Remark on the zeros of characteristic equations.

To the memory of my late friend, Tibor Szele.

By PAUL TURÁN in Budapest.

1. This paper contains some remarks concerning an important question of the classical algebra. This refers to the question, what a strip

$$(1.1) \qquad \beta_1 \leqq \Re z \leqq \beta_2$$

can be given for the $\lambda_1', \lambda_2', \ldots, \lambda_n'$ zeros of the characteristic equation

$$(1.2) \qquad \Phi(\lambda) \equiv \begin{vmatrix} b_{11}-\lambda \cdots b_{1n} \\ \vdots \qquad \vdots \\ b_{n1} \cdots b_{nn}-\lambda \end{vmatrix} = 0$$

when we know that all $\lambda_1, \lambda_2, \ldots, \lambda_n$ zeros of the characteristic equation

$$(1.3) \qquad \psi(\lambda) \equiv \begin{vmatrix} a_{11}-\lambda \cdots a_{1n} \\ \vdots \qquad \vdots \\ a_{n1} \cdots a_{nn}-\lambda \end{vmatrix} = 0$$

lie in the strip

$$(1.4) \qquad \alpha_1 \leqq \Re z \leqq \alpha_2$$

and we know an upper estimation of

$$(1.5) \qquad F \equiv \sum_{j=1}^{n} \sum_{k=1}^{n} |b_{jk} - a_{jk}|^2.$$

We assert that a lower estimation of β_1 and an upper estimation of β_2 can be given in terms of n, F and the a_{jk} numbers, at least, when F is "small enough". Obviously it is enough to deal with the lower estimation and without loss of generality we may suppose

$$(1.6) \qquad \alpha_1 = 1.$$

Let us denote

$$(1.7) \qquad \max_{j,k} |a_{jk}| = A$$

and let finally Δ be so large that

$$(1.8) \qquad e^{2\Delta} \geqq 3n\{2e(\Delta+1)\}^{2n}$$

Then we assert the

Theorem. If

$$\sum_{j=1}^{n}\sum_{k=1}^{n}|b_{jk}-a_{jk}|^2 \leqq \left(\frac{e^{-nA(\Delta^2+\Delta)}}{2n(\Delta^2+\Delta)}\right)^2, \tag{1.9}$$

then we have

$$\beta_1 \geqq 1-\frac{2}{\Delta+1}.$$

2. This theorem clearly belongs to the classical algebra; curiously enough our proof will be performed via differential-equations. A direct algebraical proof based on matrix-calculus would be of interest; such a proof is easy when the (a_{jk}) and (b_{jk}) matrices are real and symmetrical. In this case the restriction (1. 9) can be considerably lightened; it is very probable that the same is true in the general case.

The proof is based on the following theorem on differential-equations.[1])

Let the functions $w_j(z_1, z_2, \dots, z_n, t)$ for $j=1, 2, \dots, n$ and $z_\nu=\zeta_\nu+i\eta_\nu$ (ζ_ν, η_ν reell) continuous in the halfplane $t\geqq 0$ of the $(2n+1)$-dimensional $(\zeta_1, \eta_1, \dots, \zeta_n, \eta_n, t)$-space and consider the system

$$\frac{dY_j(t)}{dt}=\sum_{k=1}^{n} c_{jk}Y_k(t)+w_j(Y_1, \dots, Y_n, t) \qquad j=1, 2, \dots, n \tag{2.1}$$

with complex constants c_{jk}; let

$$\max_{j, k}|c_{jk}|=C$$

and suppose all x_j zeros of the equation

$$G(x)\equiv\begin{vmatrix} c_{11}-x & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn}-x \end{vmatrix}=0 \tag{2.2}$$

are in the half-plane $\Re x\geqq 1$. If the positive ω satisfies the inequality

$$e^{2\omega}\geqq 3n\{2e(\omega+1)\}^{2n} \tag{2.3}$$

and for the w_j-functions we have for $t\geqq 0$

$$\frac{\sum_{j=1}^{n}|w_j(z_1, z_2, \dots, z_n, t)|^2}{\sum_{j=1}^{n}|z_j|^2} \leqq \left\{\frac{1}{2n(\omega^2+\omega)}\cdot\frac{e^{-nC(\omega^2+\omega)}}{(2e(\omega+1))^n}\right\}^2, \tag{2.4}$$

then for each $(Y_1, Y_2, \dots, Y_n)$ solution of the system (2. 1) and in each interval I in $t\geqq 0$ of length $(\omega^2+\omega)$ we have a t^* with

$$\sum_{j=1}^{n}|X_j(t^*)|^2 \geqq e^{2t^*\left(1-\frac{2}{\omega+1}\right)}\sum_{j=1}^{n}|X_j(0)|^2. \tag{2.5}$$

---

[1]) To be published in *Acta Math. Acad. Sci. Hungar.* The main tool of the proof is the Theorem VIII. of my book "Eine neue Methode in der Analysis und deren Anwendungen", Budapest, Akad. Kiadó, 1953.

