On the intersection of finitely generated free groups.

In memoriam Tibor Szele.

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In a recent paper of the same title[1]) A. G. HOWSON shows that the intersection of two finitely generated subgroups of a free group is finitely generated. HOWSON shows more precisely that, if we denote the ranks of U, V, and $U \cap V$ by m, n, and N respectively, then $N \leqq 2mn - m - n + 1$. The purpose of this note is to point out that a slight modification of HOWSON's proof leads to the better bound $N \leqq 2mn - 2m - n + 1$, where we may assume $m \geqq n$. Here we exclude the trivial case that one of the groups is cyclic. In a more restricted situation, whose occurrence is however sufficiently common to warrant mentioning it, the bound is further improved to $N \leqq 2mn - 2m - 2n + 3$. These bounds are still not likely to be best possible; HOWSON shows by an example that $N = mn - m - n + 2$ may occur, and one would hope that $N \leqq mn - m - n + 2$ holds always.

1. The crucial part of HOWSON's proof is as follows: In the free group F, assumed to be of rank 2 without loss of generality, certain elements are called branch points[2]) for U. The property of $f \in F$ to be a branch point for U depends on the coset Uf only: f is a branch point for U if, and only if, each element of Uf is a branch point for U. The total number of branch points modulo U, each counted with an appropriate multiplicity, is shown to be $2m-1$ if m is the rank of U. The result then follows from a comparison of the branch points modulo $U \cap V$ with those modulo U and V respectively.

If S is a subset of the free group F generated by a and b, we say that S has s endings, if it contains s but not more than s elements whose representations as reduced words in a, b end on different letters a, b, a^{-1}, b^{-1}. Thus $0 \leqq s \leqq 4$, and $s = 0$ only if S is empty or consists of the unit element 1 only. We can then paraphrase HOWSON's definition as follows:

[1]) A. G. HOWSON, *J. London Math. Soc.* **29** (1954), 428—434.

[2]) HOWSON's main tool is DEHN's group graph.

Definition. (1) If f is an element of $F - U$, then f is a branch point of order 1 or 2 according as the coset Uf has 3 or 4 endings; if Uf has fewer than 3 endings, then f is no branch point.

(2) If $f \in U$, then f is a branch point of order 1, 2, or 3 according as U has 2, 3, or 4 endings; if U has fewer than 2 endings, it is not a branch point.

An element which is no branch point is sometimes called a branch point of order 0.

With this definition, Howson shows:

The sum of the orders of all branch points taken from a set of coset representatives of U, or briefly, the total order of branch points modulo U, is $2m-1$.

We substitute the following definition of the „order of a coset", which is the order of the branch point if $f \notin U$, but in the case of U itself is 1 less: [3])

1.1 Definition. (1) *The coset Uf $(f \neq 1)$ is of order 1 or 2 according as it has 3 or 4 different endings; otherwise it is of order 0.*

(2) *The subgroup U itself is of order -1, 0, 1, or 2 according as it has 1, 2, 3, or 4 different endings. We exclude the trivial case that U is the unit subgroup; alternatively one may allocate it the order -2, when all subsequent statements will be seen to remain valid.*

We denote the order of the coset Uf by $o(Uf)$. Clearly, the sum $O(U)$ of the orders of all cosets is one less than Howson's total order of branch points modulo U; therefore:

1.2 *If U has finite rank m, the total order of all cosets of U is $O(U) = 2m-2$. If the rank of U is infinite, $O(U)$ also is infinite.*

Howson's method of deducing the bound for N still applies to 1, 2, except that the case when a coset of negative order (necessarily U or V itself) occurs needs special consideration. We therefore show first that we may assume at least one of the two groups to have non-negative order.

If U, say, has order $o(U) = -1$, all elements $\neq 1$ of U have the same ending. Let x^{-1} be the longest common initial segment of all the reduced words representing the elements $\neq 1$ of U. Then every element $u \neq 1$ of U is represented by a reduced word of the form $u = x^{-1} u_1 x$ with $x \neq 1$. Thus $U = x^{-1} U_1 x$, where U_1 is now of non-negative order since x was taken maximal. We apply to F the automorphism ξ given by $F\xi = xFx^{-1}$. Then $U\xi = U_1$, $V\xi = V_1$, rank $(U_1 \cap V_1) = \text{rank}(U \cap V)$ and U_1 is now of non-negative order.

[3]) In this note we use the term „order of U" only in the meaning of this definition.

Clearly, we could instead have transformed F so as to ensure that V is transformed into a group of non-negative order; in other words:

1.3 *If both U and V have finite rank, we may assume that the group of least rank has non-negative order.*

2. Now let U and V be subgroups of F of finite ranks m and n respectively, where $m \geqq n$; moreover V is assumed to be of non-negative order, $o(V) \geqq 0$. Hence all cosets of U and of V, except possibly U itself, have non-negative orders.

