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The weak topology for SG-primitive ideals

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Abstract. Let A be a C^*-algebra and M a left-cancellative semigroup with unit e, and (A, M, α) a C^*-dynamical system. We define the concept of SG-primitive ideals SG-Prim(A) of A for a semigroup action α. The restriction to SG-Prim(A) of the weak topology which has been previously defined on the ideal space Id(A) is investigated. A necessary and sufficient condition for SG-Prim(A) to be a Hausdorff space is given. The classical Dauns-Hofmann's theorem is extended to a more general setting.

0. Introduction

The theory of crossed products of C^*-algebras by groups of automorphisms is a well-developed area of the theory of operator algebras. Given the importance and the success of that theory, it is natural to attempt to extend it to a more general situation by, for example, developing a theory of crossed products of C^*-algebras by semigroups of automorphisms, or even of endomorphisms. Indeed, in recent years a number of papers have appeared that are concerned with such non-classical theories of covariance algebras, see, for instance [6], [7], [8], [9], [10], [11].

In recent papers [10] and [11] Murphy introduced one such non-classical theory. There are many aspects of the extended theory developed by Murphy which are analogous to results of the original classical theory. Nevertheless, there are significant differences, in fact, perhaps rather more than one might at first expect. These differences manifest themselves not only in the kind of results that can be obtained, but also in proofs and

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methods: the analogue of a classical result may turn out to be false, or if true, may require a new proof (and often, one that is rather more difficult). For further details see [10] and [11].

Let A be a C^*-algebra. Recently, some subsets of the set $\mathrm{Id}(A)$ of all ideals in A (ideal will always mean closed two-sided ideal) were investigated, for example, the concepts of primal and minimal ideals were introduced for C^*-algebras ([1], [3]), and topologies on these ideal spaces were systematically studied in [1]. A particularly nice class of C^*-algebras is the class of quasi-standard ones ([1], [4]) which can be characterized by the fact that their minimal primal ideal spaces and their Glimm ideal spaces, arising from the complete regularization of the primitive ideals, coincide as topological spaces. The quasi-standard C^*-algebras turned out to be precisely the maximal algebras of cross sections in the sense of ([1], [4]) over the minimal primal ideal spaces.

Motivated mainly by the works ([1], [3], [5], [10], [11], [12]), in this paper we shall consider the restriction to SG-$\mathrm{Prim}(A)$ (see Section 2 for the definition) of the weak topology which has been previously defined on the ideal space $\mathrm{Id}(A)$. The weak topology τ_w is in some sense more natural and the restriction of τ_w to the primitive ideal space SG-$\mathrm{Prim}(A)$ is precisely the Jacobson (hull-kernel) topology.

This paper is organized as follows. In Section 1, we recall the basic definition and properties of the covariance algebra of the C^*-dynamical system (A, M, α) relative to a 2-cocycle θ. In Section 2 we introduce notation and some definitions, in particular, of SG-primitive ideals, and some properties of these are considered. In Section 3 we shall consider the weak topology τ_w on SG-$\mathrm{Prim}(A)$. Finally, in Section 4 we prove that the fixed point algebra of the center of a unital C^*-algebra for a left-cancellative semigroup M with unit e is isomorphic to the algebra of all the continuous functions on the SG-$\mathrm{Prim}(A)$. If M is trivial, this result is nothing but the classical Dauns-Hofmann's theorem.

1. Covariance algebras

In this section, we shall collect from [10, 11] basic definitions and properties of the covariance algebra of the C^*-dynamical system (A, M, α) relative to a 2-cocycle θ.

Let M be a semigroup, with unit denoted by e, and let A be a unital $*$-algebra. We say that a map

$$W : M \to A, \quad x \to W_x,$$

is an isometric projective homomorphism from M to A if all the elements of W_x are isometries, if $W_e = 1$, and if

$$W_{xy} = \theta_{x,y} W_x W_y, \quad x, y \in M,$$

where the $\theta_{x,y}$ are complex numbers of modulus 1. It follows from the equations

$$W_e = W_{ee} = \theta_{e,e} W_e W_e,$$

$$W_{(xy)z} = \theta_{xy,z} W_{xy} W_z = \theta_{x,y} \theta_{xy,z} W_x W_y W_z,$$

and

$$W_{x(yz)} = \theta_{x,yz} W_x W_{yz} = \theta_{x,yz} \theta_{y,z} W_x W_y W_z,$$

that we have

$$\theta_{e,e} = 1, \quad \theta_{x,y}\theta_{xy,z} = \theta_{x,yz}\theta_{y,z}. \tag{1.1}$$

We say that a function $\theta : M^2 \to \mathbb{T}$, $(x, y) \to \theta_{x,y}$ ($\mathbb{T}$ is the unit circle in $\mathbb{C}$) is a 2-cocycle of M if the equation (1.1) holds. If $A = B(H)$ for a Hilbert space H then we call (H, W) an isometric projective representation of M on H.

