Publ. Math. Debrecen
52 / 1-2 (1998), 85–101

Bands of left Archimedean semigroups

By STOJAN BOGDANOVIĆ* (Niš),
MIROSLAV ĆIRIĆ* (Niš) and BORIS NOVIKOV (Kharkov)

Abstract. Semigroups having a decomposition into a band of left (or right) Archimedean semigroups have been studied in many papers. A general theorem characterizing these semigroups has been given by M. S. Putcha in [16], and some special types of such decompositions have been studied by S. Bogdanović and M. Ćirić [3, 4, 6, 7], P. Protić [15, 16, 17], L. N. Shevrin [20] and others. In the present paper we give some new results concerning these semigroups. By Theorem 1 we simplify the above mentioned Putcha's theorem, and using this we characterize various special types of bands of left Archimedean semigroups.

1. Introduction and preliminaries

Semigroups which can be decomposed into a band of left Archimedean semigroups have been studied in many papers. M. S. Putcha in [16] proved a general theorem that characterizes such semigroups. This result we give here as the equivalence of conditions (i) and (ii) in Theorem 1. Some special decompositions of this type have been also treated in a number of papers. S. Bogdanović in [1], P. Protić in [15, 16] and S. Bogdanović and M. Ćirić in [3, 6] studied bands of left Archimedean semigroups whose related band homomorphic images belong to several very important varieties of bands. L. N. Shevrin investigated in [20] bands of nil-extensions of left groups, and S. Bogdanović and M. Ćirić investigated in [4] bands of nil-extensions of groups. Finally, bands of left simple

Mathematics Subject Classification: 20 M 10.
* Supported by Grant 04M03B of RFNS through Math. Inst. SANU.

semigroups, in the general and some special cases, were investigated by P. Protić in [19] and S. Bogdanović and M. Ćirić in [7].

In this paper we give some new results concerning decompositions into a band of left Archimedean semigroups, in the general and some special cases. By Theorem 1 we give some new characterizations of these decompositions in the general case, which are simpler than the one given by M. S. Putcha in [16]. Further we study bands of nil-extensions of left simple semigroups (Theorem 2) and bands of nil-extensions of left groups (Theorem 3). In Section 4 we investigate decompositions which corresponds to various varieties of bands. All such decompositions will be characterized by Theorems 5 and 6. Some of the results obtained in the paper generalize many results from the above mentioned papers, and some of them simplify some known results.

Throughout this paper, $\mathbb{N}$ will denote the set of positive integers, and $\mathcal{L}$ and $\mathcal{D}$ will denote the well-known Green's relations on a semigroup. For a semigroup S, $\mathrm{Reg}(S)$ and $\mathrm{LReg}(S)$ will denote the set of regular and the set of left regular elements of S, respectively, and for $a \in \mathrm{Reg}(S)$, $V(a)$ will denote the set of all inverses of a.

If A is a subset of a semigroup S, then $\sqrt{A} = \{a \in S \mid (\exists n \in \mathbb{N})\ a^n \in A\}$. A left ideal L of a semigroup S is *completely semiprime* if for any $a \in S$, $a^2 \in L$ implies $a \in L$, or equivalently, if $\sqrt{L} = L$, and it is *completely prime* if for all $a, b \in S$, $ab \in L$ implies that either $a \in L$ or $b \in L$.

The *division relations* $\mid$ and $\underset{l}{\mid}$ on a semigroup S are defined by

$$a \mid b \iff (\exists x, y \in S^1)\ b = xay, \qquad a \underset{l}{\mid} b \iff (\exists x \in S^1)\ b = xa,$$

and the relations $\longrightarrow$ and $\xrightarrow{l}$ on S are defined by

$$a \longrightarrow b \iff (\exists n \in \mathbb{N})\ a \mid b^n, \qquad a \xrightarrow{l} b \iff (\exists n \in \mathbb{N})\ a \underset{l}{\mid} b^n.$$

The relation $\xrightarrow{r}$ on S is defined dually, and the relation $\xrightarrow{t}$ is defined by $\xrightarrow{t} = \xrightarrow{l} \cap \xrightarrow{r}$. Relations $\overset{l}{\text{---}}$ and $\overset{r}{\text{---}}$ on S are defined by $\overset{l}{\text{---}} = \xrightarrow{l} \cap (\xrightarrow{l})^{-1}$ and $\overset{r}{\text{---}} = \xrightarrow{r} \cap (\xrightarrow{r})^{-1}$. For an element $a \in S$, $\Lambda_1(a) = \{x \in S \mid a \xrightarrow{l} x\}$, i.e. $\Lambda_1(a) = \sqrt{Sa}$, and the equivalence relation λ_1 on S is defined by

$$a \lambda_1 b \iff \Lambda_1(a) = \Lambda_1(b) \qquad (a, b \in S).$$

For a relation ξ on a set A, ξ^∞ denotes the transitive closure of ξ. A relation ξ on a semigroup S satisfies the *common multiple property*, shortly the *cm-property*, if for all $a, b, c \in S$, $a\,\xi\, c$ and $b\,\xi\, c$ implies $ab\,\xi\, c$. By a *quasi-order* on a set A we mean a reflexive and transitive binary relation on A.

A semigroup S is called *Archimedean* (*left Archimedean*, *weakly left Archimedean*, *t-Archimedean*) if $a \longrightarrow b$ ($a \xrightarrow{l} b$, $ab \xrightarrow{l} b$, $a \xrightarrow{t} b$), for all $a, b \in S$. A semigroup S is called *π-regular* (resp. *left π-regular*, *right π-regular*, *completely π-regular*, *intra-π-regular*) if for any $a \in S$, some power of a is regular (resp. left regular, right regular, completely regular, intra-regular). An ideal extension of a semigroup S by a nil-semigroup is called a *nil-extension* of S.

