On rings every subring of which is a multiple of the ring.

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In a previous paper we have determined all groups every cyclic subgroup of which is a power of the group [2]. We shall treat in this note a similar ring-theoretical problem.

We shall call an arbitrary ring R *cyclic* if the additive group R^+ of R is cyclic (see e. g. [1], pp. 326—327). The ring I of rational integers is obviously cyclic.

An arbitrary ring R is called *a ring with property* P if any subring S of R is a multiple nR of the ring R, where nR denotes the set of all elements nr of R (here $r \in R$ and $n \in I$). Then nR is also an ideal in the ring R. It is clear that every homomorphic image R' of a ring R with property P has likewise the property P.

We shall prove the following

Theorem. *An arbitrary ring R is cyclic if and only if it has the property P.*

PROOF. First of all we verify that a ring with property P without divisors of zero is necessarily commutative. In fact, since every subring S of R is an ideal in R, for arbitrary elements $a \neq 0$ and b of R obviously $ab = a' \in \{a\}$ holds, where $\{M\}$ denotes the subring generated by the subset M of R. By $aa' = a'a$ we obtain $a(ba - a') = (ab - a')a = 0$ and since R has no divisors of zero, $a' = ab = ba$.

Now we distinguish two cases.

I. Let R be a ring with property P whose additive group is torsion free.

If $a \neq 0$ and $b \neq 0$ are elements of R with $ab = 0$, then by $\{a\} = mR$ and $\{b\} = nR$ necessarily $\{0\} = \{a\} \cdot \{b\} = mRnR = mnR^2$. Therefore $R^2 = \{0\}$, consequently by the theorem of [2], R is a zeroring over an infinite cyclic additive group.

Now we assume that R has no divisors of zero. Then by our previous remark R is commutative. For the element $a \neq 0$ of R we have evidently $Ra \neq \{0\}$, therefore $Ra = nR$ for some natural number n. There exists there-

fore an element $b \neq 0$ of R for which $ba = ab = na$. Consider the non-void subset S of all elements c of R for which, with a certain rational integer n_c, an equation $ac = ca = n_c \cdot a$ holds. We can easily verify that S is a subring of R, consequently $S = mR$, where by $0 \neq b \in S$ evidently $m \neq 0$. Therefore $Sa = (mR)a = mnR$. But then the mapping $r \to mnr$ is an isomorphism of the additive group R^+ of R onto its subgroup $(mnR)^+$. On the other hand by the definition of S the subring $S \cdot a = mn \cdot R$ is obviously cyclic, and on the basis of the preceding group-theoretical isomorphism the ring R itself is cyclic.

II. Let R be an arbitrary ring with property P whose additive group R^+ is not torsion free. If d is an element of finite order n of the ring R, and $\{d\} = mR$, then R^+ is k-bounded, where $k \leq m \cdot n$. The elements of p-power order of R form a subring R_p, called the p-component of R, which is a direct summand of R, and has also property P. So we have to prove the theorem only for p-rings. We first observe that all subrings of R_p are:

$$R_p, \quad pR_p, \quad p^2 R_p, \ldots, \quad p^i R_p, \ldots, \quad p^{k-1} R_p \neq \{0\}, \quad p^k R_p = \{0\}.$$

Since $p^i R_p / p^{i+1} R_p$ $(i = 0, 1, \ldots, k-1)$ has no proper left ideal, it is a prime field of characteristic p, or a zeroring over an additive group of order p. In both cases every $p^i R_p^+ / p^{i+1} R_p^+$ is a group of order p, and so R_p^+ is a group of order p^k. But $p^{k-1} R_p^+ \neq 0$ shows that R_p^+ contains an element of order p^k, thus R_p^+ is cyclic.

REMARK. By the note [3] it is clear that an arbitrary ring R is cyclic if and only if every non-trivial multiple nR of R is cyclic, where nR is called trivial in the case $n = 1$, 0 or -1.

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Bibliography.

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