Modules and semi-simple rings. II.

To the memory of my unforgettable teacher and beloved friend Professor Tibor Szele.

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§ 1. Introduction.

In an earlier paper of the same title [3][1]) I gave a characterization, in the case of unitary modules, of the completely reducible modules, i. e. the modules admitting a representation as a direct sum of minimal submodules. (By a unitary module we usually mean a module furnished with a left operator domain which is a ring containing a unit element 1 such that 1 acts as the identity operator on the module.) By this characterization, the following conditions are equivalent for an arbitrary unitary R-module G:

α) G is completely reducible;

β) the order of each element ($\neq 0$) of G is the intersection of a finite number of maximal left ideals of G;

γ) every maximal independent system of elements in G is a basis of G;

δ) any submodule of G is a direct summand of G.[2])

As an application we obtained the following result: a ring R with unit element is semi-simple[3]) if and only if every unitary R-module is completely reducible.

In this second part of our paper we extend our investigations from unitary modules to arbitrary modules. We obtain, also in this more general case, a similar characterization of the completely reducible modules. Moreover, we succeed in giving a characterization of the semi-simple rings as operator

1) The numbers in brackets refer to the Bibliography at the end of this paper.

2) In the special case when R is the ring of the rational integers, this theorem evidently yields the characterizations of elementary abelian groups, i. e. of the groups which are direct sums of groups of prime order.

3) Here and in the sequel, we use, for the sake of brevity, the expression „semi-simple ring" in the classical sense, i. e. the assumption „with descending chain condition" is made tacitly throughout.

domains without postulating the existence of the unit element. This result reads as follows: *an arbitrary ring R is semi-simple if and only if, for every left ideal L of R and for each element g of an arbitrary R-module G, the submodule Lg is a direct summand of G.* By an application of this theorem we obtain a simple proof of an important theorem of O. GOLDMAN ([2], Theorem III) which is the first known characterization of the semi-simple rings as operator domains in the class of all rings.

§ 2. Preliminaries.

Let R be an arbitrary (associative) ring and G a left R-module. By a submodule resp. a homomorphism of G we mean always an R-submodule resp. an R-homomorphism. We denote by $O(g)$ the order of an element g of the module G, i. e. the set of all elements $r \in R$ with $rg = 0$. Obviously, $O(g)$ is a left ideal of R.

We call an arbitrary set $\ldots, g_\nu, \ldots$ of non-zero elements in G *independent*, if for every finite subset of this set a relation

$$r_1 g_1 + \cdots + r_n g_n = 0 \qquad (r_i \in R)$$

implies

$$r_1 g_1 = \cdots = r_n g_n = 0.$$

Since the independence so defined is a property of finite character, by virtue of Zorn's lemma an arbitrary set of elements in G contains a maximal independent subsystem. Let $S: b_1, b_2, \ldots, b_\mu, \ldots$ be an arbitrary system of elements of G. The set H of all (finite) sums of the form

$$s_1 b_{\mu_1} + \cdots + s_m b_{\mu_m} \qquad (b_{\mu_i} \in S,\ s_i \in R)$$

is a submodule of G. In this case we say that H is the submodule of G spanned by the system S. If G contains an independent subset S of elements which spans the whole module G, then S is called a *basis* of G.

Let A be an arbitrary submodule of the R-module G, and let us denote by RA the set of all finite sums with summands of the form ra $(r \in R, a \in A)$. RA is a submodule of G and also of A. If $RG = G$, we say that G is a *perfect* R-module. It is evident that an R-module is perfect if and only if it has a system of elements which spans the whole module. This implies directly that a unitary module is always perfect. The *trivial modules*, to be defined now, are of an opposite character: An R-module G is trivial if $RG = 0$. The *trivial submodule* of an arbitrary R-module G is the set of all elements $x \in G$ for which $Rx = 0$. An R-module A is called *minimal* (in another terminology: irreducible, or simple) if A contains no submodules other than A and 0. It is clear that a minimal R-module is either perfect or trivial, moreover, that an R-module which appears as a direct sum is perfect resp. trivial if and only if all its direct summands are perfect resp. trivial.