If M is left-cancellative, then isometric projective representations exist. To be specific, let H be an arbitrary non-zero Hilbert space and put $\widetilde{H} = l^2(M, H)$, the Hilbert space of all norm square-summable maps f from M to H (that is, $\sum_{x\in M} \|f(x)\|^2 < +\infty$) with the norm and scalar product given by

$$\|f\| = \Big(\sum_{x\in M} \|f(x)\|^2\Big)^{1/2}, \quad \langle f, g\rangle = \sum_{x\in M} \langle f(x), g(x)\rangle.$$

For each $x \in M$ we define an isometry W_x on $\widetilde{H}$ by setting for each element $f \in \widetilde{H}$,

$$W_x f(z) = \begin{cases} \overline{\theta}_{x,y}, & \text{if} \quad z = xy, \text{ for some } y \in M, \\ 0, & \text{if} \quad z \notin xM. \end{cases}$$

The map $W : M \to B(\widetilde{H})$, $x \to W_x$, is an isometric projective representation.

In the sequel, M will always denote a left-cancellative semigroup with unit e and θ will denote a fixed 2-cocycle of M. All isometric projective homomorphisms of M considered will be understood to have θ as associated 2-cocycle, unless the contrary is indicated in a particular context.

We call a triple (A, M, α) a C^*-dynamical system if A is a C^*-algebra and α is a homomorphism from M into the group $\mathrm{Aut}(A)$ of automorphisms of A. If B is a unital C^*-algebra, a covariant projective homomorphism (relative to θ) from (A, M, α) to B is a pair (ϕ, W), where $\phi : A \to B$ is a $*$-homomorphism and $W : M \to B$ is an isometric projective homomorphism, and ϕ, W interact via the following equation:

$$\phi\alpha_x(a)W_x = \phi(a)W_x^*, \quad a \in A,\ x \in M.$$

If $B = B(H)$ for a Hilbert space H, then we call (H, ϕ, W) a covariant projective representation of (A, M, α). Murphy has shown the following theorem:

Theorem 1.1. *If (A, M, α) is a C^*-dynamical system, then there exists a C^*-algebra $C^*_\theta(A, M, \alpha)$ and a covariant projective homomorphism (ψ, V) (relative to θ) from (A, M, α) to $M(C^*_\theta(A, M, \alpha))$ having the following universal property: For each unital C^*-algebra B and covariant projective homomorphism (ϕ, W) (relative to θ) from (A, M, α) to B, there exists a unique $*$-homomorphism $\phi \times W : C^*_\theta(A, M, \alpha) \to B$ such that*

$$\phi \times W(\psi(a)V_x) = \phi(a)W_x, \quad a \in A,\ x \in M.$$

Moreover, $C^*_\theta(A, M, \alpha)$ is generated by the elements $\psi(a)V_x$, $\forall a \in A$, $x \in M$. Up to isomorphism, these conditions uniquely determine $C^*_\theta(A, M, \alpha)$.

We call the C^*-algebra $C^*_\theta(A, M, \alpha)$ constructed in Theorem 1.1 the crossed product of A by the semigroup M under the action α (relative to the cocycle θ), or the covariance algebra of the C^*-dynamical system (A, M, α) relative to the 2-cocycle θ.

2. SG-Primitive ideals

Let A be a C^*-algebra. The set $\mathrm{Id}(A)$ of all ideals in A (ideal will always mean closed two-sided ideal) carries two important topologies, the weak topology τ_w, and the strong topology τ_s. A base for τ_w is given by the sets of the form

$$U(F) = \{I \in \mathrm{Id}(A) : J \not\subseteq I \quad \text{for all } J \in F\},$$

where F is a finite subset of $\mathrm{Id}(A)$. When restricted to $\mathrm{Prim}(A)$, the primitive ideal space of A, τ_w coincides with the Jacobson or hull-kernel topology. For $I \in \mathrm{Id}(A)$ and $F \subseteq \mathrm{Id}(A)$, $h(I)$ will denote the hull of I in $\mathrm{Prim}(A)$ and $k(F)$ the kernel of F, i.e.,

$$h(I) = \{P \in \mathrm{Prim}(A) : P \supseteq I\}, \quad k(F) = \bigcap\{J : J \in F\}.$$

Recall that $F \subseteq \mathrm{Prim}(A)$ is closed in $\mathrm{Prim}(A)$ if and only if $F = h(k(F))$. As for the strong topology, a net $(I_\lambda)_{\lambda\in\Lambda}$ is τ_s-convergent to I in $\mathrm{Id}(A)$ if and only if $\|a + I_\lambda\| \to \|a + I\|$ for all $a \in A$. For more details see [1].

Definition 2.1. Let (A, M, α) be a C^*-dynamical system. We say that a subset S of A is M-invariant if $\alpha_x^{-1}(S) \subseteq S$ for all $x \in M$. We say that (A, M, α) is simple if the only M-invariant closed ideals of A are the trivial ideals 0 and A. We say that an ideal I of A is doubly M-invariant if $\alpha_x(I) = I$ for all $x \in M$, that is, I is invariant under all the automorphisms α_x and their inverses α_x^{-1}. We say that an M-invariant ideal I of A is M-primal if whenever $n \geq 2$ and $J_1, J_2, \ldots, J_n$ are M-invariant ideals of A with zero product, then $J_i \subseteq I$ for at least one value of i.We say that an M-invariant ideal I of A is weakly M-primal if whenever $n \geq 2$ and $J_1, J_2, \ldots, J_n$ are doubly M-invariant ideals of A with zero product, then $J_i \subseteq I$ for at least one value of i. We say that an M-invariant ideal I of A is M-prime (respectively weakly M-prime) if $I \supseteq I_1 I_2$ implies $I \supseteq I_1$ or $I \supseteq I_2$ for any two M-invariant (respectively doubly M-invariant) ideals I_1 and I_2 of A.