By the following table we introduce notations for some classes of semigroups and some varieties of bands which will be used later.

Notation	*Class of semigroups*	*Notation*	*Class of semigroups*
$\mathcal{A}$	Archimedean	$\mathcal{G}$	groups
$\mathcal{A}_l$	left Archimedean	$\mathcal{N}$	nil-semigroups
$\mathcal{A}_r$	right Archimedean	$\pi\mathcal{R}$	π-regular
$\mathcal{A}_t$	t-Archimedean	$\mathcal{I}\pi\mathcal{R}$	intra π-regular
$\mathcal{A}_{wl}$	weakly left Archimedean	$\mathcal{L}\pi\mathcal{R}$	left π-regular
$\mathcal{LS}$	left simple	$\mathcal{R}\pi\mathcal{R}$	right π-regular
$\mathcal{LG}$	left groups	$\mathcal{C}\pi\mathcal{R}$	completely π-regular

Notation	*Variety of bands*	*Notation*	*Variety of bands*
O	one-element bands	B	all bands
LZ	left zero bands	LN	left normal bands
RZ	right zero bands	RN	right normal bands
RB	rectangular bands	SL	semilattices

For two classes $\mathcal{X}_1$ and $\mathcal{X}_2$ of semigroups, $\mathcal{X}_1 \circ \mathcal{X}_2$ will denote the *Mal'cev product* of $\mathcal{X}_1$ and $\mathcal{X}_2$, i.e. the class of all semigroups S on which there exists a congruence ϱ such that S/ϱ belongs to $\mathcal{X}_2$ and each ϱ-class of S which is a subsemigroup of S belongs to $\mathcal{X}_1$. If $\mathcal{X}_2$ is a subclass of B, then $\mathcal{X}_1 \circ \mathcal{X}_2$ is the class of all semigroups having a band decomposition whose related factor band belongs to $\mathcal{X}_2$ and the components belong to $\mathcal{X}_1$. Such decompositions will be called $\mathcal{X}_1 \circ \mathcal{X}_2$-*decompositions*. On the other hand, if $\mathcal{X}_2$ is a subclass of $\mathcal{N}$, then $\mathcal{X}_1 \circ \mathcal{X}_2$ is the class of all semigroups that are ideal extensions of semigroups from $\mathcal{X}_1$ by semigroups from $\mathcal{X}_2$.

For undefined notions and notations we refer to [2], [5] and [14].

2. Preliminary results

In this section we will give several preliminary results.

Various properties of relations $\xrightarrow{l}$, $\stackrel{l}{\longrightarrow\!\!\!\!\!\!-}$ and λ_1 were described by the first two authors in [3] and [11]. Here we describe yet other properties of these relations.

Lemma 1. *If a semigroup S satisfies*

$$(1) \qquad (\forall a, b \in S)\ ab \xrightarrow{l} ab^2,$$

then for any $k \in \mathbb{N}$, it satisfies

$$(2) \qquad (\forall a, b \in S)\ ab \xrightarrow{l} ab^k.$$

Proof. Suppose that S satisfies (2) for some $k \in \mathbb{N}$. Assume $a, b \in S$. By (1) it follows that $ab^k = ab^{k-1}b \xrightarrow{l} ab^{k-1}b^2 = ab^{k+1}$, that is $(ab^{k+1})^m = xab^k$, for some $m \in \mathbb{N}$, $x \in S^1$. By the hypothesis, $xab \xrightarrow{l} xab^k$, that is $(xab^k)^n = yxab$, for some $n \in \mathbb{N}$, $y \in S^1$, so $(ab^{k+1})^{mn} = yxab$. Hence, S satisfies (2) for $k+1$. Now, by induction we have that S satisfies (2) for any $k \in \mathbb{N}$. $\square$

Lemma 2. *If a semigroup S satisfies*

$$(3) \qquad (\forall a, b \in S) b^2 \xrightarrow{l} ab,$$

then it also satisfies

$$(4) \qquad (\forall a, b \in S) a^2 b \xrightarrow{l} ab.$$

PROOF. Assume $a, b \in S$. By (3) we have $a^2 \xrightarrow{l} ba$, that is $(ba)^n = xa^2$, for some $n \in \mathbb{N}$, $x \in S^1$, whence $(ab)^{n+1} = a(ba)^n b = axa^2 b$, which gives $a^2 b \xrightarrow{l} ab$. □

Several conditions equivalent to the transitivity of $\xrightarrow{l}$ were given by Theorem 5 of [11] and Theorem 6 of [3]. Here we give yet other such conditions. It is interesting to note that the transitivity of $\xrightarrow{l}$ implies its right compatibility.

Lemma 3. *The following conditions on a semigroup S are equivalent:*

(i) $\xrightarrow{l}$ *is a transitive relation on S;*

(ii) $\xrightarrow{l}$ *is a right compatible quasi-order on S;*

(iii) $\stackrel{l}{\longrightarrow} = \lambda_1$ *on S;*

(iv) $(\forall a \in S) a \lambda_1 a^2$;

(v) $(\forall a, b \in S) a \xrightarrow{l} b \;\Rightarrow\; a^2 \xrightarrow{l} b$;

(vi) $(\forall a, b \in S)(\forall k \in \mathbb{N}) b^k \xrightarrow{l} ab$;

(vii) $(\forall a, b \in S) b^2 \xrightarrow{l} ab$;

(viii) *any λ_1-class of S is a subsemigroup;*

(ix) $\sqrt{Sa}$ *is a left ideal of S, for any $a \in S$;*

(x) $\sqrt{L}$ *is a left ideal of S, for any left ideal L of S.*

PROOF. Note that the equivalence of conditions (i), (iv), (v) and (ix) is a particular case of Theorem 5 of [11], and the equivalence of (v), (vi), (vii), (ix) and (x) is the dual of Theorem 6 of [3]. Therefore, it remains to prove that the conditions (ii), (iii) and (viii) are equivalent to the remaining ones.

