Diophantine inequalities in complex quadratic fields.

Dedicated to the memory of Tibor Szele.

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§ 1. Let a, b, c, d be real numbers with $|ad-bc|=1$ and

$$f(x,y)=|(ax+by)(cx+dy)|. \tag{1}$$

A well known result by MINKOWSKI states that real numbers x, y with assigned residues (mod 1) exist such that

$$f(x,y)\leqq\frac{1}{4}, \tag{2}$$

and that the constant $\frac{1}{4}$ is best possible.

It is of interest to consider the generalization for a quadratic field $K(i\sqrt{D})$ where D is a positive square-free integer. The discriminant $-D'$ of the field is given by $D'=4D$ or D according as $D\equiv 1, 2 \pmod 4$ or $D\equiv 3 \pmod 4$. Any complex number can be expressed in the form

$$z=X+Y\left(\frac{D'+i\sqrt{D'}}{2}\right)=X+Y\omega \tag{3}$$

say, where X, Y are real. The integers in $K(i\sqrt{D})$ are given by taking X, Y to be the rational integers. When the real numbers X, Y have assigned residues (mod 1), we say that z has an assigned residue $\bmod(1,\omega)$.

Write

$$j=j(D)=\begin{cases}(1+D)^2/16D & \text{when} \quad D\equiv 3 \pmod 4\\ (1+D)/4 & \text{when} \quad D\equiv 1, 2 \pmod 4.\end{cases} \tag{4}$$

Then we can always find z with an assigned residue $z_0 \bmod (1,\omega)$ such that

$$|z|\leqq j^{\frac{1}{2}}.$$

Stated in a different form, we can find an integer z in $K(i\sqrt{D})$ such that

$$|z-z_0|\leqq j^{\frac{1}{2}}, \tag{5}$$

and this is the best possible result.

When $D \equiv 1, 2 \pmod 4$, this is obvious since real numbers x, y with assigned residues (mod 1) exist such that

$$x^2 + Dy^2 \leqq (1+D)/4.$$

When $D \equiv 3 \pmod 4$, the corresponding inequality is essentially

$$x^2 + xy + \frac{1}{4}(1+D)\,y^2 \leqq (1+D)^2/16D, \text{ or say } \xi^2 + \eta^2 \leqq (1+D)^2/16D.$$

The circle defined by this inequality contains an inscribed hexagon whose sides are the perpendicular bisectors of the lines joining the origin to the (ξ, η) points $(\pm 1, 0), \left(\pm\frac{1}{2}, \pm\frac{1}{2}\sqrt{D}\right), \left(\mp\frac{1}{2}, \pm\frac{1}{2}\sqrt{D}\right)$. This hexagon contains the points which are nearer to the origin than to any other point of the lattice based on the points $(1, 0), \left(\frac{1}{2}, \frac{1}{2}\sqrt{D}\right)$, and so contains a point (x, y) with assigned residues (mod 1). This result is a particular case of a theorem by Dirichlet for the general definite binary quadratic form. I have recently found a very simple arithmetical proof which will appear in the Schur memorial volume of the Mathematische Zeitschrift.

§ 2. The new question is to find the best possible constant $k = k(D)$ such that if now a, b, c, d are complex numbers and $|ad - bc| = 1$, then

$$|f(x, y)| \leqq k \tag{6}$$

can be satisfied by complex numbers x, y with assigned residues mod $(1, \omega)$.

If in (6), we require x, y to be integers in $K(i\sqrt{D})$, we exclude the trivial solution $x = y = 0$, and we have now to find the best possible value of k, say $l = l(D)$, such that we can solve

$$f(x, y) \leqq l \tag{7}$$

in integers x, y not both zero.

We note that such constants do exist in our applications. By a result of Mahler there are also an infinity of solutions of (6) and (7).

It has been shown that

$$\begin{gathered} l(1) = \frac{1}{\sqrt{3}}, \quad l(2) = \frac{1}{\sqrt{2}}, \quad l(3) = \frac{1}{\sqrt[4]{13}}, \quad l(7) = \frac{1}{\sqrt[4]{8}}, \\ l(11) = \frac{2}{\sqrt{5}}, \qquad l(19) = 1, \end{gathered} \tag{8}$$

and that in each case, there are an infinity of solutions. The result for $l(1)$ is due to Ford, for $l(2), l(3)$ to Perron, for $l(7)$ to Hofreiter, for $l(11)$, $l(19)$ to Poitou. Full references for (8) are given by Poitou [1].

