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On the sublattice-lattices of lattices

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Abstract. Let L and K be lattices with $\mathrm{Sub}(L) \cong \mathrm{Sub}(K)$. We prove the following statements: (I) If L is simple then $L \cong K$ or $L \cong K^d$; (II) If L satisfies a self-dual infinitary Horn sentence ψ stronger then the modular identity, then ψ holds in K. A corollary of (II) is: A modular lattice variety is closed under isomorphisms of sublattice-lattices if and only if it is self-dual.

1. Introduction

For a lattice L, let $\mathrm{Sub}(L)$ denote the lattice of all sublattices of L, and L^d the dual of L. Sublattice lattices were investigated by several authors, either from the algebraic or from the combinatoric point of view. For an overview and a comprehensive bibliography see Koh [4].

Let L and K be lattices. If $L \cong K$ or $L \cong K^d$ then $\mathrm{Sub}(L) \cong \mathrm{Sub}(K)$. The converse is not true in general. Thus, our first problem is (see Grätzer [3], p. 56, Problem I.4): Find conditions under which $\mathrm{Sub}(L)$ determines L up to isomorphism (or dual isomorphism). Our second problem (see Grätzer [3], p. 56, Problem I.8) is: Which properties of a lattice L are preserved in K under the isomorphism $\mathrm{Sub}(L) \cong \mathrm{Sub}(K)$?

Filippov [2] provided sufficient and necessary conditions on L and K (these conditions are, however, complicated) for $\mathrm{Sub}(L) \cong \mathrm{Sub}(K)$, and

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from this he deduced that modularity, distributivity and complementedness are preserved under isomorphisms of sublattice-lattices. Moreover, he showed that relatively complemented, uniquely complemented and ordinal-sum indecomposable modular lattices are determined by their sublattice-lattices. Chen, Koh and Teo [1], decomposing lattices into ordinal and delta-sums, proved the following results: Let L and K be lattices with $\mathrm{Sub}(L) \cong \mathrm{Sub}(K)$. (I) If L is sectionally complemented or L is an ordinal-sum indecomposable semimodular lattice of finite height then $L \cong K$ or $L \cong K^d$; (II) If L is a pseudocomplemented lattice of finite height then K or K^d is a pseudocomplemented lattice of finite height. The results (I) of [1] were recently sharpened by the author [5]: Let L and K be lattices with $\mathrm{Sub}(L) \cong \mathrm{Sub}(K)$. If L is weakly complemented, or L is semimodular, strongly atomic and ordinal-sum indecomposable, then $L \cong K$ or $L \cong K^d$. A new proof of Filippov's result on ordinal-sum indecomposable modular lattices was given too.

In this paper we shall prove that every self-dual infinitary Horn sentence which is stronger then the modular identity is preserved under isomorphisms of sublattice-lattices. As our first problem, using the method described in [5] we show that simple lattices are determined by their sublattice-lattices.

2. Preliminaries

In this section we review the basic definitions, notations and recall the results of [2] and [5] used in the proofs.

Let L be a lattice, $a, b \in L$, $X, Y \subseteq L$. We write $a\sigma b$ if a is comparable with b, and $a\overline{\sigma}b$ otherwise. Let $\alpha \in \{<, >, \sigma, \overline{\sigma}\}$. We write $X\alpha Y$ if $x\alpha y$ holds for every $x \in X, y \in Y$. We define $X_\alpha(a) = \{x \in X \setminus \{a\} \mid x\alpha a\}$ and $X_\alpha(Y) = \bigcap_{y\in Y} X_\alpha(y)$.

A sublattice H of L is called *prime* if $L \setminus H$ is also a sublattice and *homogeneous* if $L \setminus H = L_\sigma(H) \cup L_{\overline{\sigma}}(H)$, i.e. for any $x \in L \setminus H$ we have either $x\sigma H$ or $x\overline{\sigma}H$. Let H be a homogeneous sublattice of L. We shall use the notation $M = M(H) = \{x \in L_\sigma(H) \mid h_1 < x < h_2$ for some $h_1, h_2 \in H\}$.

Now we define the concept of the *ordinal-sum* of lattices. Let I be a chain and for every $i \in I$ let A_i be a lattice. Consider the disjoint union of the A_i's and let $x \leq y$ mean that $x \in A_i$, $y \in A_j$ and $i < j$, or $x, y \in A_i$

and $x \leq y$. We call the lattice obtained the ordinal-sum of the A_i's and denote it by $\bigoplus_{i\in I} A_i$. For $I = C_2$ (i.e., the two element chain) we use the notation $A_0 \oplus A_1$. A lattice A is said to be ordinal-sum decomposable if $A = A_0 \oplus A_1$ for some lattices A_0 and A_1, and ordinal-sum indecomposable otherwise.

