Remark on a paper of S. Gacsályi.

Dedicated to the memory of Professor Tibor Szele.

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In his paper "On pure subgroups and direct summands of abelian groups"[1]) S. GACSÁLYI proved the following theorem (Theorem 2):

A subgroup A of an abelian group G is a direct summand of G if an only if every system of linear equations[2]) *over A solvable in G possesses also a solution in A.*

S. GACSÁLYI remarked, that "for A to be a direct summand of G it is sufficient that the above requirement be satisfied for arbitrary systems of equations in a set of unknowns of a cardinality not greater than that of the group G".

In this note I give a simple proof of the theorem of GACSÁLYI. It may be remarked that from this proof it follows that the cardinality of the unknowns in consideration can be restricted to the cardinality of the factorgroup G/A and, moreover, that it is sufficient to consider equations of a very special form.

It is sufficient to prove the sufficiency of the condition. Let $C=\{g_\lambda\}$ for $\lambda \in L$ be the set of elements of G, containing one and only one element from each coset $g+A$ for $g \in G$. The cardinality of the set L is equal to that of G/A. For each pair of indices λ, μ of L there exists one and only one index $\nu \in L$ satisfying the relation

$$(g_\lambda+A)-(g_\mu+A)=g_\nu+A;$$

let $\lambda * \mu = \nu$, and put

$$a_{\lambda\mu}=g_\lambda-g_\mu-g_{\lambda*\mu}.$$

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1) *Publ. Math. Debrecen* 4 (1955), 89—92.

2) *A* system of linear equations over *A* is a set of equations of the form

$$m_1x_1+\cdots+m_kx_k=a\in A$$

where the $m_1,\ldots,m_k$ are integers and the cardinality of the equations and that of the unknowns x is arbitrary.

The elements $a_{\lambda\mu}$ belong to the group A. The following system of equations over A

(1) $$x_\lambda - x_\mu - x_{\lambda * \mu} = a_{\lambda\mu} \quad \text{for} \quad \lambda, \mu \in L$$

possesses the solution

$$x_\lambda = g_\lambda$$

in the group G. Consequently, the system (1) admits a solution

$$x_\lambda = a_\lambda$$

in the group A. We have

$$g_\lambda - g_\mu - g_{\lambda * \mu} = a_\lambda - a_\mu - a_{\lambda * \mu}$$

and thus

$$(g_\lambda - a_\lambda) - (g_\mu - a_\mu) = g_{\lambda * \mu} - a_{\lambda * \mu}.$$

The set $B = \{g_\lambda - a_\lambda\}$ for $\lambda \in L$ is a subgroup of G and in each coset $g + A$ there exists one and only one element of B; the group G is therefore the direct sum of the groups A and B.

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