It is not difficult to find upper bounds for l and PERRON has shown that

$$\text{(9)}\qquad \begin{aligned} l &< \frac{1}{\pi}\sqrt{6D}, \qquad D \equiv 1, 2 \pmod 4, \\ l &< \frac{1}{2\pi}\sqrt{6D}, \quad D \equiv 3 \pmod 4, \end{aligned}$$

and this result has been improved by other writers.

The integers x, y in (7) may not be relatively prime when the class number of the field $K(i\sqrt{D})$ exceeds unity. Then we prove that it suffices for the general case to consider only the special case when the greatest common ideal factor $\mathfrak{a}$ of x and y is $\mathfrak{m}$, an ideal with minimum norm $N(\mathfrak{m})$ in the same ideal class as $\mathfrak{a}$. It is well-known that

$$\text{(10)}\qquad N^2(\mathfrak{m}) \leqq D'.$$

Let $(x, y) = \mathfrak{a}$ and

$$f(x, y) = |(ax + by)(cx + dy)| \leqq l(D).$$

If $\mathfrak{a}, \mathfrak{b}$ are two ideals in the same ideal class and $\mathfrak{b}|\mathfrak{a}$, then $\mathfrak{a} = \mathfrak{b}(t)$ where t is an integer in $K(i\sqrt{D})$. There will be no confusion if we write $\mathfrak{a}|\mathfrak{b}$ instead of t. Hence

$$f\left(\frac{x\mathfrak{m}}{\mathfrak{a}}, \quad \frac{y\mathfrak{m}}{\mathfrak{a}}\right) \leqq \left|\frac{\mathfrak{m}}{\mathfrak{a}}\right|^2 l(D),$$

where now $\mathfrak{m}$ is the greatest common ideal divisor of the integers $x\mathfrak{m}/\mathfrak{a}, y\mathfrak{m}/\mathfrak{a}$ say, p, q. Also

$$|\mathfrak{m}/\mathfrak{a}| = N(\mathfrak{m})/N(\mathfrak{a}) \leqq 1.$$

A question that does not seem to have been considered is to find the result corresponding to (7) when we impose the condition $(x, y) = \mathfrak{m}$. I know of no results.

These results for (7) are useful for the more general case (8). We can determine two relatively prime integers (r, s) in $K(i\sqrt{D})$ such that $ps - qr = N(\mathfrak{m})$ since $\mathfrak{m} = (p, q)$ divides $N(\mathfrak{m})$.

The substitution

$$x = px' + ry', \; y = qx' + sy',$$

establishes a 1—1 correspondence between integer sets x, y and x', y'; and also for numbers x, y and x', y' with appropriate assigned residues $\bmod(1, \omega)$. It changes $f(x, y)$ into

$$f'(x', y') = |(a'x' + b'y')(c'x' + d'y')|,$$

and we can suppose that for given arbitrary small $\varepsilon > 0$,

$$|a'| \leqq \varepsilon, \; |c'| \leqq \frac{l}{\varepsilon},$$

also $|a'd' - b'c'| = N(\mathfrak{m})$.

Our problem now becomes in a slightly different form — to find the corresponding constant k' for

(11) $$f'(x', y') \leqq k',$$

where

$$|a'd' - b'c'| = N,$$

and N does not exceed the greatest of the minimum norms of the ideals in the different ideal classes. Of course, $N = 1$ when the class number is unity, and in any case N^2 does not exceed the discriminant of the field.

It has been shown that

(12) $$k(1) = \frac{1}{2}, \quad k(2) = \frac{3}{4}, \quad k(3) = \frac{1}{3}, \quad k(7) = \frac{4}{7},$$

i. e.

