(x)-\varphi_c'(x)\|_2<\delta$ $(c \in B)$. If $\varphi_c'(x) \sim$

[20]) For $\alpha, \beta \in \Delta$ we write $\beta \geqq \alpha$ if $\alpha \wedge \beta=\alpha$.

$\sim \sum_{\alpha\in\Delta} g(c,\alpha)\,\omega_\alpha(x)$, we have $g(c,\alpha)=0$ with a finite number of exceptions only. Putting $C=B\cap a_0G_0\supseteq B'$ and $\varrho'(\gamma)=(\sum_{c\in C}|g(c,\gamma)|^2)^{\frac{1}{2}}$ we get

$$|\varrho'(\gamma)-\varrho(C,\gamma)|\leqq (\sum_{c\in C}\|\varphi_c(x)-\varphi'_c(x)\|_{(2)}^{2})^{\frac{1}{2}}<\sqrt{n}\delta=\eta$$

for $\gamma\in\Delta$, where n denotes the number of elements in B. Similarly, putting $\varrho=(\sum_{c\in B'}|g(c,\gamma_0)|^2)^{\frac{1}{2}}$ we have $|\varrho-\varrho(B',\gamma_0)|<\eta$. Choosing δ sufficiently small, we can suppose $\varrho>0$ and for $\gamma\geqq\gamma_0$

$$\frac{\varrho'(\gamma)}{\varrho'(\gamma_0)}\leqq\frac{\varrho(C,\gamma_0)+\eta}{\varrho(C,\gamma_0)-\eta}\leqq 2.$$

We consider now the function $(U'\bar{U}_gU'^*)_{a_0ga_0^{-1}}$, where $U'=\sum_{c\in B}\bar{L}_{\varphi'_c(x)}\bar{U}_c$. More explicitely

$$(U'\bar{U}_gU'^*)_{a_0ga_0^{-1}}\equiv\sum_{\substack{ugv^{-1}=a_0ga_0^{-1}\\ u,v\in B}}\varphi'_u(x)\overline{\varphi'_v(xa_0ga_0^{-1})}\equiv$$
$$\equiv\sum_{\substack{ugv^{-1}=a_0ga_0^{-1}\\ u,v\in B\\ \alpha,\alpha'\leqq\bar{\alpha}}}g(u,\alpha)\overline{g(v,\alpha')}\omega_\alpha(x)\omega_{\alpha'}(xa_0ga_0^{-1}),$$

where $\bar{\alpha}$ is a suitable element in Δ. Putting $\delta^{d^{-1}}=\delta'$, and $g'=dgd^{-1}$, we have $a_0ga_0^{-1}=(\delta'^{g'^{-1}}+\delta',g')$; we assume in the following that $\gamma_0,\delta'\leqq\bar{\alpha}$. By an application of lemma 16 we fix an element $g\in G_0$ such that 1) the equality $ugv^{-1}=a_0ga_0^{-1}\,(u,v\in B)$ is possible only for $u=v\in a_0G_0$, that is for $u=v\in C=B\cap a_0G_0$, 2) $\bar{\alpha}\wedge\bar{\alpha}^{g'^{-1}}=0$, 3) $a_0ga_0^{-1}\,\bar{\in}\,G$. Since in this case $\omega_\alpha(xa_0ga_0^{-1})\equiv\omega_\alpha(xg'+\delta'+\delta'^{g'})\equiv\omega_{\alpha^{g'^{-1}}}(x+\delta'^{g'^{-1}}+\delta')\equiv\omega_{\alpha^{g'^{-1}}}(x+\delta'^{g'^{-1}})$, for $\alpha\leqq\bar{\alpha}$, we have $(U'\bar{U}_gU'^*)_{a_0ga_0^{-1}}\equiv\sum_{\substack{u\in C\\ \alpha,\alpha'\leqq\bar{\alpha}}}g(u,\alpha)\overline{g(u,\alpha')}\,\omega_\alpha(x)\omega_{\alpha'^g}(x+\delta'^{g'^{-1}})$.