For an arbitrary but fixed ring R we have a complete survey of all minimal R-modules. If the minimal R-module A is trivial then A is a (cyclic) group of prime order. In the other case, i. e. if A is perfect, then A is isomorphic to one of the factor modules R/M where M is an arbitrary maximal left ideal of R, and conversely, the module R/M is minimal for every maximal left ideal of R. Moreover, for arbitrary elements $a \neq 0$, $b \neq 0$ of a perfect minimal R-module A we have

$$A \cong R/O(a) \cong R/O(b)$$

where $O(a)$ and $O(b)$ are maximal left ideals in R, but in general $O(a) \neq O(b)$. (If R is commutative, then $O(a) = O(b)$.)

In the sequel we shall need also the following

Lemma. *An arbitrary module G is a direct sum of a finite number of minimal modules if and only if there exists a finite number of maximal submodules in G with 0 intersection.*

Proof. If G admits a representation as a direct sum of its minimal submodules $A_1, \dots, A_n$, then the intersection of the maximal submodules

$$A_2 + A_3 + \cdots + A_n,\ A_1 + A_3 + \cdots + A_n,\ \dots,\ A_1 + A_2 + \cdots + A_{n-1}$$

is 0.[4]) Conversely, suppose that $M_1, \dots, M_n$ are maximal submodules of an arbitrary operator module G, such that

$$M_1 \cap \cdots \cap M_n = 0 \tag{1}$$

and none of the M_i 's can be cancelled in (1). We show that in this case G can be represented as the direct sum of the n minimal submodules

$$A_i = M_1 \cap \cdots \cap M_{i-1} \cap M_{i+1} \cap \cdots \cap M_n \qquad (i = 1, \dots, n). \tag{2}$$

First of all we remark that the A_i 's $(i = 1, \dots, n)$ are in fact minimal. By the hypothesis that in (1) no M_i can be cancelled, we have $A_i \neq 0$, moreover, by virtue of (1) the sum of A_i and M_i is *direct*. Finally, in view of $M_i \subset A_i + M_i \subseteq G$ we have by the maximality of M_i

$$\left\{\begin{array}{l} G = A_1 + M_1 \\ G = A_2 + M_2 \\ \quad \vdots \\ G = A_n + M_n . \end{array}\right. \tag{3}$$

From this $A_i \cong G/M_i$, i. e. the minimality of A_i follows.

Now, applying the second resp. the third etc. equality in (3) to the elements of M_1 resp. $M_1 \cap M_2$ etc., we obtain

$$\begin{array}{c} M_1 = A_2 + (M_1 \cap M_2) \\ M_1 \cap M_2 = A_3 + (M_1 \cap M_2 \cap M_3) \\ \vdots \\ M_1 \cap M_2 \cap \cdots \cap M_{n-1} = A_n + (M_1 \cap M_2 \cap \cdots \cap M_n) = A_n . \end{array} \tag{4}$$

[4]) The sign $+$ is used to denote (besides the group operation) also the direct sum.

By a successive substitution of these expressions into $G = A_1 + M_1$ we obtain

$$G = A_1 + A_2 + \cdots + A_n,$$

q. e. d.

As special cases of our lemma we obtain the following results:

An abelian group is a direct sum of a finite number of groups of prime order if and only if it has a finite number of maximal subgroups with 0 *intersection.*

An arbitrary ring (considered as a module) is a direct sum of a finite number of minimal left ideals if and only if it has a finite number of maximal left ideals whose intersection is 0.[5])

§ 3. On completely reducible modules.

For an arbitrary R-module the conditions α), β), γ), δ) of § 1 are in general not equivalent. This is clearly shown e. g. by a trivial module satisfying condition α). However, for an arbitrary R-module G the following conditions are always equivalent:

1. G is completely reducible;
2. any submodule of G is a direct summand of G.

Hence it follows that every submodule and every homomorphic image of a completely reducible module is again completely reducible.

Now we restrict our considerations to the case of perfect modules. We are going to prove the following

Theorem 1. *For an arbitrary R-module G the following conditions are equivalent:*

α') *G is a completely reducible perfect module;*

β') *G is perfect and the order of each element* ($\neq 0$) *of G is the intersection of a finite number of maximal left ideals of R;*

γ') *every maximal independent system of elements in G is a basis of G;*

δ') *G is perfect and any submodule of G is a direct summand of G.*

Remark. As a module satisfying γ') is always perfect, and a trivial module can also be completely reducible, the supposition of perfectness is essential in α') and δ'). It is an open question whether or not perfectness can be omitted from β').