(13) $$k(D) = j(D).$$

PERRON has conjectured that this holds for all D, and notes that (13) cannot be replaced by $k(D) < j(D)$. For if x, y are integers in $K(i\sqrt{D})$, then

$$\left|\left(x - \frac{1+i\sqrt{D}}{2}\right)\left(y - \frac{1+i\sqrt{D}}{2}\right)\right| \geqq \left(\frac{1}{2}(1+D)^{\frac{1}{2}}\right)^2 = j(D), \; D \equiv 1, 2 \pmod 4,$$

$$\left|\left(x - \frac{1}{2} + \frac{i(D-1)}{4\sqrt{D}}\right)\left(y - \frac{1}{2} + \frac{i(D-1)}{4\sqrt{D}}\right)\right| \geqq \left(\frac{1+D}{4\sqrt{D}}\right)^2 = j(D), \; D \equiv 3 \pmod 4.$$

He also gives the estimate

(14) $$k(D) \leqq j(D)\,(D' + l(D)),$$

where $-D'$ is the discriminant of the field.

The result for $k(1)$ is due to HLAWKA, and other proofs have been given by MAHLER and PERRON, both of whom have also proved the results for $k(2), k(3)$. The result for $k(7)$ is due to SCHMETTERER [2] who follows PERRONS's method, and comments that it does not suffice to prove the conjectured result for k (11). He gives full references.

I notice that the proofs of (12) can be expressed in a form simpler than that given by other writers and that a great deal of the numerical calculation can be dispensed with. There are fewer cases to be considered and the treatment of all the known results is unified. The method does not prove the conjecture when $D = 11, 19$, and the question arises whether the conjecture holds. A modified one is suggested later on.

§ 3. Fundamental in our problem is a result concerned with congruent points in a Cassini oval. Various results are already known but the one now given and its proof are both new. The result is, however, implicit in PERRON's work for the special cases.

Lemma. *Let a be any given complex number. Then a point z with assigned residue* mod $(1, \omega)$ *is contained in the Cassini oval*

$$|z^2-a| \leqq m,$$

where

$$m = 2\sqrt{aj} \quad \text{if} \quad |a| \geqq j$$
$$(15) \qquad m = |a|+j \quad \text{if} \quad |a| < j,$$
$$\leqq \sqrt{aj}+j.$$

If (15) is not satisfied, then for arbitrary $\lambda = \lambda(p, q)$, $a = (p+iq)^2$,

$$(16) \qquad \lambda^{-1}|z-p-iq|^2+\lambda|z+p+iq|^2 > 2m.$$

Put

$$z = x+y\left(\frac{D'+i\sqrt{D'}}{2}\right)+\xi+i\eta,$$

where ξ, η are constants taken so that the coefficients of x, y in (16) vanish. Hence

$$\lambda^{-1}(\xi-p)+\lambda(\xi+p) = 0, \quad \lambda^{-1}(\eta-q)+\lambda(\eta+q) = 0,$$

and so

$$\xi = p\left(\frac{1-\lambda^2}{1+\lambda^2}\right), \qquad \eta = q\left(\frac{1-\lambda^2}{1+\lambda^2}\right).$$

Also x, y have assigned residues (mod 1). From (16),

$$\lambda^{-1}\left[\left(x+\frac{D'y}{2}\right)^2+(\xi-p)^2+\frac{D'}{4}y^2+(\eta-q)^2\right]+\lambda\left[\left(x+\frac{D'y}{2}\right)^2+(\xi+p)^2+\frac{D'y^2}{4}+\right.$$
$$\left.+(\eta+q)^2\right] > 2m$$

Substitute for ξ, η. Then

$$\left(\lambda+\frac{1}{\lambda}\right)\left[\left(x+\frac{D'y}{2}\right)^2+\frac{D'y^2}{4}\right]+\frac{4p^2(\lambda^3+\lambda)+4q^2(\lambda^3+\lambda)}{(\lambda^2+1)^2} > 2m.$$

Put $t = \lambda+\dfrac{1}{\lambda}$ and so

$$\left[\left(x+\frac{D'y}{2}\right)^2+\frac{D'y^2}{4}\right]+4\frac{(p^2+q^2)}{t} > 2m.$$

Hence by appropriate choice of x, y (mod 1) as in (5),

$$tj+\frac{4|a|}{t} > 2m.$$

If $|a| \geqq j$, we take $t = 2\sqrt{|a|/j} \geqq 2$, which gives a real value for λ, and then the contradiction

$$2\sqrt{|a|j}+2\sqrt{|a|j} > 4\sqrt{|a|j}.$$

If $|a| < j$, we take $t = 2$, and then have the contradiction

$$2j + 2|a| > 2|a| + 2j.$$

The result for $|a| < j$ is trivial since we can always take $|z^2| \leqq j$.

