An elementary combinatorial theorem with an application to axiomatic set theory.

In Memory of our self-sacrificing true friend, Tibor Szele.

By ANDRÁS HAJNAL and LÁSZLÓ KALMÁR in Szeged.

In the late SZELE we have lost not only a great algebrist but also a master in combinatorial reasoning. His first paper[1]) deals with problems of combinatorial character; but also his later papers are full of combinatorial ideas.

Combinatorial theorems are, except the classic ones, seldom interesting for their own sake; however, they often have interesting applications in diverse branches of mathematics. In SZELE's research-work, combinatorial ideas are generally used to the solution of algebraical or geometrical problems.

In the present paper, we shall prove a combinatorial theorem of elementary character. The proof is very simple, indeed, almost trivial. However, the theorem itself has an interesting application in axiomatic set theory, for it makes possible to dispense with one of the axioms of the GÖDEL axiom system for set theory.[2])

1. Let us consider *elements* $a, b, c, \dots$ of arbitrary character.[3]) Out of

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1) TIBOR SZELE, Kombinatorikai vizsgálatok az irányított teljes gráffal kapcsolatban (Combinatorial investigations concerning the oriented full graph), *Mat. Fiz. Lapok* **50** (1943), 223—256.

2) See KURT GÖDEL, The consistency of the axiom of choice and of the generalized continuum-hypothesis with the axioms of set theory, *Annals of Math. Studies* **3** (1940), 66 pages, especially pp. 3—6.

3) The elements $a, b, c, \dots$ may form a finite or an infinite set, or even a proper class, i. e. a class which is not a set in the sense of the NEUMANN—BERNAYS—GÖDEL axiomatic set theory (see J. v. NEUMANN, Eine Axiomatisierung der Mengenlehre, *J. Reine Angew. Math.* **154** (1925), 219—240, and Die Axiomatisierung der Mengenlehre, *Math. Z.* **27** (1928), 669—752; PAUL BERNAYS, A system of axiomatic set theory — Part I. *J. Symbolic Logic* **2** (1937), 65—77, Part II, *ibidem*, **6** (1941), 1—17, Part III, *ibidem*, **7** (1942), 65—89, Part IV, *ibidem*, **7** (1942), 133—145, Part V, *ibidem*, **8** (1943), 89—106, Part VI, *ibidem*, **13** (1948), 65—79 and Part VII, *ibidem*, **19** (1954), 81—96; and KURT GÖDEL, loc. cit. [2])). However, the elements $a, b, c, \dots$ themselves have to be able to be elements of a set or class, therefore, they cannot be proper classes.

these elements we form *ordered pairs* $\langle a, a\rangle, \langle a, b\rangle, \ldots$; we subjoin these to the original elements and we form again ordered pairs such as $\langle a, \langle a, b\rangle\rangle$ or $\langle\langle a, b\rangle, \langle b, c\rangle\rangle$, and so on.[4]) Every "iterated ordered pair" resulting thus, the original elements as well as the uniterated ordered pairs included is called a *complex*.[5]) Let be $x, y, z, \ldots$, without or with subscripts, variables running through the complexes. Out of these variables we can form again *expressions* such as $\langle x, y\rangle, \langle\langle x, y\rangle, z\rangle$ by (iterated) forming of ordered pairs; also the variables themselves are regarded as expressions.[6]) Every expression characterizes a particular form of complexes; e. g. all complexes are of the form x, all complexes but (some of) the elements themselves are of the form $\langle x, y\rangle$, the complexes $\langle a, a\rangle, \langle b, b\rangle, \ldots$ but also $\langle\langle a, b\rangle, \langle a, b\rangle\rangle$ and $\langle\langle a, \langle b, c\rangle\rangle, \langle a, \langle b, c\rangle\rangle\rangle$ are of the form $\langle x, x\rangle$, the complexes $\langle\langle a, b\rangle, c\rangle$ and $\langle\langle\langle a, b\rangle, \langle b, c\rangle\rangle, \langle a, c\rangle\rangle$ are of the form $\langle\langle x, y\rangle, z\rangle$.