We compute the coefficient $c_{\gamma_0+\gamma_0^{g'^{-1}}}$ of $\omega_{\gamma_0+\gamma_0^{g'^{-1}}}(x)$ in the expansion of the last expression in terms of the system $\{\omega_\alpha(x)\}$. We have by virtue of our choice of g:

$$s_{\alpha,\alpha'}=\int_X\omega_\alpha(x)\omega_{\alpha'^{g'^{-1}}}(x+\delta'^{g'^{-1}})\,\omega_{\gamma_0+\gamma_0^{g'^{-1}}}(x)\,d\mu=$$
$$=\left(\int_X\omega_\alpha(x)\omega_{\gamma_0}(x)\,d\mu\right)\left(\int_X\omega_{\alpha'^{g'^{-1}}}(x+\delta'^{g'^{-1}})\,\omega_{\gamma_0^{g'^{-1}}}(x)\,d\mu\right)=$$
$$=\delta_{\alpha,\gamma_0}\int_X\omega_{\alpha'}(x+\delta')\,\omega_{\gamma_0}(x)\,d\mu,$$

where $\delta_{\alpha,\gamma_0}=1$ if $\alpha=\gamma_0$, and $=0$ otherwise. The last integral vanishes clearly, if $\alpha'\wedge\gamma_0\neq\gamma_0$ therefore we assume in the sequel $\alpha'\geqq\gamma_0$. We put $\gamma_1=\delta'\wedge\gamma_0$, $\gamma_2=\delta'\wedge(\bar{\alpha}-\gamma_0)$ and for $\alpha\in\Delta$ we denote by $r(\alpha)$ the number of elements in the set $\{g;\alpha_g=1,\ g\in G\}$. Then

$$\int_X\omega_{\alpha'}(x+\delta')\,\omega_{\gamma_0}(x)\,d\mu=\left(\int_X\omega_{\gamma_0}(x+\gamma_1)\omega_{\gamma_0}(x)\,d\mu\right)\left(\int_X\omega_{\alpha'-\gamma_0}(x+\gamma_2)\,d\mu\right).$$

But it is easily seen, that

$$\int_X \omega_{\gamma_0}(x+\gamma_1)\,\omega_{\gamma_0}(x)d\mu = \left(\frac{q-p}{\sqrt{pq}}\right)^{r(\gamma_1)}$$

and

$$\int_X \omega_{\alpha'-\gamma_0}(x+\gamma_2)d\mu = \begin{cases} \left(\frac{q-p}{\sqrt{pq}}\right)^{r(\alpha'-\gamma_0)} & \text{if } \alpha'-\gamma_0 \leqq \gamma_2 \\ 0 \text{ otherwise.} \end{cases}$$

Summing up we get

$$S_{\alpha,\alpha'} = \begin{cases} \left(\frac{q-p}{\sqrt{pq}}\right)^{r(\gamma_1+\alpha'-\gamma_0)} & \text{if } \alpha=\gamma_0, \text{ and } \alpha'-\gamma_0 \leqq \gamma_2 \\ 0 \text{ otherwise.} \end{cases}$$

Therefore putting $\lambda = \frac{q-p}{\sqrt{pq}}$, we have

$$c_{\gamma_0+\gamma_0 g'^{-1}} = \lambda^{r(\gamma_1)}\left(\sum_{j=0}^{r(\gamma_2)} \lambda^j \left(\sum_{\substack{r(\alpha)=j\\ \alpha\leqq\gamma_2}} \left(\sum_{u\in C} g(u,\gamma_0)\overline{g(u,\gamma_0+\alpha)}\right)\right)\right) =$$
$$= W\lambda^{r(\gamma_1)}\left(\sum_{u\in C}|g(u,\gamma_0)|^2\right) \text{ where}$$
$$W = 1+\sum_{j=1}^{r(\gamma_2)} \lambda^j \left(\sum_{\substack{r(\alpha)=j\\ \alpha\leqq\gamma_2}} \left(\sum_{u\in C} \frac{g(u,\gamma_0)\overline{g(u,\gamma_0+\alpha)}}{\sum_{u\in C}|g(u,\gamma_0)|^2}\right)\right)$$