The result (15) has to be improved for our applications, and we shall in due course find a number $j' < j$ such that we can satisfy

$$(17) \qquad |z^2 - a| \leqq 2\sqrt{j|a|} \quad \text{if} \quad |a| \geqq j',$$

and of course trivially

$$|z^2 - a| \leqq j + j' \quad \text{if} \quad |a| < j'.$$

§ 4. We now enunciate the

Theorem. *Let*

$$f(x, y) = |(ax + by)(cx + dy)|,$$

where $|ad - bc| = 1$ *and* a/b *is not in* $K(i\sqrt{D})$. *Then numbers* x, y *with assigned residues* $\bmod (1, \omega)$ *exist such that*

$$f(x, y) \leqq jN,$$

provided that when j' *is as in* (17),

$$l(D)(j + j') \leqq jN,$$

and N *as defined in* (11) *is such that each ideal class contains an ideal of norm* $\leqq N$.

Further, we can satisfy for arbitrary $\varepsilon > 0$,

$$|ax + by| < \varepsilon,$$

provided that $ax + by \neq 0$ for any integers $(x, y) \neq (0, 0)$ in $K(i\sqrt{D})$.

The last proviso is essential as in obvious when we consider the particular case expressed in a slightly different form

$$f(x, y) = |(x + \vartheta y)(x + \vartheta' y + q)| \qquad q \neq 0,$$

where ϑ, ϑ' are conjugate integers and x, y variable integers in $K(i\sqrt{D})$. For if $|x| + |y| > 0$ and $|x + \vartheta y| < \varepsilon$,

$$f(x, y) = |(x + \vartheta y)(x + \vartheta' y) + q(x + \vartheta y)|$$
$$> 1 - \varepsilon |q|.$$

As already noted in (11), we may modify the enunciation by writing

$$f(x, y) = |(ax + by)(cx + dy)|,$$

where now $|ad - bc| \leqq N$, $|a| \leqq \varepsilon$, $|c| \leqq l/\varepsilon$. Write

$$ax + by = aX', \; cx + dy = Y,$$

so that

$$Ny = -acX' + aY.$$

Put $X = X' - Ny/2ac$. Then

(18) $$f(x, y) = |X'(acX' + Ny)| = |ac|\,|X^2 - N^2y^2/4a^2c^2|.$$

We note that if any value is assigned to y, we can find X' and so x with a given residue mod$(1, \omega)$. We take y with its assigned residue mod$(1, \omega)$ such that $|y| \leqq j^{\frac{1}{2}}$, and apply the lemma.

If $|N^2y^2/4a^2c^2| \geqq j$, we can satisfy

(19) $$f(x, y) \leqq |ac|\,|4jN^2y^2/4a^2c^2|^{\frac{1}{2}} \leqq jN.$$

If $|N^2y^2/4a^2c^2| < j$, we can satisfy

(20) $$f(x, y) \leqq |ac|\,(j + |N^2y^2/4a^2c^2|) \leqq |ac|\,(j + |jN^2y^2/4a^2c^2|^{\frac{1}{2}}),$$

$$\leqq j|ac| + |jN^2y^2/4|^{\frac{1}{2}},$$

and so

(21) $$f(x, y) \leqq j\left(l + \frac{1}{2}N\right).$$

Hence in any case, we can satisfy

$$f(x, y) \leqq j(l + N),$$

and this is PERRON's result (14).

If, however, (19) holds for $|N^2y^2/4a^2c^2| \geqq j'$ where $j' < j$, then when $|N^2y^2/4a^2c^2| < j'$, (20) gives

(22) $$f(x, y) \leqq |ac|\,(j + j') \leqq l(D)\,(j + j') \leqq jN$$

by our hypothesis. This of course requires that $l(D) < N$.

