On abelian groups with hereditarily generating systems.

Dedicated to the memory of Professor Tibor Szele.

By J. LOŚ, E. SĄSIADA in Torun and Z. SLOMIŃSKI in Sierpc.

A minimal generating system of an abelian group is a generating system, no element of which can be cancelled. There is no method of finding such a minimal system, even in case when it exists. Usually we start with an arbitrary generating system and by cancellation of its elements we try to obtain a minimal system. It may be shown by an easy example that this does not always lead to the proposed end. Let $p_1, p_2, \ldots$ be the sequence of all prime numbers and $m_1, m_2, \ldots$ a sequence of non-negative integers such that $m_i \neq 0$ for an infinite number of i. We form the discrete direct sum $C(m_1, m_2, \ldots) = \sum_{i=1}^{\infty} Cp_i^{m_i}$ of cyclic groups of order $p_i^{m_i}$. This group has certainly a minimal generating system, moreover, a basis. On the other hand, $C(n) = \sum_{i=1}^{n} Cp_i^{m_i}$ is a cyclic group of order $p_1^{m_1} \cdot \ldots \cdot p_n^{m_n}$, and $C(m_1, m_2, \ldots)$ is the union of all groups $C(n)$: $C(m_1, m_2, \ldots) = \bigcup_{n=1}^{\infty} C(n)$. Let a_n be a generator of the group $C(n)$; then $(a_1, a_2 \ldots)$ is a system of generators for $C(m_1, m_2 \ldots)$, but, as $C(n)$ is contained in every $C(n')$ with $n' > n$, every infinite subsystem of this system is also a generating system for $C(m_1, m_2, \ldots)$. This example directs our attention to generating systems in which every subsystem of the same power is also a generating system. Such systems will be called *hereditarily generating systems* (h. g. s.).[1])

The aim of this paper is to give a description of abelian groups which have a h. g. s. This problem is not completely solved. First we state (Theorem 1) that every group whith a h. g. s. must be countable. Next we give a necessary criterion (Lemma 3) for groups with h. g. s. and a complete solu-

[1]) Prof. VL. KOŘINEK has kindly drawn our attention to the problem of characterising groups with special generating systems. The problems discussed in this paper are due to his suggestion.

tion of the problem for torsion groups (Theorem 2). At last we state that every group with h. g. s. is of finite rank and that each divisible group of finite rank has a h. g. s. (Theorems 3 and 4). Some problems are proposed at the end of this paper.

§ 1. Terminology and notations.

We deal only with additively written abelian groups. C_n, C_{p^∞} and R^+ denote the cyclic group of order n, the generalized cyclic group of PRÜFER (p is a prime) and the additive group of rationals, respectively. The (discrete) direct sum of the groups G_ν is denoted by $\sum_\nu G_\nu$, for two groups we also write $G_1 + G_2$.

For set-theoretical operations we use $\bigcup_i X_i$ (union of the sets X_i) and $\bigcap_i X_i$ (intersection of the sets X_i).

By a generating system of a group G we understand a system (a_t) $t \in T$ of elements of the group G, such that every element of G is a linear combination of some elements a_t. A countable system is simply a sequence $a_1, a_2, \dots$. We want to point out that the elements of a generating system must not be different, in others words: $t_1 \neq t_2$ does not necessarily imply $a_{t_1} \neq a_{t_2}$. The power of the system (a_t) $t \in T$ is the power $\overline{\overline{T}}$ of the set T. Every subset T_0 of T yields a subsystem (a_t) $t \in T_0$.

§ 2. Power of the groups with h. g. s.

Theorem 1. *Every group with h. g. s. is of the power* $\leqq \aleph_0$.