Every coset of $W = U \cap V$ is uniquely the intersection $Uf \cap Vg$ of some pair of cosets of U and V respectively. The definition of the order of a coset gives immediately a relation between the order of a coset of W and the orders of the corresponding pair of cosets of U and of V.

First case: $o(U) \geqq 0$.

2.1 *If* $Wh = Uf \cap Vg$, *then* $o(Wh) \leqq \text{Min}\,[o(Uf),\ o(Vg)]$.

Only finitely many cosets of W have positive order, since only finitely many of the numbers $o(Uf)$ and $o(Vg)$ are positive.

We note that 2.1 leads to

2.2 $o(Wh) \leqq \frac{1}{2}\, o(Uf)\, o(Vg)$, *if* $\text{Max}\,[o(Uf),\ o(Vg)] = 2$;

and $o(Wh) \leqq o(Uf)\, o(Vg)$, *if both* $o(Uf)$ *and* $o(Vg)$ *are less than* 2.

However, we use at present only the rougher estimate:

2.3 $o(Wh) \leqq o(Uf)\, o(Vg)$ *for all cosets* Wh *of* W.

Summing over all Wh we obtain:

2.4 $O(W) = \sum_h o(Wh) \leqq \sum_{f,g} o(Uf)\, o(Vg) \leqq \sum_f o(Uf) \cdot \sum_g o(Vg) = O(U)\, O(V)$.

Hence by 1.2: $2N-2 \leqq (2m-2)(2n-2)$, that is

2.5 $N \leqq 2mn - 2m - 2n + 3$.

Remark. 2.2 and 2.3 show that the equality sign can hold only if all cosets of positive order of both U and V are of order 1, and the intersection of every pair of cosets of order 1 of U and V is a coset of order 1 of $W = U \cap V$. Expressed more hopefully: If it were true that the intersection of only at most half of the cosets of order 1 of U with any one fixed coset of order 1 of V could be a coset of order 1 of W, then $O(W) \leqq \frac{1}{2}\, O(U) O(V)$ would follow, giving in this case $N-1 \leqq (m-1)(n-1)$, that is the conjectured bound for N.

Second case: $o(U) = -1$.

In this case U has only one ending; hence W is either trivial or, as subgroup of U, also has order -1. Therefore, when W is not trivial, one has

2.6 $$\sum_{h \neq 1} o(Wh) = 2N-1 \quad \text{and} \quad \sum_{f \neq 1} o(Uf) = 2m-1.$$

The formulae 2. 1, 2. 2, and 2. 3 still hold whenever $Uf \neq U$; but the intersections $U \cap Vg$, where $Vg \neq V$, do not contribute to the total order of W, since they have only one ending like U itself. Thus 2. 4 is replaced by

2. 7 $$\sum_{h\neq 1} o(Wh) \leqq \sum_{f\neq 1,\, g} o(Uf)\, o(Vg) \leqq \sum_{f\neq 1} o(Uf) \cdot \sum_{g} o(Vg),$$

that is $2N-1 \leqq (2m-1)(2n-2)$.

As N is an integer, it follows that

2. 8 $$N \leqq 2mn-2m-n+1.$$

If V is cyclic, HOWSON's result gives $N \leqq m$, while 2. 5 gives $N \leqq 1$, and 2. 8 gives $N=0$, that is the correct bounds in both cases. If V is of rank $n=2$, HOWSON's result gives $N \leqq 3m-1$, while both 2. 5 and 2. 8 give $N \leqq 2m-1$. In all other cases 2. 8 is weaker than 2. 5; thus:

Whenever neither U nor V is cyclic, the rank of their intersection satisfies 2. 8.

In concrete cases, when U and V are given explicitly, for example in terms of sets of free generators, their orders are easily ascertained. It may therefore not be quite without interest to mention the bound obtained for N, when V has order 2. In that case we apply to those cosets of W which are contained in V the estimate 2. 2:

If $Wh = Uf \cap V$, then $o(Wh) \leqq \frac{1}{2} o(Uf)\, o(V) = o(Uf)$,

and this holds for all f, if $o(U) \geqq 0$, and for all $f \notin U$, if $o(U) = -1$. The sum of the orders of all cosets Vg other than V itself is now

$$\sum_{g\neq 1} o(Vg) = 2n-4.$$

If we modify 2. 4 and 2. 7 accordingly, we obtain:

$$2N-2 \leqq (2m-2)(2n-3), \text{ if } o(U) \geqq 0,$$

and

$$2N-1 \leqq (2m-1)(2n-3), \text{ if } o(U) = -1.$$

Hence:

2. 9 *If V has 4 endings, that is $o(V)=2$, then*

$$N \leqq 2mn-3m-2n+4, \textit{ if } o(U) \geqq 0,$$
$$N \leqq 2mn-3m-n+2, \textit{ if } o(U) = -1.$$

If V has rank $n=2$, both these bounds coincide, giving $N \leqq m$, which in this case is the same as the conjectured bound mentioned in the introduction.

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