Norm and hypernorm.

Dedicated to the memory of Tibor Szele.

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The norm $N(G)$ of a group G has been defined as the intersection of all the normalizers of subgroups of G. This is a characteristic subgroup of G; and it is clear that the center $Z(G)$ of G is contained in the norm $N(G)$. In general, center and norm will be different, though there exist interesting cases where these two characteristic subgroups are of necessity equal; see BAER [1], [2]. It is the aim of the present note to prove the following

Theorem: $Z(G)=1$ *implies* $N(G)=1$.

The particular interest of this result stems from the following remark. The hypercenter $H(G)$ of G could be defined as the intersection of all the normal subgroups X of G such that $Z(G/X)=1$ (and this is equivalent to defining the hypercenter as the terminal member of the ascending central chain). Similarly one would define the hypernorm of G as the intersection of all the normal subgroups X of G such that $N(G/X)=1$. Since the center is always part of the norm, the above theorem shows that $Z(G/X)=1$ if, and only if, $N(G/X)=1$; and our theorem consequently implies the equality of hypercenter and hypernorm thus improving upon a previous result of ours (see BAER [3], p. 425—426, in particular Theorem 4.19).

We turn now to the proof of our theorem; and we assume by way of contradiction the existence of a group G such that $Z(G)=1$, though $N(G)\neq 1$. We show first that

(1) $N(G)$ is an abelian torsion group.

For suppose that a and b are elements in $N(G)$ and that x is an element in G. Then there exist integers i and j such that $a^{-1}xa=x^i$ and $b^{-1}xb=x^j$. Consequently

$$(ab)^{-1}xab=x^{ji}=x^{ij}=(ba)^{-1}xba$$

so that the commutator $a^{-1}b^{-1}ab$ belongs to the center $Z(G)=1$. Hence $N(G)$ is abelian. It is a consequence of BAER [1; § 2, Satz 3] that center and norm coincide whenever the norm contains elements of order 0. But $Z(G)=1\neq N(G)$ so that $N(G)$ must be free of elements of order 0.

Since $N(G) \neq 1$ is an abelian torsion group (by (1)), there exists a prime p such that $N(G)$ contains elements of order p. The totality N of elements of order (1 or) p in $N(G)$ is a characteristic subgroup of $N(G)$ and hence of G. We note that N is abelian and that $N^p = 1 \neq N$. We denote by C the centralizer of N in G. Since N is a characteristic subgroup of G, so is C. Since N is abelian, N is part of C; and it is clear that G/C is essentially the same as the group Σ of automorphisms of N which are induced in N by elements in G. We show next:

(2) If $\sigma \neq 1$ is an automorphism in Σ, then $N^{1-\sigma}$ is a cyclic group of order p whose elements are fixed elements of σ.

It is clear that $N^{1-\sigma}$ is a subgroup, not 1, of the abelian group N, since $\sigma \neq 1$. Since σ belongs to Σ, there exists an element s in G which induces σ in N. If x is an element in N, then the norm element x belongs to the normalizer of the subgroup $\{s\}$ of G. It follows that the commutator $x^{-1}s^{-1}xs$ belongs to $\{s\}$ as well as to N. Hence

$$1 < N^{1-\sigma} \leqq N \cap \{s\}.$$

Consequently $N \cap \{s\}$ is a cyclic subgroup, not 1, of N. But $N^p = 1$ so that cyclic subgroups, not 1, of N have order p. Hence

$$N^{1-\sigma} = N \cap \{s\}$$

is a cyclic group of order p whose elements commute with s and are therefore fixed elements of σ.

If σ is an element in Σ, then we denote by $F(\sigma)$ the totality of fixed elements of σ in N. It is clear that $F(\sigma)$ is a subgroup of the abelian group N, that $F(\sigma) = N$ if, and only if, $\sigma = 1$, and that, by (2), $N^{1-\sigma} \leqq F(\sigma)$. Next we prove the following fact:

(3) If $\sigma \neq 1$ is an automorphism in Σ, then $[N:F(\sigma)] = p$.