PROOF. Suppose that (a_t) $t \in T$ is a h. g. s. for G and that $\overline{\overline{G}} = \mathfrak{m} > \aleph_0$. Then $\overline{\overline{T}} \geqq \mathfrak{m}$. Let A be the least divisible[2]) group which contains G[3]). We have $\overline{\overline{A}} > \aleph_0$ and therefore A may be represented in the form $A = A_1 + A_2 + A_3$, where A_2 and A_3 are countable divisible groups. The factor group A/A_1 is countable and therefore there exists a $g \in A_2 + A_3$ such that the set T_0 of the t's with $a_t \in g + A_1$ is of the power $\overline{\overline{T}}$. Let A' be one of the least divisible subgroups of $A_2 + A_3$ which contain g; we have $A_2 + A_3 = A' + A''$ with $A'' \neq \{0\}$, and therefore $A = A_1 + A' + A''$. The system (a_t) $t \in T_0$ is again a generating system of G and we see that each element of it belong to $A_1 + A'$, therefore $G \subset A_1 + A'$ which contradicts the supposition that A is the least divisible group containing G.

From Theorem 1 it follows that our consideration may be restricted to the hereditarily generating sequences (h. g. s.).

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[2]) The group G is divisible if $nG = G$ for all natural numbers n.

[3]) For the existence of such a unique group we refer to T. SZELE [2]. For theorems on divisible groups we use, see e. g. I. KAPLANSKY [1] pp. 7—12.

§ 3. Lemmas.

A few lemmas are needed.

LEMMA 1. *Every homomorphic image of a group with h. g. s. is also a group with h. g. s.*

The lemma is obvious. If (g_n) is a h. g. s. of G, then $(h(g_n))$ is a h. g. s. of $h(G)$.

LEMMA 2. *A finite group has a h. g. s. if and only if it is cyclic.*

If $G=\{a\}$ is a cyclic group then the sequence (a_n), with $a_n=a$ is a h. g. s. of G.

Let (a_n) be a h. g. s. of G and suppose G to be finite. Then there exists a subsequence (a_{k_n}) of (a_n) with $a_{k_i}=a_{k_j}=a$. This sequence is a generating system of G and therefore $G=\{a\}$.

LEMMA 3. *If G has a finite homomorphic image which is not a cyclic group, then G has no h. g. s.*

It follows from Lemma 1 and 2.

This lemma gives a criterion for groups without h. g. s. It follows

LEMMA 4. *A finitely generated group has a h. g. s. if and only if it is cyclic.*

Evidently, a finitely generated group which is not cyclic has a homomorphic image isomorphic with C_p+C_p.

LEMMA 5. *A group which is the union of an increasing sequence of cyclic groups has a h. g. s.*

If $G=\bigcup_{n=1}^{\infty}\{a_n\}$ then (a_n) is a h. g. s. of G.

LEMMA 6. *The groups $C(m_1, m_2, \ldots)=\sum_{n=1}^{\infty} C_{p_n}^{m_n}$, C_{p^∞} and the torsion free groups of rank one, are groups with h. g. s.*

This follows from Lemma 5.

§ 4. Torsion groups with h. g. s.

LEMMA 7. *A torsion group has a h. g. s. if and only if every primary summand of it has a h. g. s.*