It remains to show that in the estimates for $f(x, y)$ given in (19), (20), (21), (22), we can make $ax + by = aX' = aX + Ny/2c$ arbitrarily small. By the Lemma,

$$\left|X^2 - \frac{N^2y^2}{4a^2c^2}\right| \leqq \max\left(\frac{Nj^{\frac{1}{2}}|y|}{|ac|},\ j + \frac{N^2|y|^2}{4|a^2c^2|}\right),$$

and so

$$\left|a^2X^2 - \frac{N^2y^2}{4c^2}\right| \leqq \max\left(\frac{Nj|a|}{|c|},\ |a|^2j + \frac{N^2j}{4|c^2|}\right).$$

When $|a|$ is arbitrarily small, $|c|$ becomes great and so aX and hence aX' becomes arbitrarily small. The result still holds if we use j' in place of j.

§ 5. Though we have supposed the field $K(i\sqrt{D})$ to be quadratic complex, the results and proofs still hold when the field is the rational field K if we put $D = 0, N = 1, j = \frac{1}{4}$ and $l \leqq \frac{1}{2}$ in (19), (21), this giving $k \leqq \frac{1}{4}$.

Lemmas of the present type were first used for these problems of Diophantine approximation in a paper of mine [3] published in 1928. There, however, I used the cruder result $l \leqq 1$. A practically identical proof was published by PERRON [4] in 1938.

§ 6. We now consider the cases $D=1, 2, 3, 7, 11, 19$. The fields have class number one, and so here $N=1$. In all these cases $l>\frac{1}{2}$, so (20) does not suffice and we must use an appropriate j'.

We suppose $a=\xi+i\eta, M=|a|<j$, and that we cannot satisfy $|z^2-a|\leqq\sqrt{4j|a|}$. We shall show that if for all z with assigned residue mod $(1, \omega)$, $|z^2-a|>\sqrt{4ja}$, then there is a constant $j'=j'(D)$, which we wish to satisfy

$$j'\leqq j''=j(1/l-1),$$

such that $|a|<j'$. This implies that we can satisfy $|z^2-a|\leqq\sqrt{4j|a|}$ for $|a|\geqq j'$.

We can write $z=x+iy\sqrt{\Delta}$ where if $D\equiv 1, 2 \pmod 4$, $\Delta=D$, and if $D\equiv 3 \pmod 4$, $\Delta=\frac{1}{4}D$, and $0\leqq x<1, |y|\leqq\frac{1}{2}$. We have

$$(x^2-\Delta y^2-\xi)^2+(2xy\sqrt{\Delta}-\eta)^2>4j\,|a|,$$

and so

$$(x^2+\Delta y^2)^2-2\xi(x^2-\Delta y^2)-4\eta xy\sqrt{\Delta}>4jM-M^2. \tag{23}$$

We now apply an averaging process frequently used by PERRON for Cassini ovals, and recently by myself in other applications. Now (23) holds if we replace x by $x-1$, and so

$$((x-1)^2+\Delta y^2)^2-2\xi((x-1)^2-\Delta y^2)-4\eta(x-1)y\sqrt{\Delta}>4jM-M^2. \tag{24}$$

Multiply (23) by $1-x$ and (24) by x and add. On writing $X=x(1-x)$, and so $0\leqq X\leqq\frac{1}{4}$, we have

$$(1-x)x^4+x(x-1)^4=x(1-x)(x^3+(1-x)^3)=X(1-3X),$$
$$(1-x)x^2+x(x-1)^2=x(1-x)=X,$$
$$(1-x)x+x(x-1)=0.$$

Then

$$X(1-3X)+2\Delta Xy^2-2\xi X+2\xi\Delta y^2+\Delta^2y^4>4jM-M^2. \tag{25}$$

Suppose first that $\xi\leqq 0$. Then we consider here only the case $\Delta>1$. Since $|y|\leqq\frac{1}{2}$ and $\xi\Delta y^2\leqq 0$,

$$X(1-3X)+\frac{1}{2}\Delta X-2\xi X+\frac{\Delta^2}{16}>4jM-M^2.$$

Since $1-6X+\frac{1}{2}\Delta-2\xi=0$ gives $X>\frac{1}{4}$, the maximum value of the left hand side occurs when $X=\frac{1}{4}$. Hence

$$\frac{1}{16}+\frac{\Delta}{8}-\frac{\xi}{2}+\frac{\Delta^2}{16}>4jM-M^2,$$

or

$$(26)\qquad M^2-\left(4j-\frac{1}{2}\right)M+\left(\frac{\Delta+1}{4}\right)^2>0,\qquad (\Delta>1).$$