Definition 2.2. Let (A, M, α) be a C^*-dynamical system. Then the set of SG-primitive ideals SG-$\mathrm{Prim}(A)$ of a C^*-algebra A is the set of the kernels (i.e., $\ker(\phi)$) of the covariant projective representations (H, ϕ, W) of (A, M, α) (relative to θ) with irreducible representations $(H, \phi \times W)$ of the (twisted) crossed product $C^*_\theta(A, M, \alpha)$.

Proposition 2.3. *Let (A, M, α) be a C^*-dynamical system where M is an abelian semigroup, and let I be an doubly M-invariant closed ideal of A. Then*

(i) *I is weakly M-primal if and only if it is M-primal.*

(ii) *I is weakly M-prime if and only if it is M-prime.*

Proof. (i) Obviously, we need only to show the forward implication, as the reverse is trivial. Suppose that I is weakly M-primal. Let $J_1, J_2, \ldots, J_n$ be M-invariant ideals of A with trivial intersection. For $k \in \{1, 2, \ldots, n\}$, setting

$$J_{k_0} = \bigcup_{x \in M} \alpha_x(J_k),$$

it is trivially verified that J_{k_0} is an ideal of A such that $\alpha_x(J_{k_0}) = J_{k_0}$ for all $x \in M$. Hence, the closure $\overline{J}_k$ of J_{k_0} is a doubly M-invariant ideal of A. Then $\overline{J}_1$, $\overline{J}_2, \ldots, \overline{J}_n$ have trivial intersection, whence $I \supseteq \overline{J}_k$ for some k, since I is weakly M-primal. Therefore we get $J_{k_0} \subseteq I$, and hence $J_k \subseteq I$ using M-invariance of I.

(ii) It is analogous to (i), we omit its proof. □

Remark 2.4. In Proposition 2.3, the special case of (ii) when $I = 0$ is proved in [11, p. 440], from which we have borrowed this proof.

Proposition 2.5. *Let (A, M, α) be a C^*-dynamical system, and suppose that (H, ϕ, W) is a covariant projective representation of (A, M, α) with an irreducible representation $(H, \phi \times W)$ of $C^*_\theta(A, M, \alpha)$. Then*

(i) $\mathrm{Ker}(\phi)$, *kernel of ϕ, is an M-primal ideal of A.*

(ii) $\mathrm{Ker}(\phi)$ *is an M-prime ideal of A.*

Proof. For every a in $\mathrm{Ker}(\phi)$ and x in M, by the covariant equation we know

$$W_x \phi \alpha_x^{-1}(a) = \phi(a) W_x = 0.$$

This shows $\alpha_x^{-1}(a) \in \mathrm{Ker}(\phi)$, and hence $\alpha_x^{-1}(\mathrm{Ker}(\phi)) \subseteq \mathrm{Ker}(\phi)$. Suppose now that $J_1, J_2, \ldots, J_n$ are M-invariant ideals in A with trivial intersection. We must show that $J_k \subseteq \mathrm{Ker}(\phi)$ for some k. Assume that $J'_1, J'_2, \ldots, J'_n$ are ideals in $C^*_\theta(A, M, \alpha)$ generated by $J_1, J_2, \ldots, J_n$, respectively. By Theorem 1.1, it is readily verified that the J'_k's are generated by all elements of the form aV_x for $a \in J_k$ and $x \in M$. Since $J_1, J_2, \dots, J_n$ are M-invariant ideals, it immediately follows that

$$\phi \times W(J_1' J_2' \dots J_n') = \{0\}.$$

Indeed, let $a_1 \in J_1$, $a_2 \in J_2$, $\dots$, $a_n \in J_n$ and $x_1, x_2, \dots, x_n \in M$. Then, by the M-invariance of $J_1, J_2, \dots, J_n$, there exist elements $a_1' \in J_1$, $a_2' \in J_1 \cap J_2$, $\dots$, $a_{n-1}' \in \bigcap_{i=1}^{n-1} J_i$ such that

$$\alpha_{x_1}(a_1') = a_1, \alpha_{x_2}(a_2') = a_1' a_2, \dots, \alpha_{x_{n-1}}(a_{n-1}') = a_{n-2}' a_{n-1}.$$

Thus, we obtain

$$\begin{aligned}
\phi \times W(a_1 V_{x_1} a_2 V_{x_2} \dots a_n V_{x_n}) &= \phi \times W(a_1 V_{x_1}) \phi \times W(a_2 V_{x_2}) \dots \\
&\quad \phi \times W(a_n V_{x_n}) \\
&= \phi(a_1) W_{x_1} \phi(a_2) W_{x_2} \dots \phi(a_n) W_{x_n} \\
&= \phi \alpha_{x_1}(a_1') W_{x_1} \phi(a_2) W_{x_2} \dots \phi(a_n) W_{x_n} \\
&= W_{x_1} \phi(a_1') \phi(a_2) W_{x_2} \dots \phi(a_n) W_{x_n} \\
&= W_{x_1} \phi(a_1' a_2) W_{x_2} \dots \phi(a_n) W_{x_n} \\
&= W_{x_1} \phi(\alpha_{x_2}(a_2')) W_{x_2} \phi(a_3) W_{x_3} \dots \phi(a_n) W_{x_n} \\
&= W_{x_1} W_{x_2} \phi(a_2') \phi(a_3) W_{x_3} \dots \phi(a_n) W_{x_n} \\
&= W_{x_1} W_{x_2} \dots W_{x_{n-1}} \phi(a_{n-1}' a_n) W_{x_n} = 0.
\end{aligned}$$

