On the structure of ordered real vector spaces.

To the memory of my beloved teacher Professor Tibor Szele.

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§ 1. Introduction.

In the present paper we determine all simply ordered real vector spaces of countable dimension (Theorem 3. 5), and we give a total description of them (Corollary 3. 6). This result is an extension of the well-known structure theorem of real vector lattices of finite dimension ([1] Ch. XV., Theorem 1) to the case of simply ordered real vector spaces of countable dimension.[1]) Our Theorem 3. 5 ceases to be valid already in the case of simply ordered real vector spaces of dimension $2^{\aleph_0}$; nor can it be carried over to real vector lattices of countable dimension (in a suitably modified form which is adaptable to this case). From Theorem 3. 5 one immediately deduces the existence of a simply ordered real vector space of countable dimension in which any other such space can be embedded (Corollary 3. 7).

Moreover, we give a construction, by which every possible simple ordering of any real vector space can be uniquely obtained (Theorem 4. 4). This gives some insight into the structure of these ordered spaces.

The restriction that the operator domain of the vector space be the real number system, is essential throughout.

§ 2. Preliminaries.

In what follows, we shall investigate vector spaces over the field of real numbers. Only operator domains different from this will be mentioned explicitly.

We shall use the following notations. Sets are denoted by Latin capitals, their elements by small letters. V always stands for a vector space, R for the real number system; a, b, h, x, y denote vectors, i, n natural numbers, λ, μ real numbers.

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1) The numbers in brackets refer to the bibliography at the end of the paper.

I. By an *ordered vector space* we mean a vector space for the elements of which there is defined a (linear) order relation $>$ with the properties

1. $x>0$, $y>0$ imply $x+y>0$,
2. $x>0$, $\lambda>0$ imply $\lambda x>0$,
3. $x>y$ if and only if $x-y>0$.

For the set V^+ of all positive elements of an ordered vector space, i. e. for the so called *domain of positivity* of the space the following requirements always hold:

1'. $x\in V^+$, $y\in V^+$ imply $x+y\in V^+$,

2'. $x\in V^+$, $\lambda>0$ imply $\lambda x\in V^+$,

3'. $0\notin V^+$,

4'. $x\neq 0$, $x\notin V^+$ imply $-x\in V^+$.

Conversely, if a subset V^+ of a vector space has the properties 1'.—4'., then it is the domain of positivity of the space for one and only one ordering.

II. An element $x>0$ of an ordered vector space is *incomparably smaller* than $y>0$ (denoted by $x\ll y$) if and only if $\lambda x\leqq y$ holds for any λ. If, for $x>0$ and $y>0$ neither $x\ll y$ nor $y\ll x$ holds, then x and y are said to be *equivalent:* $x\sim y$. It is clear that $\ll$ defines a quasi-ordering of the set of positive elements, and that $\sim$ is an equivalence relation. The corresponding equivalence classes are termed *archimedean classes.* The set of archimedean classes becomes simply ordered, if we define the class containing x to be greater than the class containing y, if and only if $x\ll y$.

III. Let us consider the real valued functions defined on an ordered set T, taking on nonzero values at most on a finite subset of T. If $f(t)$ and $g(t)$ are functions having this property, then so are their sum $f(t)+g(t)$ and the product $\lambda f(t)$. Such a function $f(t)$ is considered positive if and only if in the first place t_0 where the value of the function is different from zero, $f(t_0)>0$ holds. The ordered vector space thus obtained will be called *the discrete lexicographically ordered function space over T.*

It is easy to determine the archimedean classes of this ordered vector space. The functions $f(t)$ assuming positive values for a fixed $t_0\in T$ but satisfying $f(t)=0$ for all $t<t_0$ form an archimedean class, and all archimedean classes can be obtained thus. Accordingly, the ordered set of the archimedean classes is similar to T. In view of this fact, the following lemma can be established without difficulty.

