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Isometric immersion of compact Riemannian manifold into E^{n+m} with mean curvature pinched

By SHIAO-SONG YANG (Chongqing)

Abstract. It is shown that an isometric immersion of an n-dimensional compact Riemannian manifold into the Euclidean space E^{n+m} with the length of mean curvature less than $m^{-1/2}r^{-1}$ can never be contained in a ball of radius r, and the estimate of the diameter of the immersion is presented.

1. Introduction

In this short paper we show that an isometric immersion of compact n-dimensional Riemannian manifold into the Euclidean space E^{n+m} can not be contained in a ball of a finite radius, provided that its length of mean curvature vector is pinched in a way by this radius. Our work is motivated by that of H. Jacobowitz [2] on the isometric immersion of compact Riemannian manifold into an Euclidean space with its sectional curvature pinched. As an application of our main theorem, we obtain another proof of nonexistence of compact minimal submanifold of Euclidean space. Finally, we give an estimate of diameter of compact immersed submanifold of Euclidean space with its mean curvature pinched.

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2. Main theorem

We now present our main theorem.

Theorem 2.1. *Let $f : M^n \to E^{n+m}$ be an isometric immersion of compact Riemannian manifold into the Euclidean space E^{n+m}, and H be the mean curvature vector of f. Suppose that H satisfies*

$$\|H\| < \frac{1}{m^{\frac{1}{2}} r},$$

where $\|H\|$ is the length of the mean curvature vector H. Then no ball of radius r can contain $f(M^n)$.

Proof. For convenience, we regard $f(M^n)$ as M^n in E^{n+m} and all the discussions are developed on M^n in E^{n+m}. Denote by Δ the Laplacian of the induced metric on M^n. Recall that for any vector fields x and y we have

$$\nabla'_x y = \nabla_x y + \Theta(x, y),$$

where ∇' is the Riemannian connection on E^{n+m}, ∇ is the Riemannian connection on M induced from ∇', and Θ is the second fundamental form of M (see [4]). If $\{e_1, \ldots, e_n\}$ is an orthonormal basis for vector fields in a neighborhood W of x in M^n (see [5], page 261), then applying in W the formula

$$\begin{aligned} \Delta\phi = \mathrm{Tr}\, \boldsymbol{D}^2\phi &= \sum_{i=1}^{n} (e_i e_i \phi - \nabla_{e_i} e_i \phi) \\ &= \sum_{i=1}^{n} \big[e_i e_i \phi - (\nabla'_{e_i} e_i - \Theta(e_i, e_i))\phi\big] \end{aligned}$$

to function $\phi = \langle f, f\rangle$, and bearing in mind that $e_i f = e_i$ in Euclidean space and

$$\begin{aligned} &e_i e_i \langle f, f\rangle - \nabla'_{e_i} e_i \langle f, f\rangle + \Theta(e_i, e_r)\langle f, f\rangle \\ &\qquad = 2 e_i \langle e_i, f\rangle - 2\langle \nabla'_{e_i} e_i, f\rangle + 2\langle \Theta(e_i, e_i), f\rangle \\ &\qquad = 2\langle \nabla'_{e_i} e_i, f\rangle + 2\langle e_i, e_i\rangle - 2\langle \nabla'_{e_i} e_i, f\rangle + 2\langle \Theta(e_i, e_i), f\rangle \\ &\qquad = 2 + 2\langle \Theta(e_i, e_i), f\rangle, \end{aligned}$$

we have

$$\Delta\langle f, f\rangle = 2n\left(1+\sum_{\alpha=1}^{m} H_\alpha\langle N_\alpha, f\rangle\right),$$

where $\langle\cdot,\cdot\rangle$ is the Euclidean metric, $\{N_\alpha\}$ local normal frame, $H_\alpha = \langle H, N_\alpha\rangle$, and $H = \frac{1}{n}\sum_{i=1}^{n}\Theta(e_i, e_i)$ (note that H can be expressed as $H = \sum_{\alpha=1}^{m} H_\alpha N_\alpha$). Because of compactness of M^n, we can take a point $x = x_0$ at which ϕ attains a maximum. At x_0 the Hessian $\boldsymbol{D}^2\phi$ must be negatively semi-definite and hence $\Delta\phi = \operatorname{Tr}\boldsymbol{D}^2\phi \leq 0$. Therefore for $x = x_0$ we have

