On a problem of old Chinese mathematics.

To the memory of my dear friend Tibor Szele.

By JÁNOS SURÁNYI in Budapest.

Dealing with questions of the history of old Chinese mathematics, CHIANG JOONG proposed the following identity arriseing from CHOO SHIH-CHIEH (Chinese mathematician of the XIII$^{\text{th}}$ century), to G. SZEKERES:

$$\sum_{i=0}^{k}\binom{k}{i}^2\binom{n+2k-i}{2k}=\binom{n+k}{k}^2.$$

This result occurs without proof in a book of LE-JEN SHOO from 1867. Alternativ proofs were found by means of analytical tools by G. SZEKERES and P. TURÁN.[1]) It is however of interest to find more simple proofs, for they can help by chance to find the ways and means by which such formulas could have been discovered.

On the basis of TURÁN [5], various simple proofs were given by LOO KENG HUA [3], L. TAKÁCS [4], G. HUSZÁR [2], L. CARLITZ [1]. Here we give (cf. [3]) the following combinatorical proof for the generalised identity:

$$(1)\qquad \sum_{i=0}^{k}\binom{k}{i}\binom{l}{i}\binom{n+k+l-i}{k+l}=\binom{n+k}{k}\binom{n+l}{l}.$$

Since the second factor vanishes on the left hand side for $i>l$ and the third one for $i>n$, the real upper bound for the index of summation is

$$m=\min(k,l,n).$$

Let us write the identity (1) in the more symmetrical form

$$\sum_{i=0}^{m}\frac{(k+l+n-i)!}{(k-i)!\,(l-i)!\,(n-i)!\,i!\,i!}=\frac{(k+l)!}{k!\,l!}\,\frac{(l+n)!}{l!\,n!}\,\frac{(n+k)!}{n!\,k!}$$

or, designing by $P(r_1, r_2, \ldots, r_u)$ the number of permutations of $r_1+r_2+\cdots+r_u$ elements, among wich there are r_1 equal pieces a_1, r_2 equal pieces $a_2, \ldots$, and

[1]) Reproduced in a paper of CHIANG JOONG in Chinese Language in the periodical entitled *Science (Ko hsye)* **11** (1938) 647—663. — See also [5]. (Numbers in brackets refer to the Bibliography at the end of this paper.)

r_u equal pieces a_u, we can write

$$(2) \qquad \sum_{i=0}^{m} P(k-i,\ l-i,\ n-i,\ i, i) = P(k, l)\, P(l, n)\, P(n, k)$$

An interpretation of the right hand side is the following. Let us take k pieees of a, l pieces of b, n pieces of c and perform all the permutations I) of the a-s and b-s, II) of the b-s and c-s and III) of the c-s and a-s. The right hand side of (2) is the number of triades containing permutations eachof the types I), II), and III). We will speak of the first, second and third permutation of a triade instead of the permutation of type I), II), or III) of it, respectively.

Among the three initial elements of the permutations of a triade either two are equal and the third different from them, or all three of them are different. In the later case only two arrangements are possible. If the first element of the first permutation is a, then that of the third permutation can be only c and the second permutation must have a b on the first place. We find similarly the other possibility b, c, a. Let us call these possibilites arrangement d resp. arrangement e.

We construct now to any triade a single permutation as follows. If two of the three initial elements are equal, then we write down the corresponding element, and then strike out the corresponding initials in the triade. In the case of an arrangement d or e however we write down a d or e and strike out all the three initial elements in the triade. We continue the same process with the remaining permutations of the triade.

The process comes to an end by exhausting all the three permutations of the triade. This is obvious if arrangements d or e do not occur during the process because one sort of elements runs out at the same time from both of the permutations containing it originally. If however also d or e must be written down, then none of the three permutations and none of the three letters a, b, c can be exhausted, unless we have written as many d-s as e-s. In fact, if there are e. g. more d-s than e-s, then we have struck out more a-s from the first permutation than from the third, so that the latter one contains some a-s. We see by the same way, that the first permutation contains some b-s and the second some c-s. In the case of more e-s than d-s the situation is quite analogous. A permutation thus obtained from a triade contains i pieces $(0 \leqq i \leqq m)$ of d and i of e besides $k-i$ of a-s, $l-i$ of b-s and $n-i$ of c-s.

Conversely it is obvious how to reconstruct a triade from a permutation of the described type, and that the corresponding triade is uniquely determined. Thus the number of the permutations just described is for a given value of i

$$P(k-i,\ l-i,\ n-i,\ i,\ i),$$

and the whole number of these permutations equals with the left hand side of (2) and this completes the proof of the identity (2).

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