Lemma 2.1. *There exists (up to an isomorphism) a one-to-one correspondence between the order types α and the discrete lexicographically ordered vector spaces V. The α belonging to V is the order type of the ordered set of its archimedean classes; the V belonging to α is the discrete lexicographically ordered function space over an arbitrary ordered set of type α.*

This uniquely determined order type α will be called the *type of the function space.*

IV. A subspace J of an ordered vector space V will be called an *ideal* of V, if the following condition is satisfied: $a \in J$, $x \in V$ and $|x| \leqq |a|$ imply $x \in J$. (By the *absolute value* $|x|$ of an element $x \neq 0$ of V we mean the positive one of the two elements $x, -x$; $|0| = 0$.) In proving Theorem 4.4, we shall make use of the important fact that of all ordered vector spaces only R fails to have non-trivial ideals. In the sequel we shall need the following simple properties of ideals. Let J be an ideal of V; the subspace $J^* \subseteq J$ is an ideal of J if and only if it is an ideal of V too. The ideals of V form a chain of sets. Union and intersection of ideals are also ideals.

V. A vector space V can be partitioned into three disjoint subsets with the aid of any of its maximal proper subspaces H, and of an element a not contained in H.[2]) One of these subsets coincides with H, another is formed by the elements $h + \lambda a$ ($h \in H$; $\lambda > 0$), and the third by the elements $h - \lambda a$ ($h \in H$; $\lambda > 0$). It is not hard to see that this partition of V is independent of the particular choice of the element a. Therefore, the two subsets different from H in this partition can be termed the two *halfspaces of V determined by H.* Each of these is obtained from the other by multiplication by -1.

§ 3. Ordered vector spaces of countable dimension.

In this § we are going to determine all ordered vector spaces of countable dimension (Theorem 3.5). The proof of this theorem will be based on Lemmas 3.1—3.4.

Lemma 3.1.[3]) *Any two linearly independent, positive and equivalent elements of an ordered vector space have a linear combination incomparably smaller than any of them.*

PROOF. Let x and y be two linearly independent, positive and equivalent elements of an ordered vector space. Then there exists a λ for which $y < \lambda x$ holds. Now, if λ_0 denotes the g. l. b. of the set of all real numbers λ having this property (it is obvious that 0 is a lower bound) then we evidently have for any $\mu > 0$

$$\left(\lambda_0 - \frac{1}{\mu}\right) x < y < \left(\lambda_0 + \frac{1}{\mu}\right) x,$$

[2]) Here and in the sequel it will be convenient to consider 0 as a proper subspace.

[3]) This is essentially Lemma 2.4 of [2].

i. e. $\mu|y-\lambda_0 x|<x$. In view of the linear independence of x and y we have $y-\lambda_0 x \neq 0$, and thus either $y-\lambda_0 x$ or $-(y-\lambda_0 x)$ is a linear combination required.

Lemma 3. 2. *If an element x of an ordered vector space is linearly independent of the system $x_1,\dots,x_n$ consisting of positive elements no two of which are equivalent, then there exists an element $x_{n+1}>0$, not equivalent to any of the elements $x_1,\dots,x_n$, for which $\{x_1,\dots,x_n,x_{n+1}\}=\{x_1,\dots,x_n,x\}$ holds.*

PROOF. If $|x|$ is not equivalent to any of the elements $x_1,\dots,x_n$, then $x_{n+1}=|x|$ obviously fulfils our requirements.