Since $\phi \times W$ is irreducible, $\mathrm{Ker}(\phi \times W)$ is prime by [12, Proposition 3.13.10]. Therefore, we know that $J_k' \subseteq \mathrm{Ker}(\phi \times W)$ for some k. On the other hand, for $a \in J_k$ and $x \in M$, by [10] and [11] we have

$$\phi(a) = \phi \times W(aV_x^* V_x).$$

Now J_k' contain $\{aV_x^* V_x\}_{x \in M}$, and thus $\phi \times W(J_k') = \{0\}$. This shows $\phi(J_k) = 0$, and hence $J_k \subseteq \mathrm{Ker}(\phi)$ for some k. (ii) is analogous to (i), we omit its proof. $\square$

Lemma 2.6. *If (A, M, α) is a C^*-dynamical system and π denotes a $*$-homomorphism from $C_\theta^*(A, M, \alpha)$ onto a C^*-algebra B, then there exists a unique covariant projective homomorphism (ϕ, W) from (A, M, α) to $M(B)$ such that $\pi = \phi \times W$.*

Proof. Uniqueness is easily checked. To see the existence, let ϕ be the restriction of π to A and define $W_x \in M(B)$ by putting

$$W_x\pi(a) = \pi(V_x a), \quad \pi(a)W_x = \pi(aV_x),$$

for all $a \in C^*_\theta(A, M, \alpha)$. For every $a \in C^*_\theta(A, M, \alpha)$, if $\pi(a) = 0$ and (u_i) is an approximate unit for $C^*_\theta(A, M, \alpha)$, then

$$\pi(V_x a) = \lim_i \pi(V_x u_i a) = \lim_i \pi(V_x u_i)\pi(a) = 0;$$
$$\pi(aV_x) = \lim_i \pi(a u_i V_x) = \lim_i \pi(a)\pi(u_i V_x) = 0.$$

This shows that W_x is well-defined. $\square$

Proposition 2.7. *Let I be an M-invariant ideal of A. Then there exists a covariant projective representation (H, ϕ, W) of (A, M, α) with an irreducible representation $(H, \phi\times W)$ of $C^*_\theta(A, M, \alpha)$ such that $I \subseteq \mathrm{Ker}(\phi)$.*

Proof. Consider a quotient C^*-algebra A/I, and define the action $\overline{\alpha}$ on M by $\overline{\alpha}_t([x]) = [\alpha_t(x)]$ for $[x] \in A/I$ and $t \in M$. Now we choose a covariant projective representation $(H, \overline{\phi}, W)$ of $(A/I, M, \overline{\alpha})$ such that $(H, \overline{\phi}\times W)$ is an irreducible representation of $C^*_\theta(A/I, M, \overline{\alpha})$ by Lemma 2.6. Define the representation ϕ of A by

$$\phi(x) = \overline{\phi}([x]).$$

Then (H, ϕ, W) is a covariant projective representation. In fact, for $x \in M$ and $a \in A$, we have

$$\phi\alpha_x(a)W_x = \overline{\phi}([\alpha_x(a)])W_x = \overline{\phi}\overline{\alpha}_x([a])W_x = W_x\overline{\phi}([a]) = W_x\phi(a).$$

Since $\phi(A) = \overline{\phi}(A/I)$, we obtain

$$\phi \times W(C^*_\theta(A, M, \alpha))'' = \overline{\phi} \times W(C^*_\theta(A/I, M, \overline{\alpha}))''.$$

Hence $(H, \phi \times W)$ is irreducible. $\square$

3. The weak topology

Let (A, M, α) be a C^*-dynamical system. We define the Jacobson topology on SG-Prim(A) as follows. For a subset F in SG-Prim(A), we define an M-invariant ideal $k(F)$ in A by

$$k(F) = \bigcap \{J : J \in F\}.$$

We call this M-invariant ideal the kernel of F. For an M-invariant ideal I, we define a set $h(I)$ by

$$h(I) = \{P \in SG\text{-Prim}(A) : P \supseteq I\}.$$

We call this set the hull of I, which is not empty by Proposition 2.7. Setting

$$\overline{F} = h(k(F))$$

we have the following

Theorem 3.1. *Suppose that (A, M, α) is a C^*-dynamical system. Then for a subset F in SG-Prim(A), the map $F \to \overline{F}$ satisfies Kuratowski's axioms:*

(i) $\overline{\phi} = \emptyset$ *where $\emptyset$ denotes the empty set.*

(ii) $F \subseteq \overline{F}$.

(iii) $\overline{\overline{F}} = \overline{F}$.

(iv) $\overline{F_1 \cup F_2} = \overline{F_1} \cup \overline{F_2}$.

Proof. (i) It is trivial.

(ii) If $I \in F$, then by $k(F) \subseteq I$ we have $I \in h(k(F)) = \overline{F}$, and hence $F \subseteq \overline{F}$.