[5]) The character of this proof shows that an analogous criterion must be fulfilled in order that an arbitrary group (ring) be the direct product (sum) of a finite number of its minimal normal subgroups (ideals).

PROOF. α') *implies* β'). Suppose that G is a direct sum of minimal R-modules A_ν:

$$G=\sum_\nu A_\nu. \tag{5}$$

Let

$$g=a_{\nu_1}+\cdots+a_{\nu_n} \qquad (0\neq a_{\nu_i}\in A_{\nu_i}).$$

By the perfectness of A_{ν_i}, $O(a_{\nu_i})=M_i$ is a maximal left ideal of R and so we have

$$O(g)=M_1\cap\cdots\cap M_n=D, \tag{6}$$

i. e. $O(g)$ is the intersection of a finite number of maximal left ideals of R.

β') *implies* γ'). Let S be an arbitrary independent system of elements in G, and H the submodule spanned by S. Since by the maximality of the system S the submodule H contains each minimal submodule of G, it is sufficient to prove that for an arbitrary element $g\neq 0$ of G Rg is contained in a submodule of G generated by minimal submodules. Namely this implies that together with all submodules Rg also RG is a part of H, and since G is perfect, i. e. $RG=G$, S is in fact a basis of G.

We are going to prove that for an arbitrary element $g\neq 0$ of G the module Rg admits a representation as a direct sum of a finite number of minimal R-modules. Suppose that the order of g can be represented in the form (6) where $M_1,\dots,M_n$ are maximal left ideals in R. It follows from (6) that

$$Rg\cong R/D$$

and

$$M_1/D\cap\cdots\cap M_n/D=0.$$

On the other hand, since in the module R/D the submodules M_i/D are maximal, by our lemma we obtain that the module Rg (which is isomorphic with R/D) is a direct sum of a finite number of minimal R-modules.

For a proof of the assertions γ') *implies* δ') and δ') *implies* α') see the proof of Theorem 1 in [3]. The proof given there for the implications γ) implies δ) and δ) implies α) remains valid also in this more general case. Finally, let us remark that the perfectness of G is an obvious consequence of γ'). This completes the proof of Theorem 1.

§ 4. Semi-simple rings as operator domains.

In this section we investigate the semi-simple rings as operator domains. By a *semi-simple ring* we mean such a ring taken in the classical sense, i. e. a ring containing no non-zero nilpotent left ideal and satisfying the descending chain condition for left ideals. According to the well-known WEDDERBURN-ARTIN structure theorem such a ring is isomorphic to a direct sum of a finite number of rings, each of which is isomorphic to the complete ring of linear transformations in a suitable finite dimensional vector space over a skew field. By another characterization, *a ring R is semi-simple if and only if every left ideal of R contains a right unit element* (see [1]). In our proofs we make use only of this second characterization of semi-simple rings.

The semi-simple rings have remarkable properties also as operator domains. An earlier related result is the following theorem by O. GOLDMANN [2]: *An arbitrary ring R is semi-simple if and only if every R-module admits a representation as a direct sum of its trivial submodule and a completely reducible (perfect) module.* Now we prove another criterion of a similar kind, an application of which will yield a simple proof of the above theorem of GOLDMAN.

Theorem 2. *An arbitrary ring R is semi-simple if and only if, for every left ideal L of R and for each element g of any R-module G, the submodule Lg is a direct summand of G.*

PROOF. Suppose that for every left ideal L of R and for each element g of the arbitrary R-module G the submodule Lg is a direct summand of G. Then we show that every left ideal of R contains a right unit element, i. e. R is a semi-simple ring. Let G be the set of all pairs (a, n), $a \in R$, n a rational integer, with the trivial definition of equality and component-wise addition. We define the product of an element $r \in R$ by $(a, n) \in G$ by

$$r(a, n) = (ra + nr, 0)$$

where nr has the obvious meaning. So G becomes a left R-module. If L is an arbitrary left ideal in R and $g = (0, 1)$ $(\in G)$, then Lg is the set of all pairs $(l, 0)$ with $l \in L$. By our hypothesis Lg is a direct summand of G, i. e.

$$G = Lg + H.$$

In this direct decomposition let

$$(0, 1) = (e, 0) + (-e, 1) \qquad (e \in L).$$

For an arbitrary element l of L the element

$$l(-e, 1) = (-le + l, 0)$$

is contained both in H and in Lg, and thus, by the properties of the direct decomposition, it is equal to zero. Thus $-le + l = 0$, i. e. $le = l$ for every $l \in L$, which shows that e is a right unit element of L.