One of the main-points of this theorem is the density of these t^*-values. For our present purposes however suffices the weaker statement that there are arbitrary big t^*-values with (2. 5). As easy to see from the proofs of the quoted Acta-paper, this can be assured requiring instead of (2. 4) only the weaker inequality

$$\frac{\sum_{j=1}^{n} |w_j(z_1, \ldots, z_n, t)|^2}{\sum_{j=1}^{n} |z_j|^2} \leqq \left(\frac{e^{-nC(\omega^2+\omega)}}{2n(\omega^2+\omega)}\right)^2. \tag{2. 6}$$

We omit a detailed repetition of the proof; the result we shall quote as the reduced Theorem.

3. In order to deduce our theorem from (2. 6) we apply it with

$$c_{jk} = a_{jk} \qquad 1 \leqq j, k \leqq n \tag{3. 1}$$

and

$$w_j(z_1, \ldots, z_n, t) \equiv \sum_{k=1}^{n} (b_{jk} - a_{jk}) z_k \tag{3. 2}$$

independently of t. Then we have

$$C \equiv A, \qquad G(\lambda) \equiv \psi(\lambda)$$

and the condition

$$\Re x_j \geqq 1 \qquad j = 1, \ldots, n$$

is owing to (1. 6) fulfilled. We may choose owing to (1. 8)

$$\omega = \Delta. \tag{3. 3}$$

The condition (2. 6) is by the choice (3. 2) fulfilled, since from the inequality of Cauchy—Buniakowski we have

$$\frac{\sum_{j=1}^{n} |w_j(z_1, \ldots, z_n, t)|^2}{\sum_{j=1}^{n} |z_j|^2} = \frac{\sum_{j=}^{n} \left| \sum_{k=1}^{n} (b_{jk} - a_{jk}) z_k \right|^2}{\sum_{j=1}^{n} |z_j|^2} \leqq$$

$$\leqq \sum_{j=1}^{n} \sum_{k=1}^{n} |b_{jk} - a_{jk}|^2 \leqq \left(\frac{e^{-nA(\Delta^2+\Delta)}}{2n(\Delta^2+\Delta)}\right)^2 = \left(\frac{e^{-nC(\omega^2+\omega)}}{2n(\omega^2+\omega)}\right)^2$$

owing to (1. 9) and (3. 3). The continuity-requirement of $w_j(z_1, \ldots, z_n, t)$ are by the choice (3. 2) evidently fulfilled and thus the reduced theorem can be applied to *each* function-system

$$Y_1(t+t_0), \ldots, Y_n(t+t_0)$$

where $(Y_1, \ldots, Y_n)$ denotes an arbitrary nonvanishing solution of the system

$$\frac{dY_j(t)}{dt} = \sum_{k=1}^{n} b_{jk} Y_k(t) \tag{3. 4}$$

$$j = 1, 2, \ldots, n$$

where t_0 is an arbitrary positive number. If t_0 is chosen so that

$$\sum_{j=1}^{n} |Y_j(t_0)|^2 > 0,$$

the reduced theorem gives the existence of a sequence

(3. 5) $$(t_0 <)\, t_1 < t_2 < \cdots \to +\infty$$

with

$$\sum_{j=1}^{n} |Y_j(t_\nu + t_0)|^2 \geqq e^{2\left(1-\frac{2}{\Delta+1}\right)t_\nu} \sum_{j=1}^{n} |Y_j(t_0)|^2$$

i. e.

(3. 6) $$\overline{\lim_{t\to+\infty}}\ \frac{1}{t} \log\left(\sum_{j=1}^{n} |Y_j(t)|^2\right) \geqq 2\left(1-\frac{2}{\Delta+1}\right).$$

On the other hand we consider a λ_1' — zero of $\Phi(\lambda)=0$ with the minimal real part. Owing to a part of a theorem of POINCARÉ—PERRON,[2]) we have an integral $(Z_1, Z_2, \ldots, Z_n)$ of (3. 4) such that

(3. 7) $$\overline{\lim_{t\to+\infty}}\ \frac{1}{t} \log\left(\sum_{j=1}^{n} |Z_j(t)|\right) = \Re\lambda_1'.$$

Let $t>0$ be now arbitrary and fixed and let j_0 be an index with

$$|Z_{j_0}(t)|^2 \geqq \frac{1}{n} \sum_{j=1}^{n} |Z_j(t)|^2.$$

Then

$$\sqrt{n}\left(\sum_{j=1}^{n} |Z_j(t)|^2\right)^{1/2} \geqq \sum_{j=1}^{n} |Z_j(t)| \geqq |Z_{j_0}(t)| \geqq$$

$$\geqq \frac{1}{\sqrt{n}}\left(\sum_{j=1}^{n} |Z_j(t)|^2\right)^{1/2},$$

i. e. from (3. 7)

$$\overline{\lim_{t\to+\infty}}\ \frac{1}{t} \log\left(\sum_{j=1}^{n} |Z_j(t)|^2\right) = 2\Re\lambda_1'.$$

Comparing this with (3. 6) we get

$$\Re\lambda_1' \geqq 1-\frac{2}{\Delta+1}$$

indeed.

(Received October 21, 1955.)

---

[2]) Cf. PERRON: Über lineare Differentialgleichungen bei denen die unabhängige Variable reel ist, *J. Reine Angew. Math.* **142** (1932), 254—270. The special case (3. 4) could have been easily treated directly too.