(iii) Without loss of generality, we may suppose $F = h(J)$ for some $J \subseteq A$. If $I \in \overline{F}$, then we have $k(F) \subseteq I$. Since $F = h(J)$, it follows that $J \subseteq I'$ for any $I' \in F$. Then

$$k(F) = \bigcap \{I' : I' \in F\} \supseteq J,$$

hence $I \supseteq J$, which means that $I \in h(J) = F$. Therefore it follows that $\overline{F} \subset F$, and then (ii) shows $\overline{F} = F$. This proves $\overline{\overline{F}} = \overline{F}$.

(iv) Assuming that $I \in \overline{F_1}$, we get $k(F_1) \subseteq I$. Then, since $k(F_1 \cup F_2)) \subseteq k(F_1)$,

$$I \in h(k(F_1 \cup F_2)) = \overline{F_1 \cup F_2}.$$

Conversely, let $I \in \overline{F_1 \cup F_2}$ and note that $k(F_1 \cup F_2) = k(F_1) \cap k(F_2)$ and $k(F_1 \cup F_2) \subseteq I$ by the definitions. Therefore $k(F_1) \cap k(F_2) \subseteq I$. $\square$

It follows that there is a unique topology on SG-Prim(A) such that for each $F \subseteq SG$-Prim(A), $\overline{F}$ is the closure of F in this topology. Now we have the following definition:

Definition 3.2. This topology is called the weak topology on SG-Prim(A). In the sequel, we shall denote by τ_w the weak topology on SG-Prim(A).

Remark 3.3.. It is obvious that, in generally, SG-Prim(A) is not homeomorphic to Prim$(C^*_\theta(A, M, \alpha))$. For example, we may consider a non-trivial C^*-dynamical system (A, M, α) where M is a partially ordered group such that (A, M, α) is simple, and $C^*_\theta(A, M, \alpha)$ is not simple (see [10], Proposition 3.3). On the other hand, SG-Prim(A) is not homeomorphic to Prim(A) either. For example, take $C(\mathbb{T})$, the continuous functions on the unit circle $\mathbb{T}$ as a C^*-algebra A, and consider $\mathbb{Z}^+$, the action by the automorphism α given by rotation through the angle $2\pi\theta$ where θ is an irrational number in $(0, 1)$. Then, by [11], $(C(\mathbb{T}), \mathbb{Z}, \alpha)$ is simple. Hence SG-Prim$(C(\mathbb{T}))$ consists of one point only. But Prim$(C(\mathbb{T}))$ is an infinite set.

Proposition 3.4. *Let (A, M, α) be a C^*-dynamical system. Then the space SG-Prim(A) is a T_0-space.*

Proof. Let I_1, I_2 be two distinct points of SG-Prim(A), so that, without loss of generality, we may suppose $I_1 \nsubseteq I_2$. Thus the set of those $I \in SG$-Prim(A) which contain I_1 is a closed subset T, such that $I_1 \in T$ and $I_2 \notin T$. $\square$

Proposition 3.5. *Let (A, M, α) be a C^*-dynamical system. If A is unital, then SG-Prim(A) is compact.*

Proof. Let $\{I_i\}$ be a family of closed subsets of SG-Prim(A) with empty intersection. If $\sum_i k(I_i) \neq A$, then there exists I in SG-Prim(A) such that $\sum_i k(I_i) \subset I$. Thus we know that $I \in h(k(I_i)) = I_i$ for all i.

This implies that $\bigcap I_i \neq \emptyset$. Therefore we obtain $\sum k(I_i) = A$. So there exists a finite subset $\{a_k\}_1^n$ of A such that $a_1 \in k(I_{i_1})$, $\ldots$, $a_n \in k(I_{i_n})$ and $a_1 + a_2 + \ldots + a_n = 1$. This implies that $k(I_{i_1}) + \ldots + k(I_{i_n}) = A$, from which it follows that $I_{i_1} \cap I_{i_2} \cap \ldots I_{i_n} = \emptyset$. $\square$

In the case when a C^*-algebra is not unital, if M is trivial then $\mathrm{Prim}(A) = SG\text{-}\mathrm{Prim}(A)$, so $SG\text{-}\mathrm{Prim}(A)$ need not be compact. However, we shall see later that $SG\text{-}\mathrm{Prim}(A)$ is locally compact. Subsequently, we consider a condition under which $SG\text{-}\mathrm{Prim}(A)$ is a Hausdorff space (without $SG\text{-}\mathrm{Prim}(A)$ being necessarily Hausdorff in the general case). First we realize each element in A as a lower semi-continuous function on $SG\text{-}\mathrm{Prim}(A)$. Let $x \in A$, and define a function f_x on $SG\text{-}\mathrm{Prim}(A)$ by

$$f_x(I) = \|x + I\|$$

for $I \in SG\text{-}\mathrm{Prim}(A)$. Then we have the following

Proposition 3.6. *Let (A, M, α) be a C^*-dynamical system and $x \in A$. Then f_x is lower semi-continuous on $(SG\text{-}Prim(A), \tau_w)$.*

