The prime radical of a polynomial ring.

Dedicated to the memory of Tibor Szele.

By NEAL H. McCOY in Northampton, Massachusetts.

If R is an arbitrary ring, by the *radical* of R we shall mean the intersection of all the prime ideals in R. By the results of [4], together with an observation of LEVITZKI [3], the radical N of R can be characterized as the set of all elements r of R with the property that every m-sequence which contains r is a vanishing m-sequence. That is, every sequence of the form

$$r, r_1 = ra_1r, r_2 = r_1a_2r_1, \dots,$$

where $a_1, a_2, \dots$ are arbitrary elements of R, is zero from some point on.

It has been proved by LEVITZKI [3] and NAGATA [5] that the radical as defined above coincides with the *lower radical* of BAER [2].

The purpose of this note is to prove the following theorem.

Theorem. *Let x be an indeterminate and denote by N and by N' the respective radicals of R and of $R[x]$. Then $N' = N[x]$.*

This result has been obtained independently by AMITSUR [1] who approached the problem from the point of view of the lower radical. Our proof, entirely different from his, will be based upon some lemmas which we proceed to establish.

Lemma 1. *Let P be a prime ideal in R and let $\bar{a}$ denote the element of the ring R/P to which a corresponds under the natural homomorphism of R onto R/P. If P' is the kernel of the homomorphism of $R[x]$ onto R/P defined by*

$$a_0x^n + \dots + a_n \to \bar{a}_n,$$

then P' is a prime ideal in $R[x]$ such that $P' \cap R = P$.

The proof of this is obvious since clearly $P' \cap R = P$, and P' is a prime ideal since $R[x]/P' \cong R/P$.

Lemma 2. *If P' is any prime ideal in $R[x]$, then $P = P' \cap R$ is a prime ideal in R.*

Suppose that a and b are elements of R such that $aRb \subseteq P$. By Theorem 1 of [4], we only need to prove that $a \in P$ or $b \in P$. Since $aRb \subseteq P \subseteq P'$, it is clear that each element of $aR[x]bR[x]$ is a sum of terms belonging to P'. Hence $aR[x]bR[x] \subseteq P'$, and since P' is a prime ideal in $R[x]$ it follows that $a \in P'$ or $b \in P'$. Thus $a \in P$ or $b \in P$, as required.

Lemma 3. *If N and N' are the respective radicals of R and of $R[x]$, then $N' \cap R = N$.*

This is an immediate consequence of the preceding lemmas. First, if $a \in N$, and P' is *any* prime ideal in $R[x]$, Lemma 2 shows that $P = P' \cap R$ is a prime ideal in R. Hence $a \in P \subseteq P'$, and thus $a \in N'$. Conversely, if $a \in N' \cap R$ and P is *any* prime ideal in R, let P' be the prime ideal in $R[x]$ defined as in Lemma 1. Since $P' \cap R = P$, it follows that $a \in P$; hence $a \in N$,

We are now ready to prove the theorem. Since, by Lemma 3, $N \subseteq N'$, it is obvious that $N[x] \subseteq N'$ in case R has a unit element. In any case, $N[x]R[x] \subseteq N'$ and it is known [4] that this always implies that $N[x] \subseteq N'$. There remains only to prove that $N' \subseteq N[x]$. Let $f(x) \in N'$, where $f(x) = a_0x^n + \cdots + a_n$. Now if P is any prime ideal in R, and P' the corresponding prime ideal in $R[x]$ defined as in Lemma 1, since $f(x) \in P'$ it is clear that $a_n \in P$. Hence $a_n \in N \subseteq N'$, and it follows that $a_0x^n + \cdots + a_{n-1}x \in N'$. Set $g(x) = a_0x^{n-1} + \cdots + a_{n-1}$. If $g(x) \notin N'$, there exists a nonvanishing m-sequence

$$g(x), \quad g_1(x) = g(x)h_1(x)g(x), \quad g_2(x) = g_1(x)h_2(x)g_1(x), \ldots .$$

However, this would imply that

$$xg(x),\ x^2g_1(x),\ x^4g_2(x), \ldots$$

is a nonvanishing m-sequence. But this is impossible since $xg(x) \in N'$. Hence we must have $g(x) \in N'$, and by the argument applied above to $f(x)$, we see that $a_{n-1} \in N$. Evidently this process can be continued to show that N contains all coefficients of $f(x)$, completing the proof.

Bibliography.

[1] S. A. Amitsur, The radical of polynomial rings, Submitted to the *Canad. J. Math.*
[2] R. Baer, Radical ideals, *Amer. J. Math.* **69** (1947), 46—58.
[3] J. Levitzki, Prime ideals and the lower radical, *Amer. J. Math.* **73** (1951), 25—29.
[4] N. H. McCoy, Prime ideals in general rings, *Amer. J. Math.* **71** (1949), 823—833.
[5] M. Nagata, On the theory of radicals in a ring, *J. Math. Soc. Japan* **3** (1951) 330—344.

(Received August 5, 1955.)