We write this as

$$(M-M_1)(M-M_2)>0,$$

where $M_1<M_2$. We know that $M_1<j$ but desire that

$$M_1\leqq j(1/l-1),\qquad M_2>j.$$

Suppose next that $\xi>0$. We consider both $\Delta\geqq 1$, $\Delta<1$. From (25), since $|y|\leqq\frac{1}{2}$,

$$(27)\qquad X(1-3X)+\frac{1}{2}\Delta X-2\xi X+\frac{1}{2}\Delta\xi+\frac{1}{16}\Delta^2>4jM-M^2.$$

We have now two subcases. When $\xi\geqq\frac{1}{4}(\Delta-1)$, the maximum value in (27) occurs when

$$1-6X+\frac{1}{2}\Delta-2\xi=0,\quad 6X=1+\frac{1}{2}\Delta-2\xi\leqq\frac{3}{2}.$$

Hence

$$\left(1+\frac{1}{2}\Delta-2\xi\right)^2\Big/12+\frac{1}{2}\Delta\xi+\frac{1}{16}\Delta^2>4jM-M^2,$$

or

$$\left(1+\frac{1}{2}\Delta\right)^2\Big/12-\left(1+\frac{1}{2}\Delta\right)\xi\Big/3+\frac{\xi^2}{3}+\frac{1}{2}\Delta\xi+\frac{1}{16}\Delta^2>4jM-M^2,$$

and so

$$\frac{4M^2}{3}-4jM+\frac{\Delta-1}{3}\xi+\left(1+\frac{1}{2}\Delta\right)^2\Big/12+\frac{1}{16}\Delta^2>0.$$

Hence on noting the term $\frac{1}{3}(\Delta-1)\xi$,

$$(28)\qquad \frac{4M^2}{3}-\left(4j-\frac{\Delta-1}{3}\right)M+\frac{\left(1+\frac{1}{2}\Delta\right)^2}{12}+\frac{1}{16}\Delta^2>0,\quad (\Delta\geqq 1)$$

$$\frac{4M^2}{3}-4jM+\frac{\left(1+\frac{1}{2}\Delta\right)^2}{12}+\frac{1}{16}\Delta^2>0,\qquad (\Delta<1).$$

In both cases we write (28) as

$$(M-M_1)(M-M_2)>0,$$

where $M_1<M_2$, and desire that

$$M_1 \leqq j\left(\frac{1}{l}-1\right) \qquad M_2>j.$$

When $\xi<\frac{1}{4}(\Delta-1)$, the maximum value in (27) occurs when $X=\frac{1}{4}$ and so

$$\frac{1}{16}+\frac{1}{8}\Delta-\frac{1}{2}\xi+\frac{1}{2}\Delta\xi+\frac{1}{16}\Delta^2>4jM-M^2,$$

or

$$M^2-4jM+\frac{1}{2}(\Delta-1)\xi+\left(\frac{\Delta+1}{4}\right)^2>0.$$

Hence since $\Delta-1>4\xi>0$,

$$M^2-\left(4j-\frac{\Delta-1}{2}\right)M+\left(\frac{\Delta+1}{4}\right)^2>0. \tag{29}$$

We write (29) as

$$(M-M_1)(M-M_2)>0,$$

where $M_1<M_2$ and desire that

$$M_1 \leqq j(1/l-1), \qquad M_2>j.$$

§ 7. We now take in turn the values $D=1,2,\dots$.

$D=1$. Here $\Delta=1$, $j=\frac{1}{2}$, $l=\frac{1}{\sqrt{3}}\cdot j''=j(1/l-1)=\frac{1}{2}(\sqrt{3}-1)=0{,}366\dots$

On replacing z by iz if need be, we may suppose $\xi \geqq 0$. Then (29) does not arise since $\xi \geqq \frac{1}{4}(\Delta-1)$, and (28) gives

$$\frac{4}{3}M^2-2M+\frac{1}{4}>0,$$

or

$$(4M-3)^2>6.$$

Clearly $M_2>1$ and so $M_1<\frac{3-\sqrt{6}}{4}<0{,}14<j''$ as desired.