Applying the Schwarz inequality and using notations introduced before, we get

$$\left|\sum_{\substack{r(\alpha)=j\\ \alpha\leqq\gamma_2}} \left(\sum_{u\in C} \frac{g(u,\gamma_0)\overline{g(u,\gamma_0+\alpha)}}{\sum_{u\in C}|g(u,\gamma_0)|^2}\right)\right| \leqq \sum_{\substack{r(\alpha)=j\\ \alpha\leqq\gamma_2}} \frac{\varrho'(\gamma_0+\alpha)}{\varrho'(\gamma_0)} \leqq 2\cdot 2^j$$

We assume now, that p and q are chosen in such a way in the construction of the measure μ, that $\lambda = \frac{q-p}{\sqrt{pq}} < \frac{1}{7}$. We have in this case $|W| \geqq 1-2\sum_{j=1}^{r(\gamma_2)}(2\lambda)^j \geqq 1-\frac{2\cdot 2\lambda}{1-2\lambda} > \frac{1}{5}$.

Therefore

$$|c_{\gamma_0+\gamma_0 g'^{-1}}| \geqq \frac{\lambda^{r(\gamma_1)}}{5}\varrho'^2(\gamma_0) = W_1\varrho'^2(\gamma_0)$$

where W_1 depends only on a_0 and γ_0.

The assumption of our lemma is $U\mathbf{P}U^* \subseteq \mathbf{P}$, therefore $(U\bar{U}_g U^*)_a \equiv 0$ if $a \overline{\in} G_0$ and $\equiv$ constant for $a \in G_0$. Thus by our choice of g $(U\bar{U}_g U^*)_{a_0 g a_0^{-1}} \equiv \equiv 0$ if $a_0 \overline{\in} G_0$ and $\equiv$ constant otherwise, but then we have $\gamma_0 \neq 0$.

So in any case

$$\varrho^2 \leqq \varrho'^2(\gamma_0) \leqq \frac{|c_{\gamma_0+\gamma_0 g'^{-1}}|}{W_1} \leqq \frac{\|(U\bar{U}_g U^*)_{a_0 g a_0^{-1}} - (U'\bar{U}_g U'^*)_{a_0 g a_0^{-1}}\|_2}{W_1} < \frac{\varepsilon}{W_1}.$$

Since ε is arbitrary, it follows necessarily $\varrho=0$, and so we arrive to a contradiction.

Thus the proof of lemma 17 is completed.

This gives immediately the

Theorem 2. *There exist factors of type* III *with singular and semi-regular maximal abelian subrings.*

Proof. (i) Let G be a countable infinite abelian group. Putting $G=G_0$ the conditions described before lemma 15 are clearly satisfied, therefore we can apply lemma 17. With the notations used there, we have $\mathbf{P}=\mathbf{R}$, and thus $\mathbf{P}$ is singular, since by virtue of this lemma $U\mathbf{P}U^*\subseteq\mathbf{P}$ for a unitary operator in $\mathbf{M}$ gives $U\in\mathbf{R}=\mathbf{P}$. Finally, lemma 14 shows, that $\mathbf{P}$ is maximal.

(ii) Consider the group G of the mappings $x\to\alpha x+\beta$, where x,α,β are rational numbers, $\alpha\neq 0$. Denoting by G_0 the abelian subgroup of the translations $x\to x+\beta$ one verifies at once, that G_0 satisfies the conditions of lemma 14 and 16. Therefore the ring corresponding to G_0 in lemma 17 is maximal abelian, and by the same lemma the ring $\mathbf{T}$ generated by the unitary transformations in $\mathbf{M}$ satisfying $U\mathbf{P}U^*\subseteq\mathbf{P}$ is $\mathbf{T}\subseteq\mathbf{R}$. But since G_0 is a normal subgroup of G, we have $\bar{U}_g\mathbf{P}\bar{U}_g^*\subseteq\mathbf{P}$ $(g\in G)$, thus $\mathbf{T}\supseteq\mathbf{R}$, hence $\mathbf{T}=\mathbf{R}$. In addition, $\mathbf{T}$ is a factor. Clearly $\mathbf{P}\neq\mathbf{R}$ which shows, that $\mathbf{P}$ is semi-regular.

So the proof of Theorem 2 is completed.

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