There exists an element $t \neq 1$ in N which is not a fixed element of σ. Then $a = t^{\sigma-1}$ is an element of order p in $N^{\sigma-1}$. The latter subgroup of N is, by (2), cyclic of order p so that $N^{\sigma-1} = \{a\}$. If x is any element in N, then $x^{\sigma-1}$ belongs to $N^{\sigma-1} = \{a\}$; and there exists consequently an integer i such that $x^{\sigma-1} = a^i$. Let $y = xt^{-i}$. Then

$$y^\sigma = yx^{\sigma-1}t^{-i(\sigma-1)} = y$$

so that y belongs to $F(\sigma)$. It follows now without any difficulty that N is the direct product of $F(\sigma)$ and of the cyclic group $\{t\}$ of order p, proving (3).

(4) If σ' and σ'' are automorphisms, not 1, in Σ such that $F(\sigma') \neq F(\sigma'')$, then $N^{\sigma'-1} = N^{\sigma''-1}$.

Because of (3) it is impossible that $F(\sigma')$ is a proper subgroup of $F(\sigma'')$. Thus there exists an element a in $F(\sigma')$ that does not belong to $F(\sigma'')$; and similarly there exists an element b in $F(\sigma'')$ that does not belong to $F(\sigma')$. Let $a^* = a^{\sigma''-1}$ and $b^* = b^{\sigma'-1}$. Since a^* and b^* are different from 1, they are of order p; and this implies, by (2),

$$N^{\sigma'-1} = \{b^*\}, \quad N^{\sigma''-1} = \{a^*\}.$$

From

$$a^{\sigma'\sigma''} = a^{\sigma''} = aa^*,$$

$\sigma' \neq \sigma''^{-1}$ and (2) we deduce that

$$N^{\sigma'\sigma''-1} = \{a^*\} = N^{\sigma''-1};$$

and from

$$b^{\sigma'\sigma''} = (bb^*)^{\sigma''} = bb^{*\sigma''}$$

since b belongs to $F(\sigma'')$ and (2) we deduce that

$$N^{\sigma'\sigma''-1} = \{b^{*\sigma''}\} = N^{(\sigma'-1)\sigma''}$$

Since $N^{\sigma''-1} \leqq F(\sigma'')$, by (2), it follows that

$$N^{\sigma'-1} = N^{(\sigma'\sigma''-1)\sigma''^{-1}} = N^{(\sigma''-1)\sigma''^{-1}} = N^{\sigma''-1},$$

as we wanted to show.

We turn now to the derivation of the desired contradiction. By our choice of N this abelian group is different from 1. Since $Z(G) = 1$, there exists to every element $x \neq 1$ in N an automorphism σ in Σ such that $x \neq x^\sigma$; and this implies in particular that $\sigma \neq 1$.

Denote now by α some automorphism, not 1, in Σ. Since $F(\alpha)$ contains the cyclic group $N^{\alpha-1}$ of order p by (2), the order of $F(\alpha)$ is at least p; and since $[N:F(\alpha)] = p$ by (3), the order of N is at least p^2. The subgroup $F(\alpha)$ contains an element $t \neq 1$. By a previous remark there exists an automorphism $\beta \neq 1$ in Σ such that t does not belong to $F(\beta)$; and thus we have found two automorphisms α and β in Σ neither of which is 1, though $F(\alpha) \neq F(\beta)$. Application of (4) shows that

$$N^{\alpha-1} = N^{\beta-1} = M;$$

and it is a consequence of (2) that M is a cyclic subgroup of order p of N.

Suppose now that $\sigma \neq 1$ is an automorphism in Σ. Since $F(\alpha)$ and $F(\beta)$ are different, $F(\sigma)$ is different from at least one of these subgroups, say $F(\sigma) \neq F(\alpha)$. By (4) again we see that $N^{\sigma-1} = N^{\alpha-1} = M$; and thus we have shown that

$$M = N^{\sigma-1} \text{ for every } \sigma \neq 1 \text{ in } \Sigma.$$

But this implies, by (2), that M is part of every $F(\sigma)$; and thus we have shown that the cyclic group M of order p belongs to $Z(G) = 1$, the desired contradiction.

Bibliography.

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(Received October 4, 1955.)