Proof. For any $\varepsilon > 0$, we set

$$\begin{aligned} Z &= \{I \in SG\text{-}\mathrm{Prim}(A) : f_x(I) \leq \varepsilon\} \\ &= \{I \in SG\text{-}\mathrm{Prim}(A) : \mathrm{Sp}(x + I) \subset [0, \varepsilon]\}, \end{aligned}$$

where $Sp(x + I)$ denotes the spectrum of $x + I$ in A/I. We must show that Z is closed. Replacing x by x^*x, we are reduced to the case where $x \in A_+$, the set of all positive elements of A. Now let $\lambda \notin [0, \varepsilon]$, and let f be a continuous function from $\mathbb{R}$ to $\mathbb{R}$ ($\mathbb{R}$ is the field of real numbers) vanishing on $[0, \varepsilon]$ with $f(\lambda) \neq 0$. Since $f(x) + I = f(x + I)$ by [13, 1.8.4], we have

$$Sp(f(x) + I) = Sp(f(x + I)) = f(Sp(x + I)) \subset f([0, \varepsilon]).$$

Thus we see that $\lambda \notin Sp(x + I)$, and then $I \in Z$. This means that Z is closed. $\square$

Proposition 3.7. *Let (A, M, α) be a C^*-dynamical system, and let I be a proper M-invariant ideal of A. For every $x \in A$, there exists $J \in h(I)$ such that $\|x + J\| = \|x + I\|$.*

Proof. Consider a quotient C^*-algebra A/I and define the action $\overline{\alpha}$ on M by $\overline{\alpha}_J(x + I) = \alpha_J(x) + I$ for $x + I \in A/I$ and $J \in M$. Regarding $x+I$ as an element of the multiplier algebra $M(C^*_\theta(A/I, M, \overline{\alpha}))$ of a covariance algebra $C^*_\theta(A/I, M, \overline{\alpha})$, by [5, Lemma 3.3.6] we can choose an irreducible representation (H, ρ) of $M(C^*_\theta(A/I, M, \overline{\alpha}))$ satisfying $\|\rho(x+I)\| = \|x + I\|$. It follows that $\rho|_{C^*_\theta(A/I,M,\overline{\alpha})}$ is an irreducible representation of $C^*_\theta(A/I, M, \overline{\alpha})$, which is written as $\overline{\phi}\times W$ via some covariant projective representation $(H, \overline{\phi}, W)$ of $(A/I, M, \overline{\alpha})$. If we define a representation (H, ϕ) of A by $\phi(x) = \overline{\phi}(x + I)$, it immediately follows that (H, ϕ, W) is a covariant projective representation of (A, M, α). Since

$$\phi \times W(C^*_\theta(A, M, \alpha))'' = \overline{\phi} \times W(C^*_\theta(A/I, M, \overline{\alpha}))'',$$

we have $\mathrm{Ker}(\phi) \in SG\text{-}\mathrm{Prim}(A)$ and $I \subset \mathrm{Ker}(\phi)$. Furthermore, we have

$$\|x + I\| = \|\phi(x)\| = \|\overline{\phi}(x + I)\| = \|\rho(x + I)\| = \|x + I\|.$$

□

Proposition 3.8. *Let (A, M, α) be a C^*-dynamical system, let $a \in A$ and $\varepsilon > 0$, and put*

$$N = \{I \in SG\text{-}Prim(A) : \|a + I\| \geq \varepsilon\}.$$

Then N is compact in $(SG\text{-}Prim(A), \tau_w)$.

Proof. Replacing a by a^*a, we are reduced to the case when $a \in A_+$. Let $a \in A_+$. First we have

$$\begin{aligned} N &= \{I \in SG\text{-}\mathrm{Prim}(A) : \|x + I\| \geq \varepsilon\} \\ &= \{I \in SG\text{-}\mathrm{Prim}(A) : \mathrm{Sp}(x + I) \cap [\varepsilon, \infty) \neq = \emptyset\}. \end{aligned}$$

Consider a positive continuous function f on $\mathbb{R}$ with $f(0) = 0$ such that $f = 1$ on $[\varepsilon, \infty)$ and $0 \leq f < 1$ on $(-\infty, \varepsilon)$. Then we have

$$N = \{I \in SG\text{-}\mathrm{Prim}(A) : \|f(x + I)\| = 1\}.$$

Let $\{N_i\}$ be a decreasing filtering family of closed nonempty subsets of SG-Prim(A) such that $N \cap N_i \neq \emptyset$ for each i, and then let $I_i \in N \cap N_i$. If we put $J_i = k(N_i)$, then we have

$$1 = \|f(x + I_i\| \leq \|f(x) + J_i\| \leq 1.$$

Since $\{J_i\}$ is an increasing filtering family of M-invariant ideals of A, the above implies that $\|f(x) + J\| = 1$ for $J = \overline{\bigcup J_i}$. Since J is a proper M-invariant ideal of A, it follows that there exists an element P in $h(J)$ such that $\|f(x) + P\| = 1$, which implies $P \in N$. On the other hand, since P contains J_i for all i, we have $P \in h(J_i) = h(k(N_i)) = N_i$. Hence we see that $N \cap (\bigcap N_i) \neq \emptyset$. □

Theorem 3.9. *Let (A, M, α) be a C^*-dynamical system. Then (SG-Prim$(A), \tau_w$) is a Hausdorff space if and only if the function $I \to \|a + I\|$ is continuous on (SG-Prim$(A), \tau_w$) for every $a \in A$.*

Proof. Suppose that (SG-Prim$(A), \tau_w$) is a Hausdorff space. By Proposition 3.6, we have only to show that the function $I \to \|a + I\|$ is upper semi-continuous. In fact, since $\{I \in SG\text{-Prim}(A) : \|a + I\| < \varepsilon\}$ is the complement of $\{I \in SG\text{-Prim}(A) : \|a+I\| \geq \varepsilon\}$, it is an open subset by Proposition 3.8. Thus the function $a \to \|a + I\|$ is upper semi-continuous.