$$\sum_{\alpha}^{m} H_\alpha\langle N_\alpha, f\rangle \leq -1.$$

Now from Schwarz inequality, we have

$$-1 \geq -\left(\sum_{\alpha=1}^{m} H_\alpha^2\right)^{\frac{1}{2}}\left(\sum_{\alpha=1}^{m}\langle N_{\alpha=1}, f\rangle^2\right)^{\frac{1}{2}} \geq -m^{\frac{1}{2}}\left(\sum_{\alpha=1}^{m} H_\alpha^2\right)^{\frac{1}{2}}\langle f, f\rangle^{\frac{1}{2}}$$

or

$$\left(\sum_{\alpha=1}^{m} H_\alpha^2\right)^{\frac{1}{2}} \geq \frac{1}{(m\langle f, f\rangle)^{\frac{1}{2}}}\,.$$

To complete the proof, suppose that $f(M^n)$ can be contained in a ball of radius r. Without loss of generality, assume that the centre of the ball is at the origin. Then $\langle f, f\rangle \leq r^2$ on M^n, and consequently, at $x = x_0$,

$$\|H\| = \left(\sum_{\alpha=1}^{m} H_\alpha^2\right)^{\frac{1}{2}} \geq \frac{1}{m^{\frac{1}{2}} r}\,,$$

thus leading to a contradiction, which completes the proof.

Corollary 2.2 ([6]). *Let $f : M^n \to E^{n+1}$ be a compact isometric immersion and the mean curvature H of M^n satisfies*

$$-\frac{1}{r} < H < \frac{1}{r}.$$

Then no ball of radius r can contain $f(M^n)$.

As the Tompkins' result [2] on nonexistence of isometric embedding of the n-dimensional flat torus into the Euclidean space E^{2n-1} is a special case of H. JACOBOWITZ's theorem [2], the following well-known result is a special case of Theorem 2.1 in this paper.

Corollary 2.3 ([1]). *There exists no isometric minimal immersion of compact Riemannian manifold into the Euclidean space.*

3. Diameter of isometric immersion with mean curvature pinched

In this section we investigate the relationship between the diameter of isometric immersion of compact Riemannian manifold and its mean curvature vector in Euclidean space.

Definition 3.1. The diameter d_f of the isometric immersion $f : M^n \to E^{n+m}$ is defined as follows:

$$d_f(M^n) = \max_{x,y \in M} \|f(x) - f(y)\|,$$

where $\| \cdot \|$ is the Euclidean distance.

Clearly this definition is not intrinsic, because we are concerned with how M^n is placed in its ambient space. To our pleasure we find Jung's covering theorem is very useful here.

Jung's theorem ([3]). *Each subset of E^n of diameter $\leq d$ lies in a ball of radius $\leq (\frac{n}{2n+2})^{\frac{1}{2}} d$.*

Now we give a result on the diameter of $f(M^n)$ in E^{n+m}.

Theorem 3.2. *Let f be an isometric immersion of compact n-dimensional Riemannian manifold into Euclidean space E^{n+m}. Suppose that the mean curvature vector H satisfies*

$$\|H\| = \left(\sum_{\alpha=1}^{m} H_\alpha^2 \right)^{\frac{1}{2}} \leq c.$$

Then

$$d_f(M^n) \geq \left[(2n+2)/mn \right]^{\frac{1}{2}} c^{-1}.$$

PROOF. If this inequality does not hold, then we have

$$d_f(M^n) < \left[\frac{2n+2}{mn} \right]^{\frac{1}{2}} (m^{\frac{1}{2}} c)^{-1}.$$

By Jung's theorem, $f(M^n)$ can be contained in a ball of radius $[n/(2n+2)]^{\frac{1}{2}} \cdot [(2n+2)/mn]^{\frac{1}{2}} c^{-1} = (m^{\frac{1}{2}} c)^{-1}$. Now from Theorem 2.1, H should satisfy

$$\|H\| > \frac{1}{m^{\frac{1}{2}}(m^{\frac{1}{2}}c)^{-1}} = c,$$

giving rise to a contradiction. Thus we complete the proof. □

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SHIAO–SONG YANG
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA
HEFEI ANHUI 230026
and
CENTRE FOR NONLINEAR DYNAMICS
CHONGQING INSTITUTE OF OSTS AND TELECOMMUNICATIONS
CHONGQING 630065
P.R. CHINA
Correspondence address:
B01–318/ NORTH CAMPUS
UNIV. OF SCI. AND TECHN. OF CHINA
HEFEI ANHUI 230026
P.R. CHINA

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