If $|x|$ is equivalent to one of $x_1,\dots,x_n$, then we can suppose that x_1 is the least one of these elements for which an equivalent of the form $\lambda_1 x_1+\cdots+\lambda_n x_n+\lambda x>0$ $(\lambda\neq 0)$ still exists. Since x_1 and this $\lambda_1 x_1+\cdots+ +\lambda_n x_n+\lambda x$ are linearly independent, there exists by Lemma 3. 1 a linear combination $\mu_1 x_1+\mu(\lambda_1 x_1+\cdots+\lambda_n x_n+\lambda x)$ of them, incomparably smaller than any of the two components. Clearly, $\mu\neq 0$ must hold. Thus, in view of the minimality of x_1, the element $(\mu_1+\mu\lambda_1)x_1+(\mu\lambda_2)x_2+\cdots+(\mu\lambda_n)x_n+ +(\mu\lambda)x>0$ cannot be equivalent to any of the $x_1,\dots,x_n$. We see therefore that

$$(\mu_1+\mu\lambda_1)x_1+(\mu\lambda_2)x_2+\cdots+(\mu\lambda_n)x_n+(\mu\lambda)x$$

can be chosen for x_{n+1}.

Lemma 3. 3. *Any ordered vector space of countable dimension has a basis consisting of positive elements no two of which are equivalent.*

PROOF. Let $b_1,\dots,b_n,\dots$ be a basis of the ordered vector space V of countable dimension. We inductively define a sequence $b_1^*,\dots,b_n^*,\dots$ of pairwise inequivalent positive elements satisfying $\{b_1^*,\dots,b_n^*\}=\{b_1,\dots,b_n\}$ for any natural number $n\leqq \dim V$. By this equality $b_1^*,\dots,b_n^*,\dots$ is an independent generating system of V.

Let $b_1^*=|b_1|$. Suppose that the elements $b_1^*,\dots,b_n^*$ $(n<\dim V)$ are already defined, and are pairwise inequivalent positive elements satisfying the relation $\{b_1^*,\dots,b_n^*\}=\{b_1,\dots,b_n\}$. Then b_{n+1} is independent also of the system $b_1^*,\dots,b_n^*$, and thus, by Lemma 3. 2, there exists an element $b_{n+1}^*>0$ — which we choose for the $n+1$-st term of our sequence — equivalent to none of the elements $b_1^*,\dots,b_n^*$, and satisfying $\{b_1^*,\dots,b_n^*,b_{n+1}^*\}=\{b_1^*,\dots,b_n^*,b_{n+1}\}$. Hence, by our induction hypothesis, $\{b_1^*,\dots,b_n^*,b_{n+1}^*\}=\{b_1,\dots,b_n,b_{n+1}\}$ follows. This completes the proof of the lemma.

Lemma 3. 4. *A linear combination $\lambda_1 x_1+\cdots+\lambda_n x_n$ of the elements $x_1\gg\cdots\gg x_n$ of an ordered vector space is >0 if and only if the first coefficient different from zero is >0.*

PROOF. Suppose that the first nonzero coefficient λ_i of the linear combination $\lambda_1 x_1 + \cdots + \lambda_n x_n$ is positive. For $i = n$ our statement is obviously true, so we may suppose $i < n$. In view of $x_i \gg x_{i+1}, \ldots, x_i \gg x_n$ we have the inequalities

$$x_i > \left[(i-n)\frac{\lambda_{i+1}}{\lambda_i}\right] x_{i+1}, \ldots, x_i > \left[(i-n)\frac{\lambda_n}{\lambda_i}\right] x_n,$$

and by adding these, we obtain

$$(n-i)x_i > (i-n)\left(\frac{\lambda_{i+1}}{\lambda_i} x_{i+1} + \cdots + \frac{\lambda_n}{\lambda_i} x_n\right).$$

From this, by our hypotheses, $\lambda_1 x_1 + \cdots + \lambda_n x_n > 0$ immediately follows.

Conversely, if in the linear combination $\lambda_1 x_1 + \cdots + \lambda_n x_n$ the first nonzero coefficient is negative, multiplication by -1 leads back to the case just treated: $-\lambda_1 x_1 - \cdots - \lambda_n x_n > 0$, and consequently $\lambda_1 x_1 + \cdots + \lambda_n x_n < 0$.

The identity $0x_1 + \cdots + 0x_n = 0$ completes the proof of the lemma.

