On a problem in set theory.

To the memory of Professor Tibor Szele.

By G. FODOR in Szeged.

Let E be a given non countable set of power $\mathfrak{m}$ and suppose that there exists a relation R between the elements of E such that for any $x \in E$, the power of the set $H(x)$ of elements $y \in E\,(y \neq x)$ for which xRy holds, is smaller than a given cardinal number $\mathfrak{n}$ which is smaller than $\mathfrak{m}$. Two distinct elements x and y of E are called *independent* if neither xRy nor yRx holds. We say that a subset of E is a *free* set if any two points of this subset are independent. Let $Z(x)$ denote, for every $x \in E$, the set of $y \in E$ $(y \neq x)$ for which yRx holds; further, let $Z[F]$ denote, for every $F \subseteq E$ the set $\bigcup_{x \in F} Z(x)$.

The following problem is due to RUZIEWICZ [1]:

Does there always exist a free subset of power $\mathfrak{m}$ *of* E?

The answer to this problem is affirmative if $\mathfrak{n} = \aleph_0$ and $\mathfrak{m}$ is either of the form $2^{\mathfrak{q}}$ or of the form $\aleph_{\alpha+1}$ [2], [3], and also if $\mathfrak{m}$ is a regular cardinal number or if $\mathfrak{m}$ is a countable sum of cardinals smaller than $\mathfrak{m}$ [4], [5], finally, in the general case, assuming the generalized continuum hypothesis [6].

We prove the following

Theorem. *If* $\mathfrak{m}$ *is a singular cardinal number and if for every* $x \in E$ *the power of the set* $Z(x) = \{y \in E : yRx\}$ *is smaller than* $\mathfrak{m}$, *then the answer to the problem is affirmative.*

PROOF.[1]) Let φ be the initial number of $\mathfrak{m}$ and φ^* the smallest ordinal number ψ for which φ is confinal to ψ. There exist regular cardinal numbers $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_\xi, \ldots$ $(\xi < \varphi^*)$ such that $\max\{\mathfrak{n}, \bar{\varphi}^*\} < \mathfrak{m}_\xi < \mathfrak{m}$ $(\xi < \varphi^*)$, $\mathfrak{m}_\alpha > \mathfrak{m}_\beta$ for $\alpha > \beta$ and

$$\mathfrak{m} = \sum_{\xi < \varphi^*} \mathfrak{m}_\xi .$$

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1) The proof is similar to the proof of a theorem of DUSHNIK, MILLER and ERDŐS. ([7] p. 606, Theorem 5,22.)

Define the free sets N_η $(\eta < \varphi^*)$ by transfinite induction as follows. Let M_1 be an arbitrary subset of power $\mathfrak{m}_1$ of E. Since $\mathfrak{m}_1$ is regular and $\mathfrak{n} < \mathfrak{m}_1$, by the result of S. PICCARD [4], M_1 contains a free subset G_1 of power $\mathfrak{m}_1$. Let $G_{1\xi}$ be the set of elements x of G_1 for which $\overline{\overline{Z(x)}} \leqq \mathfrak{m}_\xi$. It is obvious that

$$G_1 = \bigcup_{\xi < \varphi^*} G_{1\xi}.$$

Since $\mathfrak{m}_1$ is regular and $\bar{\varphi}^* < \mathfrak{m}_1$, there exists therefore an ordinal number $\xi_1 < \varphi^*$ such that the power of $G_{1\xi_1}$ is $\mathfrak{m}_1$. It is obvious that the power of the set

$$Z[G_{1\xi_1}] = \bigcup_{x \in G_{1\xi_1}} Z(x)$$

is not greater than $\mathfrak{m}_1 \cdot \mathfrak{m}_{\xi_1}$. Put

$$N_1 = G_{1\xi_1}.$$

Let α be a given ordinal number $1 < \alpha < \varphi^*$ and suppose that the free set N_ξ is defined for every $1 \leqq \xi < \alpha$ such that $\overline{\overline{N_\xi}} = \mathfrak{m}_\xi$ and $\overline{\overline{Z[N_\xi]}} < \mathfrak{m}$. Since $\alpha < \varphi^*$, we have that the power of the set

$$E_\alpha = E - (\bigcup_{\xi < \alpha} (N_\xi \cup Z[N_\xi]),$$

is $\mathfrak{m}$. Let M_α be an arbitrary subset of power $\mathfrak{m}_\alpha$ of E_α. Let now N_α be defined starting from M_α in the same way as N_1 is defined starting from the set M_1.

Put

$$B_\xi = N_\xi - \bigcup_{\gamma < \xi} (\bigcup_{x \in N_\gamma} H(x)) \qquad (\xi < \varphi^*).$$

By the hypothesis $\overline{\overline{H(x)}} < \mathfrak{n}$ and $\max\{\mathfrak{n}, \overline{\varphi}^*\} < \mathfrak{m}_\eta$ for every $\eta < \varphi^*$. It follows that

$$\overline{\overline{\bigcup_{x \in N_\gamma} H(x)}} \leqq \mathfrak{n} \cdot \mathfrak{m}_\gamma = \mathfrak{m}_\gamma.$$

Since $\mathfrak{m}_\gamma < \mathfrak{m}_\xi$ $(\gamma < \xi)$ and $\bar{\xi} < \bar{\varphi}^* < \mathfrak{m}_\xi$, we have by the regularity of $\mathfrak{m}_\xi$ that

$$\overline{\overline{\bigcup_{\gamma < \xi} (\bigcup_{x \in N_\gamma} H(x))}} < \mathfrak{m}_\xi.$$

Consequently $\overline{\overline{B_\xi}} = \mathfrak{m}_\xi$. Let

$$B = \bigcup_{\xi < \varphi^*} B_\xi.$$

It is obvious that $\overline{\overline{B}} = \mathfrak{m}$.

Now we prove that B is a free set. Indeed, let y and z be two distinct elements of B. If $y \in B_\alpha$ and $z \in B_\alpha$ for some $\alpha < \varphi^*$, then it follows, by the definition of N_α and the relation $B_\alpha \subseteqq N_\alpha$, that y and z are independent. Let now $y \in B_\alpha$ and $z \in B_\sigma$ $(\alpha \neq \sigma)$. Obviously we may suppose that $\alpha < \sigma$. It follows from

$$B_\sigma \subseteqq E_\sigma = E - \bigcup_{\xi < \sigma} (N_\xi \cup Z[N_\xi])$$

that z non Ry. On the other hand, we have by the equality

$$B_\sigma = N_\sigma - \bigcup_{\xi<\sigma} \left(\bigcup_{x \in N_\xi} H(x) \right)$$

that y non Rz. Thus the set B is free, and so our theorem is proved.

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