Since we wish only to satisfy $\frac{3-\sqrt{6}}{4}<\frac{1}{2}\left(\frac{1}{l}-1\right)$, it would have been sufficient to take for l the crude estimate $l<2/(5-\sqrt{6})$.

$D=2$. Here $\Delta=2$, $j=\frac{3}{4}$, $l=1/\sqrt{2}$, $j''=\frac{3}{4}(\sqrt{2}-1)=0{,}310\dots$.

Suppose first $\xi \leqq 0$. Then from (26),

$$M^2 - \frac{5}{2}M + \frac{9}{16} > 0, \qquad \left(M - \frac{1}{4}\right)\left(M - \frac{9}{4}\right) > 0.$$

Hence $M_1 = \frac{1}{4} < 0,310$.

Suppose next $\xi > 0$. Then (28) gives

$$\frac{4M^2}{3} - \frac{8M}{3} + \frac{7}{12} > 0, \qquad (4M-1)\left(M - \frac{7}{4}\right) > 0,$$

and $M_1 = \frac{1}{4}$.

Finally (29) gives

$$M^2 - \frac{5M}{2} + \frac{9}{16} > 0,$$

etc.

$D = 3$. Here $\Delta = \frac{3}{4}$, $j = \frac{1}{3}$, $l = \frac{1}{\sqrt[4]{13}}$, $j'' = \frac{1}{3}(\sqrt[4]{13} - 1) = 0,299\ldots$

When $\xi > 0$, (28) for $\Delta < 1$ gives

$$\frac{4M^2}{3} - \frac{4}{3}M + \frac{121}{768} + \frac{9}{256} > 0,$$

or

$$M^2 - M + \frac{37}{256} > 0.$$

Clearly $M_2 > \frac{1}{2}$ and so $M_1 < \frac{74}{256} < 0,29$.

Suppose next that $\xi \leqq 0$. Since $\Delta < 1$, we cannot apply (26) but use (25) directly. We consider the two cases $-\xi \leqq \frac{1}{16}$, $-\xi > \frac{1}{16}$. If $-\xi \leqq \frac{1}{16}$, (25) remains true if we omit the term $2\xi\Delta y^2$ and replace y^2 by $\frac{1}{4}$. The modified expression takes its greatest value when $6X = \frac{11}{8} - 2\xi$, and then

$$\left(\frac{11}{8} - 2\xi\right)^2 \Big/ 12 + \frac{9}{256} > \frac{4M}{3} - M^2, \qquad \frac{121}{768} - \frac{11\xi}{24} + \frac{\xi^2}{3} + \frac{9}{256} > \frac{4M}{3} - M^2,$$

or since $-\xi \leqq \frac{1}{16}$,

$$\frac{4M^2}{3} - \frac{4M}{3} + \frac{121}{768} + \frac{11}{384} + \frac{9}{256} > 0, \qquad 4M^2 - 4M + \frac{170}{256} > 0.$$

Clearly $M_2 > \frac{3}{4}$ and so $M_1 < \frac{170}{768} < \frac{1}{4} < 0{,}29$. Next let $-\xi > \frac{1}{16}$. We consider the two subcases $\xi \leqq -\frac{1}{4}$, $\xi > -\frac{1}{4}$. For the first, since $1-6X+\frac{3}{2}y^2-2\xi=0$ gives $X \geqq \frac{1}{4}$. the left hand side of (25) has its greatest value as a function of X when $X=\frac{1}{4}$, and so

$$\frac{1}{16}+\frac{3}{8}y^2-\frac{1}{2}\xi+\frac{3}{2}\xi y^2+\frac{9}{16}y^4 > \frac{4}{3}M-M^2.$$

Since the coefficient of y^2 is not positive, and $y^2 \leqq \frac{1}{4}$,

$$\frac{1}{16}-\frac{1}{2}\xi+\frac{9}{256} > \frac{4M}{3}-M^2,$$

or

$$M^2-\frac{5}{6}M+\frac{25}{256} > 0.$$

Clearly $M_2 > \frac{5}{12}$ and so $M_1 < \frac{60}{256} < 0{,}29$. Suppose finally that $\xi > -\frac{1}{4}$. We write (25) as

$$X(1-3X)+\frac{3}{2}Xy^2-2\xi X+\left(\frac{3}{2}\xi+\frac{3}{8}\right)y^2+\frac{9}{16}y^4-\frac{3}{8}y^2 > \frac{4}{3}M-M^2.$$

