A theorem on description adequacy.

In memory of my good friend, Tibor Szele.

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1. Introduction. There are six well-known propositions concerning the adequacy (defined in Section 3 below) of certain classes of nets [2], [3][1]) in a topological space to describe the topology of the space. They may be stated:

(P1). The set of (countable) sequences in a locally separable space (and, hence, in particular, a metric space) adequately describes the topology of the space [5].

(P2). The set of phalanxes in a topological space adequately describes the topology of the space [4].

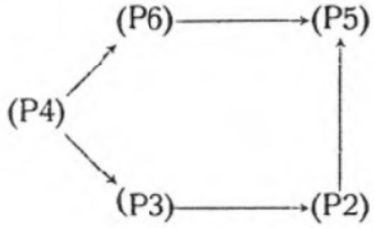
(P3). The set of topophalanxes in a topological space adequately describes the topology of the space [4].

(P4). The set of ultraphalanxes in a topological space adequately describes the topology of the space [4].

(P5). The set of all nets in a topological space adequately describes the topology of the space [2].

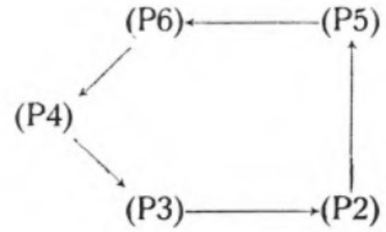
(P6). The set of ultranets in a topological space adequately describes the topology of the space [2].

(P1) is a trivial consequence of the axiom of choice. It is also clear that one has immediately the following implication diagramm:



1) Numbers in square brackets refer to the Bibliography concluding the note.

The implication (P5)→(P6) is a consequence of the proposition [2] that every net has a subnet which is an ultranet[2]). In Section 2 below, it is shown that every net has a subnet which is a phalanx. Since every subnet of an ultranet is an ultranet, this yields the implication (P6)→(P4). Thus, one has the implication cycle:



so that all five propositions may be deduced from the validity of the simplest; namely, (P5).

Now, proofs of (P1) to (P6) all employ, explicitly or otherwise, some form of the axiom of choice. It is then a natural idea to construct an adequate class of nets in a topological space directly from the mapping whose existence is a form of the axiom of choice; namely, the selection operator. That this is possible, and, indeed, simple, is shown in Section 3 below.

2. By *topological space*, hereafter called simply *space* and denoted S, we shall mean a non-null T_1 space [1], [5].

A *directed set* is simply a partially ordered set in which each pair of elements (and, hence, each non-null finite subset) has an upper bound. A subset of a directed set is *residual* if it contains all elements above some given one. A subset of a directed set is *cofinal* if it contains an upper bound for every finite non-null subset of the directed set.

A *net* [2], [3] is any mapping of a directed set into a set. If $\mathfrak{n}:\Delta\rightarrow S$ is a net, one refers to the net $\mathfrak{n}$ in S based on Δ.

A *stack* [4] is the directed set composed of all non-null finite subsets of some set, the partial ordering being taken as set-theoretic inclusion.

A net based on a stack is called a *phalanx* [4].

If $\mathfrak{n}:\Delta\rightarrow S$ is a net and $T\subset S$ one says [3]:

1. $\mathfrak{n}$ *decides for* T if $\mathfrak{n}$ maps some residual subset of Δ into T.

2. $\mathfrak{n}$ *decides against* T if $\mathfrak{n}$ decides for $S-T$.

3. $\mathfrak{n}$ *decides about* T if either 1. or 2. subsists.

4. $\mathfrak{n}$ is *undecided about* T (or fails to decide about T) if neither 1. nor 2. subsists.

Clearly, 4. subsists if and only if both T and $S-T$ contain images under $\mathfrak{n}$ of cofinal subsets of Δ.

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[2]) Kelley uses the term *universal net* rather than ultranet.