Conversely, suppose that the function $a \to \|a + I\|$ is continuous for every $a \in A$. Let us take I_1 and I_2 from SG-Prim(A) with $I_1 \neq I_2$. Then we may assume that $I_2 \not\subseteq I_1$. So, we can choose a positive element a in I_2 with $a \notin I_1$. Set $\varepsilon = \|a + I_1\|/2$. Then

$$\{I \in SG\text{-Prim}(A) : \|a + I\| > \varepsilon\}, \quad \text{and}$$
$$\{I \in SG\text{-Prim}(A) : \|a + I\| < \varepsilon\}$$

are open subsets by continuity of the function $a \to \|a + I\|$. Since they are disjoint neighbourhoods in SG-Prim(A) of I_1 and I_2 respectively, SG-Prim(A) is a Hausdorff space. □

Theorem 3.10. *Suppose that (A, M, α) is a C^*-dynamical system. Then (SG-Prim$(A), \tau_w$) is locally compact.*

Proof. Let $I \in SG$-Prim(A) and let W be a neighbourhood of I in SG-Prim(A). Set $J = k(SG\text{-Prim}(A)\ W)$. Since $J \not\subseteq I$, we can choose a positive element a in J with $a \notin I$ and $\|a + I\| > 1$. Let

$$N = \{P \in SG\text{-Prim}(A) : \|a + P\| > 1\}.$$

By Proposition 3.6, using the lower semi-continuity of the function $a \to \|a+I\|$, N is an open subset of $(SG\text{-Prim}(A), \tau_w)$. Since $I \subseteq N$, N is a compact neighbourhood of I by Proposition 3.8. $\square$

4. The Dauns-Hofmann type theorem

We finish this paper with a discussion of spaces of continuous functions. Let us denote by $C^b(X)$ the C^*-algebra of all bounded continuous complex-valued functions on the topological space X.

Let (A, M, α) be a C^*-dynamical system. For $I \in SG\text{-Prim}(A)$, there is a covariant projective representation (H, ϕ, W) of (A, M, α) with an irreducible representation $(H, \phi \times W)$ of $C^*_\theta(A, M, \alpha)$ such that the kernel of ϕ is I. We denote by ϕ'' the normal extension of ϕ to the enveloping von Neumann algebra A'' of A (see [12, Theorem 3.7.7]). If α''_t denotes the double transpose of α_t for each $t \in M$, then the map $t \to \alpha''_t$ is a homomorphism of M into the automorphism group $\mathrm{Aut}(A'')$ of A''. Now denote by Z the centre of A''. Since $\phi''(Z) \subseteq \phi(A)'' \cap \phi(A)'$, we easily see that

$$\phi''(Z^{\alpha''}) \subseteq \phi(A)'' \cap \phi(A)' \cap W'_M,$$

where $W'_M = \{W_t : \forall t \in M\}'$, and $Z^{\alpha''}$ is the fixed point algebra of Z for α''. Thus, we have

$$\phi''(Z^{\alpha''}) \subseteq \phi \times W(C^*_\theta(A, M, \alpha))'' \bigcap \phi \times W(C^*_\theta(A, M, \alpha))' = \mathbb{C} \cdot 1.$$

Therefore, for an element $x \in Z^{\alpha''}$ there exists a bounded complex-valued function f_x on $SG\text{-Prim}(A)$ such that $\phi''(x) = f_x(I) \cdot 1$ for $\phi \in SG\text{-Prim}(A)$. If x is also a positive element in A, then we have

$$f_x(I) = \|x + \mathrm{Ker}\,\phi\| = \|x + I\|.$$

Now we have the following

Proposition 4.1. *Let A be a unital C^*-algebra with centre Z, let (A, M, α) be a C^*-dynamical system and Z^α a fixed point algebra of Z for α. Then for each positive element $x \in Z^\alpha$, f_x is continuous on $(SG\text{-Prim}(A), \tau_w)$.*

Proof. Choosing some positive number λ, we consider a positive element $y = -x + \lambda \cdot 1$ in Z^α. Then, since $\phi(y) = -\phi(x) + \lambda \circ 1$ for

$\mathrm{Ker}(\phi) = I \in SG\text{-Prim}(A)$, we see that $\|y + I\| = -\|x + I\| + \lambda \cdot 1$, that is, $f_y(I) = -f_x(I) + \lambda$. On the other hand, since constant functions are continuous on SG-Prim(A), we conclude by Proposition 3.6 that $f_y - \lambda \cdot 1$ is lower semi-continuous. This means that f_x is upper semi-continuous. Consequently, f_x is continuous by Proposition 3.6. □

Notation 4.2. Let A be a C^*-algebra. We denote by A_{sa}, $(A_{sa})^m$, and $(A_{sa})_m$, the set of all self-adjoint elements of A the set of strong limits (in A'') of monotone increasing nets from A_{sa}, and the set of strong limits (in A'') of monotone decreasing nets from A_{sa}, respectively.