Hence

$$X(1-3X)+\frac{3}{8}X-2\xi X+\frac{3\xi}{8}+\frac{3}{32} > \frac{4M}{3}-M^2.$$

The maximum value of the left hand side occurs when $X=\frac{1}{4}$ since $1-6X+\frac{3}{8}-2\xi=0$ gives $X > \frac{1}{4}$ since $-2\xi > \frac{1}{8}$. Hence

$$\frac{1}{16}+\frac{3}{32}-\frac{\xi}{2}+\frac{3\xi}{8}+\frac{3}{32} > \frac{4}{3}M-M^2,$$

or

$$M^2-\left(\frac{4}{3}-\frac{1}{8}\right)M+\frac{1}{4} > 0.$$

Clearly $M_2 > \frac{29}{48} > 0{,}299$. Also $M_1 < 0{,}299$ since

$$0{,}09+\frac{0{,}3}{8}+0{,}25-\frac{1{,}196}{3} < 0.$$

$D=7$. Here $\Delta=\frac{7}{4}$, $j=\frac{4}{7}$, $l=\frac{1}{\sqrt[4]{8}}$, $j''=\frac{4}{7}(\sqrt[4]{8}-1)=0{,}38\ldots$

When $\xi\leqq 0$, (26) gives

$$M^2-\frac{25}{14}M+\frac{121}{256}>0.$$

Clearly $M_2>1$ and $M_1<0{,}38$ since $0{,}144-0{,}67+0{,}5<0$. Suppose next that $\xi>0$. Then (28) gives

$$\frac{4M^2}{3}-\left(\frac{16}{7}-\frac{1}{4}\right)M+\frac{225}{768}+\frac{49}{256}>0,$$

or

$$4M^2-\frac{171}{28}M+\frac{372}{256}>0.$$

Clearly $M_2>1$ and $M_1<\frac{93}{256}<0{,}38$.

Finally (29) gives

$$M^2-\left(\frac{16}{7}-\frac{3}{8}\right)M+\frac{121}{256}>0.$$

$$M^2-\frac{107}{56}M+\frac{121}{256}>0.$$

Clearly $M_2>\frac{3}{2}$ and so $M_1<\frac{242}{768}<0{,}38$.

$D=11$. Here $\Delta=\frac{11}{4}$, $j=\frac{9}{11}$, $l=\frac{2}{\sqrt{5}}$, $j''=\frac{9}{11}\left(\frac{1}{2}\sqrt{5}-1\right)=0{,}097$.

From (26), (28), (29), it suffices if $M_1<\frac{1}{2}$ and so the value of j'' is too small for the method to succeed. But it happens that the homogeneous minimum corresponding to $l=\frac{2}{\sqrt{5}}$ is isolated, i. e. if a special set of homogeneous linear forms is excluded and so of course corresponding inhomogeneous forms, then l may be replaced by a smaller number, say l'. The present method succeeds if

$$M_1\leqq j\left(\frac{1}{l'}-1\right) \quad \text{or} \quad l'\leqq 1\Big/\frac{M_1}{j}+1.$$

Here we require $l'\leqq 1\Big/\frac{11}{18}+1\leqq 0{,}62\ldots$, but I do not know whether this holds.

$D=19$. Here $\Delta=\frac{19}{4}$, $j=\frac{25}{19}$, $l=1$. It suffices if $M_1\leqq 0{,}8$ and if $l=1$ is isolated, the method succeeds if $l'\leqq 1\Big/\frac{15{,}2}{25}+1\leqq 0{,}6$.

§ 8. When D is such that the class number exceeds unity, further difficulties arise not only with the value of N but also because the value of l is not known. Thus when $D=6$, $N=1$ or 2, $j=\frac{7}{4}$. Mr. BIRCH informs me that he has reason to believe that $l=\frac{2}{\sqrt{3}}$. Then (19) gives $k\leqq\frac{7}{2}$ and (21) gives $k\leqq\frac{7}{4}\left(\frac{2}{\sqrt{3}}+1\right)$. Hence we have the estimate

$$k(6)\leqq\frac{7}{4}\left(\frac{2}{\sqrt{3}}+1\right).$$

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