If $\mathfrak{n}: \Delta \to S$ and $\mathfrak{m}: \Gamma \to S$ are nets and if there is a net $\gamma: \Gamma \to \Delta$ which decides for every residual subset of Δ and if $\gamma\mathfrak{n} = \mathfrak{m}$, then $\mathfrak{m}$ is called a *subnet* [2] of $\mathfrak{n}$.

Clearly, every subnet of a net decides for the same sets as does the net itself.

A net which decides about every subset of its range is called an *ultranet*.

A phalanx which is an ultranet is called an *ultraphalanx* [4].

If S is a space, a phalanx in S is called a *topophalanx* [4] if it decides about every open set.

If S is a space and $p \in S$ and $\mathfrak{n}$ is a net in S, one says [2] that $\mathfrak{n}$ *converges* to p, or that $\mathfrak{n}$ has *limit* p, provided $\mathfrak{n}$ decides for every open set of which p is a member.

Proposition. *Every net has a subnet which is a phalanx.*

Proof. Let $\mathfrak{n}: \Delta \to S$ be a net. Consider the stack Γ of Δ. Define $\gamma: \Gamma \to \Delta$ by ordinary induction as follows: If $\alpha \in \Gamma$ and α contains exactly one point, let $\alpha\gamma$ be that point. Suppose γ has been defined for all members of Γ which contain less than $n > 1$ elements. If $\xi \in \Gamma$ has exactly n elements, let $\xi\gamma$ be any upper bound of the (necessarily finite) set $\{\eta\gamma\}$ where η ranges over all non-null proper subsets of ξ. Define then $\mathfrak{m} = \gamma\mathfrak{n}$.

3. Let S be any non-null set and let $\mathfrak{S}$ be the set of all non-null subsets of S. A *selection operator* in S is a mapping $f: \mathfrak{S} \to S$ having the property that for each $T \in \mathfrak{S}$, $Tf \in T$. The existence of a selection operator for each non-null S is a rudimentary form of the axiom of choice.[3])

If f is a selection operator in S and if $\mathfrak{A} \subset \mathfrak{S}$ and $\mathfrak{A}$ is directed by inverse inclusion (that is, the set-theoretic product of any two members of $\mathfrak{A}$ contains a member of $\mathfrak{A}$), then $f/\mathfrak{A}$ shall be called a *selector net* in S.

A class $\mathfrak{N}$ of nets in a space S is said to *adequately describe* the topology of S provided the following proposition is valid: If $T \subset S$ and $p \in S$, then p is an accumulation point of T if and only if there is a net in $T - \{p\}$ which is a member of $\mathfrak{N}$ and which has limit p.

Proposition. *If S is a space and if f is a selection operator in S, then the selector nets derived from f adequately describe the topology of S.*

Proof. Let $T \subset S$ and $p \in S$. If there is any net, and, in particular, a selector net, in $T - \{p\}$ with limit p, then p is an accumulation point of S by (P5). Suppose, alternatively, that p is an accumulation point of T. Then $T - \{p\}$ is non-null; indeed, infinite. Let $\mathfrak{A}$ be the set $\{G \cap (T - \{p\})\}$ where G ranges over all open sets of which p is a member. Since $\mathfrak{A}$ is actually

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[3]) Cf. Tibor Szele, On Zorn's lemma, *Publ. Math. Debrecen* **1** (1950), 254—257 and Garrett Birkhoff, Lattice theory (*Revised Ed.*), *New York*, 1947.

a groupoid under set-theoretic product, $f/\mathfrak{A}$ is a selector net whose values lie in $T-\{p\}$. If $p\in G$ and G is open, $f/\mathfrak{A}$ maps $\{H\cap(T-\{p\})\}$ into G where H ranges over all open sets with $p\in H\subset G$. Thus, $f/\mathfrak{A}$ decides for G. Hence, $f/\mathfrak{A}$ converges to p.

Bibliography.

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(Received August 24, 1955.)