We will establish the following sequences of implications: (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) and (vii) $\Rightarrow$ (ii) $\Rightarrow$ (viii) $\Rightarrow$ (iv).

(i) $\Rightarrow$ (iii). This follows by Lemma 6 of [11].

(iii) $\Rightarrow$ (iv). This is obvious.

(vii) $\Rightarrow$ (ii). By the equivalence of the conditions (vii) and (i) we have that $\xrightarrow{l}$ is a quasi-order. Assume that $a \xrightarrow{l} b$, for $a, b \in S$, and assume an arbitrary $c \in S$. Then $b^n = xa$, for some $n \in \mathbb{N}$, $x \in S^1$, and by (vii) and Lemma 2 we have that $b^{2k} c \xrightarrow{l} bc$, for any $k \in \mathbb{N}$. Assume $k \in \mathbb{N}$

such that $2k > n$. Then $(bc)^m = yb^{2k}c = yb^{2k-n}xac$, for some $m \in \mathbb{N}$, $y \in S^1$, whence $ac \xrightarrow{l} bc$. Hence, $\xrightarrow{l}$ is right compatible.

(ii) $\Rightarrow$ (viii). Clearly, λ_1 is a right congruence on S. Let A be a λ_1-class of S and let $a, b \in A$. Then $b\,\lambda_1\,a$, whence $b\,\lambda_1\,b^2\,\lambda_1\,ab$, since λ_1 is a right congruence, and hence $ab \in A$.

(viii) $\Rightarrow$ (iv). This is obvious. $\square$

We will see that the following lemma is one of the crucial results of the paper:

Lemma 4. *Let ξ be a band congruence on a semigroup S. Then the following conditions are equivalent:*

(i) *ξ is contained in $\xrightarrow{l}$;*

(ii) *ξ is contained in λ_1;*

(iii) *any ξ-class is a left Archimedean semigroup.*

Proof. (i) $\Rightarrow$ (iii). Let A be a ξ-class of S and let $a, b \in A$. Then $a^2\,\xi\,b$, whence $a^2 \xrightarrow{l} b$, that is $b^n = xa^2$, for some $n \in \mathbb{N}$, $x \in S^1$. Seeing that ξ is a band congruence, $xa\xi xa^2 = b^n\xi b$, so $xa \in A$ and $b^n = (xa)a \in Aa$. Therefore, $A \in \mathcal{A}_l$.

(iii) $\Rightarrow$ (ii). Assume an arbitrary pair $(a, b) \in \xi$. Let $c \in \Lambda_1(a)$, that is $a \xrightarrow{l} c$. Then $c^n = xa$, for some $n \in \mathbb{N}$ and $x \in S^1$, and $xa, xb \in A$, where A is a ξ-class of S. Since $A \in \mathcal{A}_l$, then there exist $m \in \mathbb{N}$ and $y \in S^1$ such that $(xa)^m = yxb$. Therefore, $c^{mn} = (xa)^m = yxb$, so $b \xrightarrow{l} c$ and $c \in \Lambda_1(b)$. Thus, $\Lambda_1(a) \subseteq \Lambda_1(b)$. Similarly we prove $\Lambda_1(b) \subseteq \Lambda_1(a)$. Hence, $\Lambda_1(a) = \Lambda_1(b)$, so $(a, b) \in \lambda_1$. This proves (ii).

(ii) $\Rightarrow$ (i). This is obvious. $\square$

Lemma 5. *The following conditions on a semigroup S are equivalent:*

(i) *$(\forall a, b \in S)ab^2 \xrightarrow{l} ab$;*

(ii) *$(\forall a, b \in S)a \xrightarrow{l} b \;\Rightarrow\; ba \xrightarrow{l} b$;*

(iii) *$\xrightarrow{l}$ satisfies the cm-property on S;*

(iv) *for any left ideal L of S, $\sqrt{L}$ is an intersection of completely prime left ideals of S.*

Proof. (i) $\Rightarrow$ (iii). Let $a, b, c \in S$, $a \xrightarrow{l} c$ and $b \xrightarrow{l} c$. Then $c^n = xa = yb$, for some $n \in \mathbb{N}$, $x, y \in S^1$, and by (i), $(yb)^m = zyb^2$, for some $m \in \mathbb{N}$, $z \in S^1$, whence

$$c^{nm} = (yb)^m = zyb^2 = z(yb)b = zuab \in Sab,$$

so $ab \xrightarrow{l} c$.

(iii) $\Rightarrow$ (ii). Let $a, b \in S$ and $a \xrightarrow{l} b$. Then $b \xrightarrow{l} b$ and $a \xrightarrow{l} b$, whence $ba \xrightarrow{l} b$, by (iii).

(ii) $\Rightarrow$ (i). Let $a, b \in S$. Then $b \xrightarrow{l} ab$, so by (ii), $ab^2 \xrightarrow{l} ab$.

(iii) $\Rightarrow$ (iv). Since (i) $\Leftrightarrow$ (ii), then by Lemma 3 we have that $\xrightarrow{l}$ is transitive, that is $\xrightarrow{l} = \xrightarrow{l}^{\infty}$, so by Theorem 5 of [11], for each left ideal L of S, $\sqrt{L}$ is a completely semiprime left ideal of S, and by Theorem 2 of [11], it is an intersection of completely prime left ideals of S.