Conversely, let R be a semi-simple ring. We are going to show that for an arbitrary left ideal L of R and for an arbitrary element g of any R-module G, Lg is a direct summand of G. Let e be a right unit element of R. (The existence of such an element follows from the semi-simplicity of R.) First of all we show that e is a *unit* element of R. Indeed by

$$s(r - er) = sr - sr = 0 \qquad (r, s \in R)$$

0 is the only right unit element in the left ideal J consisting of all elements $r-er$ $(r \in R)$, and thus $J=0$; this implies that e is also a left unit element in R.

Let now G be an arbitrary R-module. Making use of the well-known Peirce-decomposition, we represent G as a direct sum of its trivial submodule G_0 and of a unitary R-module G_1:

$$G = G_0 + G_1.$$

Now, if L is a left ideal of R, g an element of G, which, by the direct decomposition can be written in the form

$$g = g_0 + g_1 \qquad (g_0 \in G_0,\ g_1 \in G_1),$$

then $Lg = Lg_1$. It is therefore sufficient to show that any submodule M (in particular: Lg_1) of the unitary R-module G_1 is a direct summand of G_1. The proof of this can be obtained as follows. Let K be a submodule of G_1 maximal with respect to the property of having with M intersection 0. The proof of our theorem will be completed as soon as we show that the direct sum $M+K$ contains any element h of G_1. The elements r of R with $rh \in M+K$ form a left ideal Q in R. Let e^* be a right unit element in Q. Then the cyclic submodule $R(h-e^*h)$ generated by the element $h-e^*h$ has no non-zero element in common with $M+K$, for $r(h-e^*h)$ is equal to zero if $r \in Q$; if, however $r \notin Q$, then by the definition of Q, this element is not contained in the module $M+K$. We have therefore $R(h-e^*h)=0$, by the maximality of K. Now, since G_1 is a unitary R-module, this implies $h-e^*h=0$, i. e. $h=e^*h \in M+K$. This completes the proof of Theorem 2.[6])

As an application of the preceding theorem we have the following

Theorem 3. *A ring R is semi-simple if and only if every R-module G admits a representation as a direct sum of its trivial submodule and an R-module for which one of the four conditions α'), β'), γ'), δ') in Theorem 1 is satisfied.*

Since by Theorem 1 conditions α')—δ') are equivalent for an arbitrary R-module G, our theorem comprises the theorem of O. GOLDMAN mentioned above.

In order to prove this theorem, let us first suppose that the ring R is such that every R-module G admits a representation

$$G = G_0 + G_1$$

where G_0 is the trivial submodule of G and G_1 is an R-module for which condition δ') is satisfied. It is sufficient to show that for a left ideal L of R and for any element g of G, Lg is a direct summand of G, since, by

[6]) It is possible to simplify the second half of the proof of this theorem by a reference to Theorem 2 in [3]. we prefer however to give for this theorem a proof complete in itself.

Theorem 2, this implies the semi-simplicity of the ring R. Let g have by the above decomposition a representation

$$g = g_0 + g_1 \qquad (g_0 \in G_0, g_1 \in G_1).$$

Then $Lg = Lg_1 \subseteq G_1$ and so, by condition δ'), Lg is a direct summand of G.

On the other hand, if R is semi-simple, then by the Peirce-decomposition any R-module can be obtained as a direct sum of its trivial submodule and of a unitary R-module, where the unitary direct summand is a completely reducible R-module.

REMARK. After having completed the above paper, the author succeeded in proving the following theorem, giving another characterization of the completely reducible modules:

An arbitrary R-module G is completely reducible if and only if

a) *the intersection of G and its maximal submodules is* 0, *and*

b) *G satisfies the descending chain condition for cyclic submodules.*

The proof of this theorem will be published elsewhere.

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