Assume that A_0 and A_1 are bounded lattices. The *delta-sum* $A_0 \Delta A_1$ is the lattice obtained from $A_0 \oplus A_1$ by identifying 1_{A_0} and 0_{A_1}.

We shall make use of the following earlier results:

Lemma 1 ([2]). *If H is a homogeneous sublattice of L and $x \in L_{\overline{\sigma}}(H)$, then $x \vee y = x \vee z$ and $x \wedge y = x \wedge z$ for any $y, z \in H$.*

Lemma 2 ([2]). *If* $\mathrm{Sub}(L) \cong \mathrm{Sub}(K)$ *and L has no proper*[1] *homogeneous prime sublattices, then $L \cong K$ or $L \cong K^d$.*

Lemma 3 ([5]). *Let L be an ordinal-sum indecomposable lattice and H a proper homogeneous prime sublattice of L. Then $L_{\overline{\sigma}}(H) \neq \emptyset$.*

By an infinitary Horn sentence we mean a formula

$$\psi : \qquad \bigwedge_{i\in I} p_i = q_i \implies p = q,$$

where p_i, q_i, p, q are lattice terms. The equation $p = q$ is said to be the conclusion of ψ.

3. Simple lattices

This section is devoted to proving the following theorem:

Theorem 4. *Let L and K be lattices with* $\mathrm{Sub}(L) \cong \mathrm{Sub}(K)$. *If L is simple then $L \cong K$ or $L \cong K^d$.*

First we need two lemmata on the structure of lattices having homogeneous prime sublattices. In these lemmata we write shortly “sublattice” for “sublattice or empty”, and we use these “sublattices” freely as ordinal summands.

[1]A sublattice H of L is called proper if $H \neq L$ and $|H| > 1$.

Lemma 5. *Let L be a lattice and $H(\neq \emptyset)$ a homogeneous prime sublattice of L. Then $L_\sigma(H)$, $L_<(H)$, $L_>(H)$ and M are sublattices, and $L_\sigma(H) = L_<(H) \oplus M \oplus L_>(H)$.*

Proof. $L_<(H), L_>(H)$ and M are sublattices by the definition and since H is prime. Clearly $L_\sigma(H) = L_<(H) \cup M \cup L_>(H)$ and $L_<(H) < M < L_>(H)$, proving the last statement, which trivially implies that $L_\sigma(H)$ is a sublattice. $\square$

Lemma 6. *Let L be a lattice and $H(\neq \emptyset)$ a homogeneous prime sublattice of L. Then $H' = H \cup M$ is also a homogeneous prime sublattice of L and $M(H') = \emptyset$.*

Proof. Let $a \in H$ and $b \in M$. Then there exist elements $c, d \in H$ such that $c < b < d$, and so $a \wedge c \leq a \wedge b < a \vee b \leq a \vee d$. As H and M are sublattices, this means that H' is sublattice. By the transitivity of $<$ we have $L_<(H) < H' < L_>(H)$ and $L_{\overline{\sigma}}(H)\overline{\sigma}H'$. Hence

$$L_<(H') = L_<(H), \quad L_>(H') = L_>(H) \quad \text{and} \quad L_{\overline{\sigma}}(H') = L_{\overline{\sigma}}(H),$$

therefore H' is homogeneous and $M(H') = \emptyset$. To prove that H' is prime, it is enough to show that for $a, b \in L \setminus H'$, $a \vee b, a \wedge b \notin M$, as H is prime. Since $\{a \vee b, a \wedge b\}\sigma\{a, b\}$ and $<$ is transitive, neither $a \vee b \in M$ nor $a \wedge b \in M$ will hold if a or b is an element of $L_{\overline{\sigma}}(H)$. But if $a, b \in L_<(H) \cup L_>(H)$ then by $L_\sigma(H) = L_<(H) \oplus M \oplus L_>(H)$ we obtain $a \vee b, a \wedge b \notin M$. $\square$

Proof of Theorem 4. Suppose indirectly that L is simple but neither $L \cong K$ nor $L \cong K^d$. Then by Lemma 2 there exists a proper homogeneous prime sublattice H of L. Consider $H' = H \cup M(H)$ and define a binary relation θ on L:

$$x\theta y \iff x, y \in H' \quad \text{or} \quad x = y.$$

We claim that θ is a nontrivial congruence. θ is clearly an equivalence. To show the substitution property, assume that $a\theta b$, $a \neq b$ and $c \in L$. We have to prove that $(a \vee c)\theta(b \vee c)$ and $(a \wedge c)\theta(b \wedge c)$. The first two assumptions imply that $a, b \in H'$. If $c \in H'$ then $a \vee c, b \vee c \in H'$. If $c \in L_<(H')$ then $a \vee c = a, b \vee c = b \in H'$ and if $c \in L_>(H')$ then $a \vee c = c = b \vee c$. Finally, if $c \in L_{\overline{\sigma}}(H')$ then Lemma 1 implies that $a \vee c = b \vee c$. A similar argument holds for $\wedge$, and so θ is a congruence.