Theorem 3. 5. *Any ordered vector space of countable dimension is isomorphic to a discrete lexicographically ordered function space.*

PROOF. Let $b_1^*, \ldots, b_n^*, \ldots$ be a basis, consisting of pairwise inequivalent positive elements of the ordered vector space V of countable dimension; the existence of such a basis is stated in Lemma 3. 3. Let T denote the set of the elements $b_1^*, \ldots, b_n^*, \ldots$ with an ordering inverse to that in V. To any element x of V we assign a uniquely determined real valued function f defined on T: Let $f(b_n^*)$ be the coefficient of b_n^* in the decomposition of x in terms of the basis $b_1^*, \ldots, b_n^*, \ldots$. Clearly, the mapping $x \to f$ is an isomorphism of V onto the discrete lexicographically ordered function space defined over T; the monotonity of this mapping is a consequence of Lemma 3. 4.

Corollary 3. 6. *There exists (up to an isomorphism) a one-to-one correspondence between the countable order types α and the ordered vector spaces V of countable dimension. The α corresponding to V is the order type of the ordered set of its archimedean classes; the V corresponding to α is the discrete lexicographically ordered function space of type α.*

This follows immediately from Theorem 3. 5 and Lemma 2. 1.

Corollary 3. 7. *There exists an ordered vector space of countable dimension in which any other such space can be imbedded. The discrete lexicographically ordered function space defined over the ordered set of the rational numbers has this property.*

This is a simple consequence of Theorem 3. 5, in view of the fact that any countable ordered set is similar to some subset of the ordered set of all rationals.

§ 4. Ordered vector spaces of arbitrary dimension.

We shall see in § 5 that for ordered vector spaces of non-countable dimension Theorem 3. 5 does not hold in general. Our Theorem 4. 4 gives however some insight into the structure of these spaces.

First of all we present the method by which, as we shall show, any ordering (more exactly: the corresponding domain of positivity) of a vector space of arbitrary dimension can uniquely be constructed.

Let us consider a maximal chain of subspaces of an arbitrary vector space. If $H \subset H'$ are two members of this chain and one is contained as a maximal proper subspace in the other, then we single out one and only one of the two halfspaces of H' determined by H, and form the union of all such halfspaces. The procedure which leads us to this union will be called for the sake of brevity *construction* (A).

Lemma 4. 1. *Any subset of a vector space obtained by the construction* (A) *is the domain of positivity belonging to an ordering of the vector space.*

Proof. Let us suppose that the subset V^+ of the vector space V has been obtained by the construction (A). The maximal chain of subspaces yielded by the first step in this procedure will be denoted by $\mathfrak{K}$.

1. Let $x \in V^+$, $y \in V^+$. We show that in this case $x+y \in V^+$ also holds. $\mathfrak{K}$ has members H_1, H_1', H_2, H_2' such that H_1 is a maximal proper subspace in H_1', and similarly H_2 in H_2', and $x \notin H_1$, $x \in H_1'$, $y \notin H_2$, $y \in H_2'$.

If $H_1 = H_2$ then $H_1' = H_2'$. In this case x belongs to the same halfspace of H_1' determined by H_1 as y, i. e. $x = h + \lambda y$ $(h \in H_1;\ \lambda > 0)$. This implies $x+y = h + (\lambda+1)y$ and this proves that $x+y$ belongs to the same halfspace of H_1' as y, and thus $x+y \in V^+$.

Now, if $H_1 \neq H_2$, then we can suppose $H_1 \subset H_2$. From this $H_1' \subseteq H_2$ follows, and by $x \in H_1'$ this implies that $x+y$ belongs to the same halfspace determined by H_2 of H_2', as y, and therefore $x+y \in V^+$.

2. Let $x \in V^+$ and $\lambda > 0$. We show that in this case $\lambda x \in V^+$. $\mathfrak{K}$ has members H, H', such that H is a maximal proper subspace in H' and $x \notin H$, $x \in H'$. λx evidently belongs to the same halfspace of H' determined by H as x, and thus $\lambda x \in V^+$.