PROOF. Let $T=T_{p_1}+T_{p_2}+\cdots$ be the decomposition of the torsion group T into primary summands. It follows from Lemma 1 that if T has a h. g. s., then every T_{p_i} has also a h. g. s. Let $(a_1^{(i)}, a_2^{(i)}, \ldots)$ now be a h. g. s. of T_{p_i}. It will be shown that the sequence $a_n=a_n^{(1)}+a_n^{(2)}+\cdots+a_n^{(n)}$ is a h. g. s. of T. In fact, let (a_{k_n}) be a subsequence of (a_n). For every $a_{k_n}=a_{k_n}^{(1)}+a_{k_n}^{(2)}+\cdots+a_{k_n}^{(k_n)}=a_{k_n}^{(1)}+b_{k_n}$ the orders of the elements $a_{k_n}^{(1)}$ and $b_{k_n}=a_{k_n}^{(2)}+\cdots+a_{k_n}^{(k_n)}$ are relatively prime and therefore there exists integers k and l such that $k\cdot O(a_{k_n}^{(1)})+l\cdot O(b_{k_n})=1$. We have: $a_{k_n}^{(1)}=k\cdot O(a_{k_n}^{(1)})\,a_{k_n}^{(1)}+l\cdot O(b_{k_n})\,a_{k_n}^{(1)}=l\cdot O(b_{k_n})\,a_{k_n}^{(1)}$; $O(b_{k_n})\,a_{k_n}=O(b_{k_n})\,a_{k_n}^{(1)}$ yields $a_{k_n}^{(1)}=l\cdot O(b_{k_n})a_{k_n}$. Therefore each element of $a_{k_1}^{(1)}, a_{k_2}^{(1)}, \ldots$ belongs to the group generated by the sequence (a_{k_n}), and, as $(a_i^{(1)})$ is a h. g. s. of T_{p_1}, the whole group T_{p_2} is contained in the group generated by (a_{k_n}). Analogously we prove that each T_{p_i} is contained in the group generated by (a_{k_n}) and so (a_{k_n}) is a generating system of T. This proves that (a_n) is a h. g. s. of T.

LEMMA 8. *A countable (or finite) primary p-group T_p has a h. g. s. if and only if it is of the form $T_p=T_p'+T_p''$, where T_p' is a cyclic group or $T_p'=\{0\}$ and T_p'' is a divisible group.*

PROOF. In view of Lemma 3 it remains to show that each group of the mentioned form has a h. g. s. As it is known, a countable divisible p-group is of the form $T_p''=\sum_\nu C_{p^\infty}^{(\nu)}$, where ν runs over all natural numbers or $\nu=1,2,\ldots,m$. It is evident that we can assume (in view of Lemma 1) that ν runs over all natural numbers. Let $C_{p^\infty}^{(\nu)}=\{c_1^{(\nu)},c_2^{(\nu)},\ldots\}$ where $pc_{i+1}^{(\nu)}=c_i^{(\nu)}$ for $i,\nu=1,2,\ldots$ $(c_0^{(\nu)}=0)$ and let $T_p'=C_{p^s}=\{c\}$. We have $O(c_i^{(\nu)})=p^i$ and $O(c)=p^s$. The sequence $(c_1^{(\nu)},c_2^{(\nu)},\ldots)$ is a h. g. s. of $C_{p^\infty}^{(\nu)}$. We put $a_n=c+$ $+c_{2^n}^{(1)}+c_{2^{n-1}}^{(2)}+\cdots+c_2^{(n)}+c_1^{(n+1)}$ and show that (a_n) is a h. g. s. of T_p. Let (a_{k_n}) be a subsequence of (a_n) and let $a\in C_{p^\infty}^{(1)}$. We choose a k_i such that $k_i\geqq s$ and $2^{k_i}>O(a)$. Then $b=2^{k_i-1}a_{k_i}=2^{k_i-1}c_{2^{k_i}}^{(1)}\in C_{p^\infty}^{(1)}$ and $O(b)\geqq O(a)$. Therefore $a\in\{b\}$ and so we have demonstrated that $C_{p^\infty}^{(1)}$ is contained in the group generated by the system (a_{k_i}). Now, applying the same method to the sequence $a'_{k_i}=a_{k_i}-c_{2^{k_i}}^{(1)}$ we obtain that $C_{p^\infty}^{(2)}$ is contained in the group generated by (a'_{k_i}), etc. We conclude that the whole group T_p'' is contained in the group generated by (a_{k_n}) and as $c=a_{k_1}-(c_{2^{k_1}}^{(1)}+\cdots+c_1^{(k_1-1)})$, it follows that $T_p'=\{c\}$ is contained also in this group.

From Lemmas 7 and 8 it follows:

Theorem 2. *A torsion group T is a group with h. g. s. if and only if*

$$T=D+C(m_1,m_2,\ldots)$$

holds, where D is a divisible countable group (or $D=\{0\}$) and $m_1,m_2,\ldots$ is an arbitrary sequence of non-negative integers.