Proposition 4.3. *Let (A, M, α) be a C^*-dynamical system and Z the centre of A''. Then the map $x \to f_x$ is injective from $Z^{\alpha''} \cap ((A_{sa})^m \cup (A_{sa})_m)$ into the set of real bounded functions on SG-Prim(A).*

Proof. We know by Lemma 2.6 that any irreducible representation of $C^*_\theta(A, M, \alpha)$ is of the form $(H, \phi \times W)$ with some covariant isometric representation (H, ϕ, W) of (A, M, α). Since $(\phi \times W)''|_A = \phi$, the map $x \to f_x$ is the restriction to $Z^{\alpha''} \bigcap ((A_{sa})^m \bigcup (A_{sa})_m)$ of the atomic representation of $C^*_\theta(A, M, \alpha)''$. Since A_{sa} is contained in the multiplier algebra of $C^*_\theta(A, M, \alpha)$, we see from [12, Theorem 3.12.9] that $(A_{sa})^m \subseteq (\overline{(C^*_\theta(A, M, \alpha)_{sa}})^m$ and $(A_{sa})_m \subset (\overline{(C^*_\theta(A, M, \alpha)_{sa})_m}$. Here we denote by $\overline{C^*_\theta(A, M, \alpha)}$ the unital C^*algebra obtained by adjunction of 1 to $C^*_\theta(A, M, \alpha)$. Since the set of universally measurable elements in $C^*_\theta(A, M, \alpha)''$ is a vector space containing $(\overline{(C^*_\theta(A, M, \alpha)_{sa}})^m = -\overline{((C^*_\theta(A, M, \alpha)_{sa})}_m$ (see [12, Proposition 4.3.13]), $(A_{sa})^m \cup (A_{sa})_m$ is contained in the set of universally measurable elements in $(C^*_\theta(A, M, \alpha))''$. Since the atomic representation is faithful on the set of universally measurable elements by [12, Theorem 4.3.15], we get the desired result. □

Proposition 4.4. *Let A be a unital C^*-algebra with centre Z, let (A, M, α) be a C^*-dynamical system and Z^α a fixed point algebra of Z for α. Then the map $x \to f_x$ is surjective from Z^α onto the algebra of continuous complex-valued functions on SG-Prim(A).*

Proof. To show that the map is surjective, let f be a continuous function on SG-Prim(A). Without loss of generality, suppose first that $0 \le f \le 1$. We set

$$f_n = \sum_{k=1}^{2^n} 2^{-n} \chi_{n_k},$$

where χ_{n_k} denotes the characteristic function of the set $\{I \in SG\text{-Prim}(A): f(I) > k2^{-n}\}$ for every k and n with $1 \leq k \leq 2^{-n}$. Since every such set is open in SG-Prim(A), there is by the definition of the weak topology an M-invariant closed ideal I_{n_k} in A such that $SG\text{-Prim}(A)\backslash h(I_{n_k})$ corresponds to the above open set. Let p_{n_k} be the central open projection in A'' corresponding to I_{n_k} (see [12]). Then $p_{n_k} \in Z^{\alpha''} \bigcap (A_{sa})^m$ and $f_{p_{n_k}} = \chi_{n_k}$. In fact, if $I \in h(I_{n_k})$ then we have $I_{n_k} \subset I$. For ϕ with $\mathrm{Ker}(\phi) = I$, since $\phi''(I''_{n_k}) = \{0\}$, we have $\phi''(p_{n_k}) = 0$. If $I \in SG\text{-Prim}(A)\backslash h(I_{n_k})$ then we have $I_{n_k} \not\subseteq I$. Thus we conclude that $\phi(I_{n_k}) \neq \{0\}$ with $\mathrm{Ker}(\phi) = I$, which means that $\phi''(p_{n_k}) \neq 0$. Since $\phi''(p_{n_k})$ is a projection, we obtain that $\phi''(p_{n_k}) = 1$. Define

$$x_n = 2^{-n} \sum_{k=1}^{2^n} p_{n_k}.$$

Then $x_n \in Z^{\alpha''} \bigcap (A_{sa})^m$ and $f_{x_n} = f_n$. Since $f_n \nearrow f$, the sequence $\{x_n\}$ must increase to an element $x \in Z^{\alpha''} \bigcap (A_{sa})^m$ such that $f_x = f$.

If f is not positive, we may have to modify it by adding a scalar. Now we consider $-f + 1$ instead of f. Since $0 \leq -f + 1 \leq 1$, it follows from the above arguments that there exists an element $y \in Z^{\alpha''} \bigcap (A_{sa})^m$ such that $f_y = -f + 1$. This implies that $x = 1 - y$. Therefore, by [12, 4.4.7], we assert that

$$x \in Z^{\alpha''} \bigcap (A_{sa})^m \bigcap (A_{sa})_m = Z^{\alpha''} \bigcap A_{sa},$$

which means $x \in Z^{\alpha}$. □

Theorem 4.5. *Let (A, M, α) be a C^*-dynamical system, where A is a unital C^*-algebra with centre Z and M is a monoid. Then there is a $*$-isomorphism ϕ of Z^α onto $C^b(SG\text{-Prim}(A), \tau_w)$.*

Proof. Since $(\phi \times W)''|_A = \phi$, the map $x \to f_x$ is the restriction to Z^α of the atomic representation of $(C^*_\theta(A, M, \alpha))''$. Since the atomic representation is faithful on the multiplier algebra and A belongs to the multiplier algebra of $C^*_\theta(A, M, \alpha)$, the homomorphism $x \to f_x$ is injective. The result is now immediate from Proposition 4.4. □

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