(iv) $\Rightarrow$ (iii). Let $a \in S$. By (iv), $\sqrt{Sa}$ is a completely semiprime left ideal of S, so by Theorem 5 of [11], $\xrightarrow{l}$ is transitive, i.e. $\xrightarrow{l} = \xrightarrow{l}^{\infty}$. Now, by Theorem 2 of [11], $\xrightarrow{l}$ satisfies the *cm*-property. □

For an equivalence relation ξ on a semigroup S, by $\xi^{\flat}$ we denote the greatest congruence relation on S contained in ξ. It is well-known that

$$\xi^{\flat} = \{(a, b) \in \xi \mid (\forall x, y \in S^1)\ (xay, xby) \in \xi\}.$$

On a semigroup S we define a relation η by

$$a\eta b \ \Leftrightarrow\ (\forall x \in S^1)\ xa \overset{l}{\text{---}} xb.$$

This relation (or, more precisely, its dual) was introduced by P. Protić in [17], who also proved that η is a congruence relation on any semigroup. Here we prove the following:

Lemma 6. *On any semigroup S, $\eta = \lambda_1^{\flat}$.*

Proof. Assume an arbitrary pair $(a, b) \in \eta$. If $c \in \Lambda_1(a)$, that is $c^n = xa$, for some $x \in S^1$, $n \in \mathbb{N}$, then by $a\eta b$ we have that $xa \overset{l}{\text{---}} xb$, so $(xa)^m \in Sxb$, for some $m \in \mathbb{N}$, which yields $c^{nm} \in Sb$, so $c \in \Lambda_1(b)$. Thus we proved $\Lambda_1(a) \subseteq \Lambda_1(b)$. Similarly we prove $\Lambda_1(b) \subseteq \Lambda_1(a)$. Therefore, $a\,\lambda_1\,b$, which means that $\eta \subseteq \lambda_1$.

Let ϱ be an arbitrary congruence relation on S contained in λ_1. Assume an arbitrary pair $(a, b) \in \varrho$. Then for any $x \in S^1$ we have that

$$(xa, xb) \in \varrho \subseteq \lambda_1 \subseteq \overset{l}{\text{---}},$$

whence it follows that $(a,b) \in \eta$. Therefore, $\varrho \subseteq \eta$, which was to be proved. This completes the proof of the lemma. □

At the end of this section we quote four theorems proved by the first two authors in [6]. They will be used in our further work.

Theorem A. *A semigroup S belongs to $\mathcal{A}_{wl} \circ SL$ if and only if* $a \longrightarrow b \;\Rightarrow\; ab \xrightarrow{\;l\;} b$, *for all $a, b \in S$.*

Theorem B. $\mathcal{A}_{wl} = \mathcal{A}_l \circ RB = \mathcal{A}_l \circ RZ$. *A semigroup S belongs to $\mathcal{A}_{wl}$ if and only if $\xrightarrow{\;l\;}$ is a symmetric relation on S.*

Theorem C. $\mathcal{L}\pi\mathcal{R} \cap \mathcal{A}_{wl} = \mathcal{I}\pi\mathcal{R} \cap \mathcal{A}_{wl} = (\mathcal{LS} \circ \mathcal{N}) \circ RB = (\mathcal{LS} \circ \mathcal{N}) \circ RZ$.

Theorem D. $\pi\mathcal{R}\cap\mathcal{A}_{wl} = \mathcal{R}\pi\mathcal{R}\cap\mathcal{A}_{wl} = \mathcal{C}\pi\mathcal{R}\cap\mathcal{A}_{wl} = (\mathcal{LG}\circ\mathcal{N})\circ RB = (\mathcal{LG} \circ \mathcal{N}) \circ RZ$.

3. Decomposition theorems: The general case

As we noted before, the first characterization of bands of left Archimedean semigroups was given by M. S. Putcha in [16], and this result we quote in the next theorem as the equivalence of conditions (i) and (ii). Moreover, we give several new characterizations of semigroups having such a decomposition.

Theorem 1. *The following conditions on a semigroup S are equivalent:*

(i) $S \in \mathcal{A}_l \circ B$;

(ii) $(\forall a \in S)(\forall x, y \in S^1)\, xay \overset{l}{\text{---}} xa^2y$;

(iii) η *is a band congruence on* S;

(iv) $(\forall a, b \in S)\, a^2b \xrightarrow{\;l\;} ab \;\&\; ab \xrightarrow{\;l\;} ab^2$;

(v) $(\forall a, b \in S)\, ab \overset{l}{\text{---}} ab^2$.

Proof. (i) $\Leftrightarrow$ (ii). This is Theorem 1 of [16].

(iii) $\Rightarrow$ (i). This follows by Lemma 4.

(iv) $\Rightarrow$ (v). Assume $a, b \in S$ such that $a \longrightarrow b$, that is $b^m = xay$, for some $m \in \mathbb{N}$, $x, y \in S^1$. By (iv) we have $(xa)^2y \xrightarrow{\;l\;} xay$, that is $(xay)^n = z(xa)^2y = zxab^m$, for some $n \in \mathbb{N}$, $z \in S^1$. On the other hand, by Lemma 1, $zxab \xrightarrow{l} zxab^m$, that is $(zxab^m)^k = uzaxb$, for some $k \in \mathbb{N}$, $u \in S^1$, which gives $b^{mnk} = uzxab$, that is $ab \xrightarrow{l} b$. Now, by Theorem A, S is a semilattice Y of weakly left Archimedean semigroups $S_\alpha, \alpha \in Y$.

Assume $a, b \in S$. Then $ab \xrightarrow{l} ab^2$ in S, and $ab, ab^2 \in S_\alpha$, for some $\alpha \in Y$, so by Lemma 11 (c) of [11], $ab \xrightarrow{l} ab^2$ in S_α. By Theorem B, $\xrightarrow{l}$ is a symmetric relation on S_α, whence $ab^2 \xrightarrow{l} ab$.

(v) $\Rightarrow$ (iv). This follows by Lemma 2.