As any simple lattice is ordinal-sum indecomposable, we can use Lemma 3, which together with Lemma 6 gives $L_{\overline{\sigma}}(H') = L_{\overline{\sigma}}(H) \neq \emptyset$. Thus H' is a proper sublattice, and since it is a congruence class of θ, θ is a nontrivial congruence, a contradiction. $\square$

4. Horn sentences in ordinal-sums of lattices

The result that ordinal-sum indecomposable modular lattices are determined by their sublattice lattices was presented originally in the following way:

Theorem ([2]). *Let L and K be lattices and suppose that L is modular. Then $\mathrm{Sub}(L) \cong \mathrm{Sub}(K)$ if and only if K can be obtained from L by dualizations and permutations of the ordinal summands of L.*

This has a straightforward corollary pertinent to our second problem:

Corollary 8. *Suppose that ϕ is a self-dual lattice property which implies modularity. If $(\forall i \in I : \phi(A_i)) \iff \phi(\bigoplus_{i\in I} A_i)$, then ϕ is preserved under isomorphisms of sublattice-lattices.*

In this section we consider properties that can be characterized by infinitary Horn sentences.

Lemma 9. *Let A and B be bounded lattices. Then $A\Delta B \in \mathbf{SP}(A, B)$.*

Proof. The map $\alpha : A\Delta B \to A \times B$,

$$x\alpha = \begin{cases} (x, 0_B), & \text{if } x \in A, \\ (1_A, x), & \text{if } x \in B \end{cases}$$

is an embedding. $\square$

Theorem 10. *Let A and B be lattices and ψ an infinitary Horn sentence which is not equivalent to the identity $x = y$. If ψ holds in A and B then ψ holds in $A \oplus B$.*

Proof. Suppose first that A and B are bounded lattices satisfying ψ. Then $A \oplus B = A\Delta C_2\Delta B \subseteq \mathbf{SP}(A, B, C_2)$ by Lemma 9. Since ψ trivially holds in C_2, it holds in $A \oplus B$ too.

Now let A and B be arbitrary lattices satisfying ψ, and assume that ψ does not hold in $A \oplus B$. As the conclusion of ψ has only finitely many variables, there exists a finitely generated sublattice $C = [x_0, x_1, \ldots, x_n]$ of $A \oplus B$ such that ψ fails in C. Let $D = [\{x_i \mid x_i \in A\}]$ and $E = [\{x_i \mid x_i \in B\}]$. Clearly, D and E are bounded lattices satisfying ψ, and $C = D \oplus E$. Consequently, ψ holds in C, a contradiction. $\square$

Using induction, we can generalize this theorem from two ordinal summands to finitely many ones, and since the conclusion of an infinitary Horn sentence contains only finitely many variables, we have

Corollary 11. *If an infinitary Horn sentence which is not equivalent to the identity $x = y$ holds in the lattices A_i $(i \in I)$ then it holds also in $\bigoplus_{i\in I} A_i$.*

Combining Corollary 8 and Corollary 11 we obtain

Theorem 12. *Let L and K be modular lattices with* $\mathrm{Sub}(L) \cong \mathrm{Sub}(K)$ *and let ψ be a self-dual infinitary Horn sentence. If ψ holds in L then it holds in K too.*

Remark. If ψ is a non self-dual infinitary Horn sentence then it is not preserved under isomorphisms of sublattice-lattices. Indeed, let A be a lattice such that ψ holds in A but not in A^d. Then $\mathrm{Sub}(A \oplus A) \cong \mathrm{Sub}(A \oplus A^d)$, ψ holds in $A \oplus A$ but it does not hold in $A \oplus A^d$. Moreover, $A \oplus A^d$ is self-dual, hence ψ is not satisfied in $(A \oplus A^d)^d$ either.

A special case of our second problem is (see Grätzer [3], p. 56, Problem I.5): Which equational classes $\mathcal{V}$ are closed under isomorphisms of sublattice-lattices, i.e., $L \in \mathcal{V}$ and $\mathrm{Sub}(L) \cong \mathrm{Sub}(K)$ imply $K \in \mathcal{V}$? This was known for the trivial varieties, and the classes $\mathcal{M}$ resp. $\mathcal{D}$ of modular resp. distributive lattices. By Theorem 12 and the Remark after it we have

Corollary 13. *A variety contained in $\mathcal{M}$ is closed under isomorphisms of sublattice-lattices if and only if it is self-dual.*

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