It is easy to see that these are the only groups, which have no finite, non-cyclic homomorphic images. Thus for torsion groups the condition of our Lemma 3 is not only necessary but also sufficient.

§ 5. Some results on torsion-free groups with h. g. s.

Theorem 3. *Every torsion-free group with h. g. s. is of finite rank.*

PROOF. Let G be a torsion-free group of infinite rank and let A be the east divisible group which contains G. It follows that

$$G \subset A = R_1^+ + R_2^+ + \cdots.$$

Suppose that (a_k) is a h. g. s. of G. Each a_i may be written in the from $a_i = a_1^{(i)} + \cdots + a_{n_i}^{(i)}$, where $a_j^{(i)} \in R_j^+$ and $a_{n_i}^{(i)} \neq 0$. Let (a_{k_i}) be a subsequence of (a_k) such that $n_1 < n_{k_1} < n_{k_2} < \ldots$ It is easy to see that no non-zero linear combination of some a_{k_i} belong to $R_1^+ + \cdots + R_{n_1}^+$ and therefore a_1 does not belong to the group generated by (a_{k_i}). So (a_k) is not a h. g. s. of G which contradicts the assumption.

Theorem 4. *Each divisible, torsion-free group of finite rank has a h. g. s.*

PROOF. As it is known, these are the groups of the form $G_n = \sum_{i=1}^{n} R_i^+$ (direct sums of n groups R^+). We proceed by induction on n. First we represent the elements g in G_n as sequences of rational numbers: $g = \langle r_1, r_2, \ldots, r_n \rangle$ and then we prove that $g_k^{(n)} = \langle \frac{1}{k!}, \frac{1}{k!^2}, \ldots, \frac{1}{k!^n} \rangle$ is a h. g. s. of G_n. Obviously this is true for $n=1$. Suppose that it is true for $n-1 \geqq 1$ and let $g_{k_i}^{(n)}$ be a subsequence of $(g_k^{(n)})$. It is easy to see that an element $\alpha\langle 1, 0, \ldots, 0 \rangle$, where α is an integer different from zero can be obtained as a linear combination of some $g_{k_i}^{(n)}$'s. We put $a_{k_i} = \alpha \cdot k_i! \, g_{k_i}^{(n)} - \alpha \langle 1, 0, \ldots, 0 \rangle = \langle 0, \frac{\alpha}{k_i!}, \ldots, \frac{\alpha}{k_i!^{n-1}} \rangle$. It follows from the assumption that $b_{k_i} = \langle 0, \frac{1}{k_i!}, \ldots, \frac{1}{k_i!^{n-1}} \rangle$ is a generating system of $R_2^+ + \cdots + R_n^+ = \alpha(R_2^+ + \cdots + R_n^+)$, therefore (a_{k_i}) is also a generating system of this group. As $g_{k_i}^{(n)} - b_{k_i} = \langle \frac{1}{k_i!}, 0, \ldots, 0 \rangle$ it may be concluded that $(g_{k_i}^{(n)})$ is a generating system of the whole group G_n.

§ 6. Remarks and problems.

As it has been shown, condition of Lemma 3, i. e. then non-existence of a non-cyclic homomorphic image, is a sufficient condition for the existence of a h. g. s. for countable torsion groups. For torsion-free groups this condition is not sufficient, the group must be of finite rank. The question: „is the condtion of lemma 3 sufficient for torsion-free groups of finite rank?" is not yet solved. Z. SŁOMINSKI has shown [4]) that it is sufficient also for fully

[4]) The demonstration will appear in a separate paper.

decomposable torsion-free groups of finite rank (i. e. for the direct sum of a finite number of subgroups of R^+). This is obviously a generalization of our Theorem 4.

For mixed groups the following problem may be proposed: Let T be the torsion group of G and suppose that G/T and T both have h. g. s. Has G under this assumption also a h. g. s.?

Bibliography.

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