(v) $\Rightarrow$ (ii). Clearly, $b^2 \xrightarrow{l} ab$, for all $a, b \in S$, so by Lemma 3, $\overset{l}{\text{---}}$ is a right congruence. Assume $a, b, c \in S$. By (v) and (iv) we have $ab \overset{l}{\text{---}} ab^2$ and $ab \overset{l}{\text{---}} a^2b$, and since $\overset{l}{\text{---}}$ is a right congruence, then $abc \overset{l}{\text{---}} ab^2c$. Hence, (ii) holds.

(ii) $\Rightarrow$ (v). This is clear.

(iii) $\Rightarrow$ (i). This follows by Lemma 4.

(v) $\Rightarrow$ (iii). This follows by Lemma 6. $\square$

As a consequence of the previous theorem we obtain the next corollary. Note that the characterization of semigroups from $\mathcal{A}_t \circ B$ given here is simpler than the one given by M. S. Putcha in [16].

Corollary 1. *A semigroup S belongs to $\mathcal{A}_t \circ B$ if and only if $a^2b \overset{r}{\text{---}} ab \overset{l}{\text{---}} ab^2$, for all $a, b \in S$.*

The concept of π-regularity, in its various forms, appeared first in Ring theory, as a natural generalization of the regularity. In Semigroup theory this concept attracts great attention both as a generalizations of the regularity and a generalization of the finiteness and the periodicity. On the other hand, there are specific relations between the π-regularity and the Archimedeaness, as was shown by M. S. Putcha in [15]. That motivates us to investigate $\mathcal{A}_l \circ B$-decompositions of π-regular semigroups.

We do it first for intra π-regular and left π-regular semigroups. It is interesting to note that for left π-regular semigroups only a half of the Condition (v) of Theorem 1 is enough.

Theorem 2. *The following conditions on a semigroup S are equivalent:*

(i) $S \in \mathcal{L}\pi\mathcal{R} \cap \mathcal{A}_l \circ B$;

(ii) $S \in \mathcal{I}\pi\mathcal{R} \cap \mathcal{A}_l \circ B$;

(iii) $S \in (\mathcal{LS} \circ \mathcal{N}) \circ B$;

(iv) *$S \in \mathcal{L}\pi\mathcal{R}$ and $ab^2 \xrightarrow{l} ab$, for all $a, b \in S$;*

(v) $(\forall a, b \in S)(\exists n \in \mathbb{N})(ab)^n \in S(ab^2)^n$.

Proof. (iii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (ii). This is trivial.

(ii) $\Rightarrow$ (i). Since $\mathcal{I}\pi\mathcal{R}\cap\mathcal{A}_l\circ B \subseteq \mathcal{I}\pi\mathcal{R}\cap\mathcal{A}_{wl}\circ SL = (\mathcal{I}\pi\mathcal{R}\cap\mathcal{A}_{wl})\circ SL = (\mathcal{L}\pi\mathcal{R}\cap\mathcal{A}_{wl})\circ SL = \mathcal{L}\pi\mathcal{R}\cap\mathcal{A}_{wl}\circ SL$, by Theorems A, B and C, then (iii) implies (ii).

(i) $\Rightarrow$ (iii). As known, each component of a band decomposition of a left π-regular semigroup is also left π-regular. By this and by Theorem 4.1 of [15] we obtain (i).

(iii) $\Rightarrow$ (v). Let S be a band I of semigroups S_i, $i \in I$, and for each $i \in I$, let S_i be a nil-extension of a left simple semigroup K_i. Then for all $a, b \in S$, $ab, ab^2 \in S_i$, for some $i \in I$, and $(ab)^n, (ab^2)^n \in K_i$, for some $n \in \mathbb{N}$, whence $(ab)^n \in K_i(ab^2)^n \subseteq S(ab^2)^n$.

(v) $\Rightarrow$ (iv). This is obvious.

(iv) $\Rightarrow$ (i). By Theorem 1 of [10], S is a semilattice Y of Archimedean semigroups S_α, $\alpha \in Y$. I was proved by Theorem 1 of [8] that $\mathcal{A}\cap\mathcal{L}\pi\mathcal{R} = (\mathcal{LS}\circ RZ)\circ\mathcal{N}$, so for any $\alpha \in Y$, S_α is a nil-extension of a semigroup K_α which is a right zero band I_α of left simple semigroups K_i, $i \in I_\alpha$.

Assume $\alpha \in Y$, $i \in I_\alpha$, and set $S_i = \sqrt{K_i}$. Further, let $i, j \in I_\alpha$, $a \in S_i$, $b \in S_j$, and assume $m \in \mathbb{N}$ such that $b^m \in K_j$. By (iv) and by Lemma 1, $ab^{m+1} \xrightarrow{l} ab$ in S, so by Lemma 11 (c) of [11], $(ab)^n = xab^{m+1}$, for some $n \in \mathbb{N}$, $x \in S_\alpha^1$. Assume $k \in \mathbb{N}$ such that $(ab)^k \in K_\alpha$. Then

$$(ab)^{k+n} = (ab)^k(xab)b^m \in K_\alpha S_\alpha K_i \subseteq K_\alpha K_i \subseteq K_i,$$

so $ab \in S_j$. Hence, for any $\alpha \in Y$, S_α is a right zero band I_α of semigroups S_i, $i \in I_\alpha$, and for any $i \in I_\alpha$, S_i is a nil-extension of a left simple semigroup K_i. Now, by Theorem B, for any $\alpha \in Y$, $\xrightarrow{l}$ is a symmetric relation on S_α, and as in the proof of (iv) $\Rightarrow$ (v) of Theorem 1 we obtain that $ab \overset{l}{\text{---}} ab^2$, for all $a, b \in S$. Hence, by Theorem 1 we obtain (ii). $\square$

For π-regular semigroups we have the following:

Theorem 3. *The following conditions on a semigroup S are equivalent:*

(i) *$S \in \mathcal{R}\pi\mathcal{R}\cap\mathcal{A}_l\circ B$;*

(ii) *$S \in \pi\mathcal{R}\cap\mathcal{A}_l\circ B$;*

(iii) $S \in \mathcal{C}\pi\mathcal{R} \cap \mathcal{A}_l \circ B$;

(iv) $S \in (\mathcal{LG} \circ \mathcal{N}) \circ B$;

(v) $S \in \pi\mathcal{R}$ *and* $ab^2 \xrightarrow{l} ab$, *for all* $a, b \in S$;

(vi) $(\forall a, b \in S)(\exists n \in \mathbb{N})(ab)^n \in (ab)^n S(ab^2)^n$.