3. Clearly $0 \notin V^+$.

4. Let $x \neq 0$ be an element of V not contained in V^+. We show that in this case $-x \in V^+$. Let H denote the union of those members of the chain $\mathfrak{K}$ which do not contain x (the 0 is such a subspace), and let H' denote the intersection of those members of the chain $\mathfrak{K}$ which contain x (such is the whole space V). $\mathfrak{K}$ is a maximal chain of subspaces, and thus it contains together with any subspaces their union and their intersection too; thus $H \in \mathfrak{K}$ and $H' \in \mathfrak{K}$. Clearly $H \subset H'$. Moreover, if $H^* \in \mathfrak{K}$ is a subspace for which

$H \subseteq H^* \subseteq H'$, then $H^* = H$ or $H^* = H'$ according as $x \notin H^*$ or $x \in H^*$ holds. Thus $\mathfrak{K}$ has no member lying between H and H', and by the maximality of $\mathfrak{K}$ this implies that H is a maximal proper subspace in H'. Of the two halfspaces of H' determined by H the one containing x cannot be marked out because of $x \notin V^+$, therefore the other, namely the one containing $-x$ is a subset of V^+. Thus $-x \in V^+$.

Lemma 4.2.[4]) *The domain of positivity of any ordered vector space can be obtained by the construction* (A).

PROOF. First we show that the ideals of an ordered vector space V form a maximal chain of subspaces. Suppose the contrary, i. e. that V has a subspace S which is no ideal but can be compared with any ideal with respect to the subset-relation. The union J of the ideals of V not containing S and the intersection J' of the ideals containing S are also ideals and evidently $J \subset S \subset J'$. J is a maximal proper ideal in J', for if $J \subseteq J^* \subseteq J'$ holds for some ideal J^*, then in the case $S \supset J^*$ we have $J^* = J$ and for $S \subset J^*$ we have $J^* = J'$. Thus, by the isomorphism theorem the fact that of all ordered vector spaces only the space R has no non-trivial ideals, implies that J is a maximal proper subspace in J'. This contradicts however $J \subset S \subset J'$.

Let now the ideal H be a maximal proper subspace in the ideal H'. One of the two halfspaces of H' determined by H contains a positive element x, for any of these halfspaces is obtained from the other by multiplication by -1. All elements of the halfspace containing x are positive. As a matter of fact, in the case $h + \lambda x \leqq 0 (h \in H; \lambda > 0)$, $|\lambda x| = \lambda x \leqq -h = |-h|$ would imply $\lambda x \in H$, for H is an ideal; this however is impossible because of $x \notin H$. One sees also that all elements of the other halfspace must be negative.

Let us single out for any pair $H \subset H'$ of consecutive ideals the halfspace consisting of positive elements; the union of these halfspaces will be denoted by V^*. V^* contains all positive elements of V. Indeed, let $0 < x \in V$. As V^* has been obtained by the construction (A), in view of Lemma 4.1 either x or $-x$ is contained in V^*. $-x \in V^*$ is however impossible, for V^* contains only positive elements of V, and thus $x \in V^*$. We see that V^* is the set of positive elements of V, and this completes the proof of our assertion.

Lemma 4.3. *There is only one uniquely determined way of obtaining the domain of positivity of an ordered vector space by the construction* (A).

PROOF. Let us denote by $\mathfrak{K}$ the maximal chain of subspaces obtained in the course of constructing the domain of positivity of the ordered vector space V.

[4]) This lemma can be proved in a simpler way by making use of the main embedding theorem 3.1 of [2].