Proof. (iv) $\Rightarrow$ (iii) and (vi) $\Rightarrow$ (v). This is clear.

(i) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iii). This can be proved similarly as (ii) $\Rightarrow$ (i) of Theorem 2, using Theorems C and D.

(iii) $\Rightarrow$ (iv). This follows by the arguments similar to the ones used in (i) $\Rightarrow$ (iii) of Theorem 2.

(iv) $\Rightarrow$ (vi). This can be proved similarly as (iii) $\Rightarrow$ (v) of Theorem 2, using Lemma 1.1 of [9].

(v) $\Rightarrow$ (ii). Let $a \in \mathrm{Reg}(S)$, $a' \in V(a)$. Then $a'a^2 \xrightarrow{l} a'a$, whence $a \in \mathrm{LReg}(S)$, so S is left π-regular, and by Theorem 2, $S \in \mathcal{A}_l \circ B$. $\square$

Some other characterizations of semigroups from $(\mathcal{LG} \circ \mathcal{N}) \circ B$ one can obtain by the results concerning their dual semigroups, given by L. N. Shevrin in [20].

Corollary 2 [4]. *The following conditions on a semigroup* S *are equivalent:*

(i) $S \in (\mathcal{G} \circ \mathcal{N}) \circ B$;

(ii) $S \in \mathcal{I}\pi\mathcal{R} \cap \mathcal{A}_t \circ B$;

(iii) $S \in \pi\mathcal{R} \cap \mathcal{A}_t \circ B$;

(iv) $S \in \pi\mathcal{R}$ *and* $a^2b \xrightarrow{r} ab$ & $ab^2 \xrightarrow{l} ab$, *for all* $a, b \in S$.

4. Decomposition theorems: Special kinds of bands

Our next goal is to characterize semigroups from $\mathcal{A}_l \circ V$, for an arbitrary variety of bands V.

The lattice LVB of all varieties of bands was studied by P. A. Birjukov, C. F. Fennemore, J. A. Gerhard, M. Petrich and others. Here we use the characterization of LVB given by J. A. Gerhard and M. Petrich in [12]. They defined inductively three systems of words as follows:

$$G_2 = x_2x_1, \qquad H_2 = x_2, \qquad I_2 = x_2x_1x_2,$$

$$G_n = x_n\overline{G}_{n-1}, \qquad H_n = x_n\overline{G}_{n-1}x_n\overline{H}_{n-1}, \qquad I_n = x_n\overline{G}_{n-1}x_n\overline{I}_{n-1},$$

Figure 1.

(for $n \geq 3$), and they shown that the lattice LVB can be represented by the graph given in Figure 1.

Let us give some additional explanations concerning the graph from Figure 1. Throughout this section, for a semigroup identity $u = v$, by $[u = v]$ we will denote the variety of bands determined by this identity. In other words, this is a shortened notation for the semigroup variety $[x^2 = x,\, u = v]$. For a word w, $\overline{w}$ denotes the *dual* of w, that is, the word obtained from w by reversing the order of the letters in w. In the graph from Figure 1 we have labelled only the nodes which represent varieties of bands that will appear in our further investigations.

The central point of this section is the following theorem:

Theorem 4. *Let V be an arbitrary variety of bands. Then*

$$LZ \circ V = \begin{cases} LZ & \text{if } V \in [O, LZ]; \\ RB & \text{if } V \in [RZ, RB]; \\ [G_2 = I_2] & \text{if } V \in \big[SL, [G_2 = I_2]\big]; \\ [G_3 = I_3] & \text{if } V \in \big[RN, [G_3 = H_3]\big]; \\ [G_{n+1} = I_{n+1}] & \text{if } V \in \big[[\overline{G}_n = \overline{I}_n], [G_{n+1} = I_{n+1}]\big],\ n \geq 2; \\ [G_{n+1} = H_{n+1}] & \text{if } V \in \big[[\overline{G}_n = \overline{H}_n], [G_{n+1} = H_{n+1}]\big],\ n \geq 3. \end{cases}$$

Proof. Consider the congruence η on a band S. Since $\lambda_1 = \overset{l}{\overline{\quad}} = \mathcal{L}$ on S, then $\eta = \mathcal{L}^\flat$. It is known that the Green relation $\mathcal{L}$ on S is defined by $(a,b) \in \mathcal{L} \ \Leftrightarrow\ ab = a \ \&\ b = ba$, whence we conclude that

$$(a,b) \in \eta \ \Leftrightarrow\ (\forall x \in S^1)\ xa = xaxb \ \&\ xb = xbxa. \tag{5}$$

But, if $xa = xaxb$ and $xb = xbxa$, for any $x \in S$, then for $x = a$ we have $a = ab$, and for $x = b$ we have $b = ba$, so the condition (5) is equivalent to

$$(a,b) \in \eta \ \Leftrightarrow\ (\forall x \in S)\ xa = xaxb \ \&\ xb = xbxa. \tag{6}$$

Let $[V_1, V_2]$ be some of the intervals of LVB which appears in the formulation of the theorem. We will prove:

$$S \in V_2 \ \Leftrightarrow\ S/\eta \in V_1, \tag{7}$$

for any band S.

Case 1: $[V_1, V_2] = [O, LZ]$. This case is trivial.

Case 2: $[V_1, V_2] = [RZ, RB]$. In this case the assertion (7) is an immediate consequence of the construction of a rectangular band.

Case 3: $[V_1, V_2] = \big[SL, [G_2 = I_2]\big]$. This case was considered in the dual of Proposition II 3.12 of [14].

Case 4: $[V_1, V_2] = \big[RN, [G_3 = H_3]\big]$. This case was considered in the dual of Proposition II 3.8 of [14].

Case 5: $[V_1, V_2] = \big[\overline{G}_2 = \overline{I}_2, [G_3 = I_3]\big]$. This case was considered in the dual of Proposition II 3.5 of [14].

Note that in all of these cases the Green relation $\mathcal{L}$ is a congruence, i.e. $\eta = \mathcal{L}$. In other words, for a band S we have that $\mathcal{L}$ is a congruence on S if and only if $S \in [G_3 = I_3]$.

Case 6: $[V_1, V_2] = \big[\overline{G}_n = \overline{I}_n, [G_{n+1} = I_{n+1}]\big]$, $n \geq 3$. Here we have that $V_2 = [x_{n+1}\overline{G}_n = x_{n+1}\overline{G}_n x_{n+1}\overline{I}_n]$.

Let S be an arbitrary band. Suppose first that $S \in V_2$. For $1 \leq i \leq n$ let the letter x_i get a value a_i in S. Then the words $\overline{G}_n$ and $\overline{I}_n$ get some values u and v in S, respectively. To prove that $S/\eta \in V_1 = [\overline{G}_n = \overline{I}_n]$, it is enough to prove that $(u, v) \in \eta$.

Assume an arbitrary $a \in S$. If the letter x_{n+1} assumes in S a value a, then by $S \in V_2$ it follows $au = auav$. Since the words $\overline{G}_n$ and $\overline{I}_n$ have the same letters, then $(u, v) \in \mathcal{D}$ and $(au, av) \in \mathcal{D}$. But, any $\mathcal{D}$-class of S is a rectangular band, whence by $au = auav$ it follows $avau = avauav = av$. Therefore, $au = auav$ and $av = avau$, for any $a \in S$, whence $(u, v) \in \eta$, which was to be proved.

Conversely, assume that $S/\eta \in V_1$. For $1 \leq i \leq n+1$ let the letter x_i get an arbitrary value a_i in S. Then the words $\overline{G}_n$ and $\overline{I}_n$ get some values u and v in S, respectively, and $(u, v) \in \eta$, since $S/\eta \in V_1 = [\overline{G}_n = \overline{I}_n]$. But, by $(u, v) \in \eta$ it follows that $a_{n+1}u = a_{n+1}ua_{n+1}v$, by (6), whence we conclude that $S \in [x_{n+1}\overline{G}_n = x_{n+1}\overline{I}_n] = V_2$. This completes the proof of this case.

Case 7: $[V_1, V_2] = \big[\overline{G}_n = \overline{H}_n, [G_{n+1} = H_{n+1}]\big]$, $n \geq 3$. This case is analogous to the previous one.

Considering all the cases we have completed the proof of the theorem. □

Note that some related results were obtained by E. V. Sukhanov in [21] and F. Pastijn in [13].

By a straightforward verification we prove the following lemma:

Lemma 7. *Let $\mathcal{X}$ be a class of semigroups and let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two classes of bands. Then $\mathcal{X} \circ (\mathcal{B}_1 \circ \mathcal{B}_2) \subseteq (\mathcal{X} \circ \mathcal{B}_1) \circ \mathcal{B}_2$.*

A particular case of the previous lemma is the well-known result of A. H. Clifford from 1954 that asserts that $\mathcal{X} \circ B = \mathcal{X} \circ (RB \circ SL) \subseteq (\mathcal{X} \circ RB) \circ SL$, for an arbitrary class $\mathcal{X}$ of semigroups. For the class $\mathcal{G}$ of all groups, $\mathcal{G} \circ B = \mathcal{G} \circ (RB \circ SL)$ is the class of all semigroups that are bands of groups, and $(\mathcal{G} \circ RB) \circ SL$ is the class of all semigroups that are unions of groups. As known, these classes are different, so $\mathcal{G} \circ (RB \circ SL) \subsetneqq (\mathcal{G} \circ RB) \circ SL$. This proves that the inclusion in Lemma 7 can be proper.

Using the above theorem and lemma we prove the following

Theorem 5. *Let V be an arbitrary variety of bands. Then*

$$\mathcal{A}_l \circ V = \begin{cases} \mathcal{A}_l & \text{if } V \in [O, LZ]; \\ \mathcal{A}_l \circ RZ & \text{if } V \in [RZ, RB]; \\ \mathcal{A}_l \circ SL & \text{if } V \in \big[SL, [G_2 = I_2]\big]; \\ \mathcal{A}_l \circ RN & \text{if } V \in \big[RN, [G_3 = H_3]\big]; \\ \mathcal{A}_l \circ [\overline{G}_n = \overline{I}_n] & \text{if } V \in \big[[\overline{G}_n = \overline{I}_n], [G_{n+1} = I_{n+1}]\big],\ n \geq 2; \\ \mathcal{A}_l \circ [\overline{G}_n = \overline{H}_n] & \text{if } V \in \big[[\overline{G}_n = \overline{H}_n], [G_{n+1} = H_{n+1}]\big],\ n \geq 3. \end{cases}$$