First we show that any member H of this chain $\mathfrak{K}$ is an ideal in V. Let $a \in H$ and suppose that for an element x of V not contained in H the inequality $|x| \leqq |a|$ holds. $\mathfrak{K}$ has members H_1 and H_1', such that H_1 is a maximal proper subspace of H_1', and $|x| \notin H_1$, $|x| \in H_1'$. We see that $H \subseteqq H_1$ for, in view of $x \notin H$, $H_1' \subseteq H$ cannot hold. Therefore $|x|$ and $|a| - |x|$ belong to the two different halfspaces of H_1' determined by H_1, and thus $|a| - |x| < 0$, in contradiction to our hypothesis. So $|x| \leqq |a|$ can hold only for elements x belonging to H, and this proves that H is an ideal in V.

From this it follows that $\mathfrak{K}$ is the set of all ideals of V. Namely, all ideals of V form a chain, of which the maximal chain of subspaces $\mathfrak{K}$ can form a part only if the two chains coincide. Thus $\mathfrak{K}$ is uniquely determined by V.

If $H \in \mathfrak{K}$, $H' \in \mathfrak{K}$ and H is a maximal proper subspace in H', then of the two halfspaces of H' determined by H one and only one is contained by the construction in the domain of positivity of V. Thus V uniquely determines the marked out halfspaces too.

Theorem 4. 4. *The domain of positivity belonging to any ordering of a vector space can be obtained by the construction* (A), *in one and only one way. The domains of positivity are the only sets obtainable by this construction.*

The proof of this theorem is given by Lemmas 4. 1—4. 3.

§ 5. Remarks and examples.

I. Theorem 3. 5 cannot be extended even to ordered vector spaces of dimension $2^{\aleph_0}$.

Example 5. 1. Let us consider the set of all sequences $(\lambda_1, \dots, \lambda_n, \dots)$ of real numbers. The addition of sequences and their multiplication by a real number shall be defined in the usual way, i. e. by termwise addition viz. multiplication. A sequence will be termed positive if and only if its first non-zero member is positive. The set of the archimedean classes of this ordered vector space V of dimension $2^{\aleph_0}$ has countable cardinality (the archimedean classes can be obtained in a similar way, as in § 2, III.). Thus if V were isomorphic to a discrete lexicographically ordered function space, then, by Lemma 2. 1 V would be a vector space of countably infinite dimension; this, however does not hold.

II. It is known that the finite dimensional vector lattices can be built up from R by forming direct and lexicographical products (see [1], Ch. XV., Theorem 1). In this connection (simply) ordered vector spaces and archimedean vector lattices are in a certain sense duals of each other, since they are obtained by using only lexicographical product, viz. only direct product.

Theorem 3.5 runs to the effect that any ordered vector space of countable dimension is isomorphic to a discrete lexicographical product of R's. Accordingly, one would expect that any archimedean vector lattice of countable dimension is isomorphic to a discrete direct product of R's. (A discrete direct product of R's is defined in an analogous way as a discrete lexicographically ordered function space, with the only difference that the ordering of the set T in § 2, III. is not taken into account, and a function $f(t)$ is considered to be $\geqq 0$ if and only if $f(t) \geqq 0$ for any $t \in T$.)

The following example disproves this conjecture: there exists an archimedean vector lattice of countably infinite dimension, not isomorphic to a discrete direct product of R's.

Example 5.2. Consider the set of sequences $(\lambda_1, \ldots, \lambda_n, \ldots)$ of real numbers in which all terms are equal, except for a finite number of them. Operations on these sequences will be defined in the usual way. Contrarily to the convention adapted in Example 5.1 we put that $(\lambda_1, \ldots, \lambda_n, \ldots) \geqq 0$ if and only if $\lambda_1 \geqq 0, \ldots, \lambda_n \geqq 0, \ldots$. Thus we obtain an archimedean vector lattice V. V is a vector space of countably infinite dimension, namely the sequences $(1, \ldots, 1, \ldots)$, $(1, 0, \ldots)$, $(0, 1, 0, \ldots)$, $(0, 0, 1, 0, \ldots)$ form a basis of V.