Proof. One verifies easily that $\mathcal{A}_l \circ LZ = \mathcal{A}_l$. Further, let $[V_1, V_2]$ be some of the intervals of the lattice LVB which appears in the formulation of the theorem, and let $V \in [V_1, V_2]$. By Theorem 4 we have that $V_2 = LZ \circ V_1$, whence

$$\mathcal{A}_l \circ V_1 \subseteq \mathcal{A}_l \circ V \subseteq \mathcal{A}_l \circ V_2 = \mathcal{A}_l \circ (LZ \circ V_1) \subseteq (\mathcal{A}_l \circ LZ) \circ V_1 = \mathcal{A}_l \circ V_1,$$

using Lemma 7. Therefore, $\mathcal{A}_l \circ V_1 = \mathcal{A}_l \circ V = \mathcal{A}_l \circ V_2$, which was to be proved. □

Finally, we prove the following:

Theorem 6. *Let V be an arbitrary variety of bands and let S be a semigroup. Then $S \in \mathcal{A}_l \circ V$ if and only if $S/\eta \in V$.*

Proof. Let $S \in \mathcal{A}_l \circ V$. Then there exists a congruence ξ on S such that $S/\xi \in V$ and any ξ-class of S is in $\mathcal{A}_l$. By Lemma 4 we have $\xi \subseteq \lambda_1$, and by Lemma 6, $\xi \subseteq \eta$. Therefore, S/η is a homomorphic image of S/ξ and $S/\xi \in V$, whence $S/\eta \in V$, which was to be proved.

Conversely, if $S/\eta \in V$, then by Lemma 4 we have that any η-class is in $\mathcal{A}_l$, and hence, $S \in \mathcal{A}_l \circ V$. □

Note that the corresponding results can be obtained for bands of left simple semigroups and bands of left groups.

Acknowledgement. The authors are very indebted to the referee, whose valuable comments and suggestions considerably increased the quality of the paper.

References

[1] S. Bogdanović, Semigroups of Galbiati-Veronesi, Algebra and Logic, *Zagreb*, 1984, 9–20.

[2] S. Bogdanović, Semigroups with a system of subsemigroups, *Inst. of Math.*, *Novi Sad*, 1985.

[3] S. Bogdanović and M. Ćirić, Semigroups in which the radical of every ideal is a subsemigroup, *Zb. rad. Fil. fak. Niš, Ser. Mat.* **6** (1992), 129–135.

[4] S. Bogdanović and M. Ćirić, Semigroups of Galbiati-Verovesi IV (*Bands of nil-extensions of groups*), *Facta Univ. (Niš), Ser. Math. Inform.* **7** (1992), 23–35.

[5] S. Bogdanović and M. Ćirić, Semigroups, *Prosveta*, *Niš*, 1993. (in *Serbian*)

[6] S. Bogdanović and M. Ćirić, Semilattices of weakly left Archimedean semigroups, *FILOMAT (Niš)* **9:3** (1995), 603–610.

[7] S. Bogdanović and M. Ćirić, A note on left regular semigroups, *Publ. Math. Debrecen* **49** / 3–4 (1996), 1–7.

[8] S. Bogdanović and M. Ćirić, Semilattices of left completely Archimedean semigroups (*to appear*).

[9] S. Bogdanović and B. Stamenković, Semigroups in which S^{n+1} is a semilattice of right groups (*Inflations of a semilattices of right groups*), *Note di Matematica* **8** no. 1 (1988), 155–172.

[10] M. Ćirić and S. Bogdanović, Decompositions of semigroups induced by identities, *Semigroup Forum* **46** (1993), 329–346.

[11] M. Ćirić and S. Bogdanović, Semilattice decompositions of semigroups, *Semigroup Forum* **52** (1996), 119–132.

[12] J. A. Gerhard and M. Petrich, Varieties of bands revisited, *Proc. London Math. Soc.* **58** (3) (1989), 323–350.

[13] F. Pastijn, The lattice of completely regular semigroup varieties, *J. Austral. Math. Soc. (Ser. A)* **49** (1990), 24–42.

[14] M. Petrich, Lectures in semigroups, *Akad. Verlag*, *Berlin*, 1977.

[15] M. S. Putcha, Semilattice decompositions of semigroups, *Semigroup Forum* **6** (1973), 12–34.

[16] M. S. Putcha, Bands of t-Archimedean semigroups, *Semigroup Forum* **6** (1973), 232–239.

[17] P. Protić, The band and the semilattice decompositions of some semigroups, *Pure Math. and Appl. Ser. A* **2** no. 1–2 (1991), 141–146.

[18] P. Protić, On some band decompositions of semigroups, *Publ. Math. Debrecen* **45** / 1–2 (1994), 205–211.

[19] P. Protić, Bands of right simple semigroups, *Publ. Math. Debrecen* **47** / 3–4 (1995), 311–313.

[20] L. N. Shevrin, Theory of epigroups II, *Mat. Sbornik* **185** no. 9 (1994), 153–176.

[21] E. V. Sukhanov, The groupoid of varieties of idempotent semigroups, *Semigroup Forum* **14** (1977), 143–159.

STOJAN BOGDANOVIĆ
UNIVERSITY OF NIŠ
FACULTY OF ECONOMICS
TRG JNA 11, 18000 NIŠ
YUGOSLAVIA

E-mail: sbogdan@archimed.filfak.ni.ac.yu

MIROSLAV ĆIRIĆ
UNIVERSITY OF NIŠ
FACULTY OF PHILOSOPHY
ĆIRILA I METODIJA 2, 18000 NIŠ
YUGOSLAVIA

E-mail: mciric@archimed.filfak.ni.ac.yu

BORIS NOVIKOV
UNIVERSITY OF KHARKOV
DEPARTMENT OF MATHEMATICS
310178 KHARKOV
UKRAINE

E-mail: novikov@relay.univer.kharkov.ua

(Received July 18, 1996; revised April 7, 1997)