Suppose V to be isomorphic to a discrete direct product of R's over a set T. Then V must have a basis $b_1, \ldots, b_n, \ldots$ such that $x \geqq 0$ holds for an element $x \in V$ if and only if all its components relative to this basis are $\geqq 0$ (such a basis is given by the images of those functions $f(t)$, for which $f(t) = 1$ holds at a single point $t \in T$, and $f(t) = 0$ otherwise). This basis contains an element (we may suppose this to be the sequence $b_1 = (\lambda_1^*, \ldots, \lambda_n^* \ldots)$) with an infinity of nonzero members. Now, let $\lambda_n^* \neq 0$, i. e. because of $b_1 > 0$, $\lambda_n^* > 0$. In this case $b_1' = (0, \ldots, 0, \lambda_n^*, 0, \ldots) \in V$ is a positive multiple of a basis element b: $b_1' = \mu b$, as we infer without difficulty from our hypothesis. Considering now the element $a = b_1 - \mu b$, we get the following contradiction. $a > 0$, for the sequence b_1 has an infinity of nonzero members, whereas μb has but one such member, which is equal to the member of corresponding index in b_1. On the other hand, b_1 and b are two distinct basis elements, and therefore in the sequence of components of a relative to the basis $b_1, \ldots, b_n, \ldots$ there occurs also a negative number, namely $-\mu$. This contradicts our assumption.

III. If, in Theorems 3.5, 3.6, 4.4 we replace the ordered field of reals by another nonisomorphic ordered field, then these theorems cease to remain valid. This is an easy consequence of the following assertion:

If all two-dimensional ordered vector spaces over an ordered field F are isomorphic, then F is isomorphic to the ordered field of real numbers.

PROOF. Let S be an arbitrary proper subset of F, containing with each of its elements all smaller elements of F too. We define the set of positive elements of the vector space formed by the pairs (α_1, α_2) $(\alpha_1 \in F, \alpha_2 \in F)$ as follows:

for $\alpha_2 > 0$ $(\alpha_1, \alpha_2) > 0$ if and only if $\alpha_1 \alpha_2^{-1} \notin S$;

for $\alpha_2 < 0$ $(\alpha_1, \alpha_2) > 0$ if and only if $\alpha_1 \alpha_2^{-1} \in S$;

$(\alpha, 0) > 0$ if and only if $\alpha > 0$.

It is not difficult to verify that this set satisfies the requirements 1'.—4'. listed in § 2, I.

The ordered vector space thus obtained is by hypothesis isomorphic to the lexicographical product of F with itself. Thus it contains a 1-dimensional subspace, any positive element of which is incomparably smaller than any positive element not belonging to the subspace.

The set of pairs $(\alpha, 0)$ $(\alpha \in F)$ cannot be such a subspace. Namely in the contrary case, if we make also the additional requirement $(0, 1) > 0$, we have $(1, 0) \ll (0, 1)$, i. e. $(-\alpha, 1) > 0$ for any element $\alpha \in F$. This means that $-\alpha \notin S$, which is impossible. If, on the other hand, $(0, 1) < 0$, then a comparison with the pair $(0, -1) > 0$ leads to the contradiction $S = F$.

Thus there exists either a pair $(\alpha, 1) > 0$, or a pair $(\alpha, -1) > 0$, incomparably smaller than $(1, 0)$. Let $(\alpha, 1) > 0$; then $\alpha \notin S$. Consider now any element β of F, smaller than α. $(\alpha, 1) \ll (1, 0)$ implies $(\alpha - \beta)^{-1}(\alpha, 1) < (1, 0)$, and from this $\beta \in S$ follows. So α is a smallest one of the elements of F, not belonging to S. In case of $(\alpha, -1) > 0$ we see by a similar argument that $-\alpha$ is a greatest one among the elements of S.

Thus we have proved that S possesses a l. u. b., and consequently $F \cong R$.

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