The expressions

$$x, \langle x, y\rangle, \langle x, \langle y, z\rangle\rangle, \langle x, \langle y, \langle z, u\rangle\rangle\rangle, \ldots$$

(also with any other variables instead of $x, y, z, u, \ldots$) are called *sequences*; they are denoted also by

$$\langle x\rangle, \langle x, y\rangle, \langle x, y, z\rangle, \langle x, y, z, u\rangle, \ldots$$

respectively. In other words, the sequence $\langle x_1, \ldots, x_n\rangle$ is defined for $n = 1, 3, 4, \ldots$ by induction as follows:

$$\text{(1)} \qquad \begin{cases} \langle x_1\rangle = x_1, \\ \langle x_1, x_2, \ldots, x_{n+1}\rangle = \langle x_1, \langle x_2, \ldots, x_{n+1}\rangle\rangle. \end{cases}$$

The complexes $x_1, \ldots, x_n$ are called the *components* of the sequence $\langle x_1, \ldots, x_n\rangle$. We note for subsequent application

$$\text{(2)} \qquad \langle x_1, x_2, x_3, \ldots, x_n\rangle = \langle x_1, x_2, \langle x_3, \ldots, x_n\rangle\rangle.$$

Indeed, we have by (1)

$$\langle x_1, x_2, x_3, \ldots, x_n\rangle = \langle x_1, \langle x_2, x_3, \ldots, x_n\rangle\rangle = \\ = \langle x_1, \langle x_2, \langle x_3, \ldots, x_n\rangle\rangle\rangle = \langle x_1, x_2, \langle x_3, \ldots, x_n\rangle\rangle.$$

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[4]) The case is not excluded that e. g. the ordered pair $\langle a, b\rangle$ (or $\langle a, \langle a, b\rangle\rangle$) is one of the original elements $c, d, \ldots$; of course, in such case the subjunction of the ordered pair in question to the original elements does not take place.

[5]) An exact definition of the notion of a complex is the following. Let us call each of the original elements a complex of order 0; and, assuming defined the notion of a complex of order m for $m < n$, define a complex of order n (> 0) as an ordered pair $\langle x, y\rangle$ where x and y, but not $\langle x, y\rangle$, are complexes of an order less than n. Or, alternatively, the class of complexes can be defined as the intersection of all classes containing the original elements as well as, together with x and y, the pair $\langle x, y\rangle$ as elements. (Both definitions involve some difficulty, in the case the original elements form a proper class, when expressed in axiomatic set theory; however, this difficulty can be got over by appropriate devices.)

[6]) I. e. the notion of an expression can be defined analogously to that of a complex, by giving to the variables $x, y, z, \ldots, x_1, \ldots$ the rôle of the elements $a, b, c, \ldots$

2. We consider *operations* such as

(3) $$x \to \langle y, x\rangle$$

or

(4) $$\langle x, y\rangle \to \langle y, x\rangle$$

or

(5) $$\langle x, y, z\rangle \to \langle z, x, y\rangle;$$

in an operation any two expressions may stand on the left-hand and right-hand sides of the arrow. The operation (4), e. g., has to be interpreted as applicable to any complex of the form $\langle x, y\rangle$ ($\langle\langle a, b\rangle, a\rangle$, say,) and transforming it into the corresponding complex $\langle y, x\rangle$, where x and y denote, of course, the same complexes as in $\langle x, y\rangle$ (in our case, into $\langle a, \langle a, b\rangle\rangle$). Operations as (3), containing a variable on the right-hand side of the arrow which does not figure on the left-hand side, are to be interpreted as one-many transformations. E. g., the operation (3) is applicable to any complex x and it transforms it into any complex of the form $\langle y, x\rangle$ with the same x and an arbitrary complex y; e. g., it transforms the complex $\langle a, b\rangle$ into any of the complexes $\langle a, \langle a, b\rangle\rangle, \langle b, \langle a, b\rangle\rangle, \langle c, \langle a, b\rangle\rangle, \ldots, \langle\langle a, a\rangle, \langle a, b\rangle\rangle$, $\langle\langle a, b\rangle, \langle a, b\rangle\rangle, \ldots, \langle\langle a, \langle a, a\rangle\rangle, \langle a, b\rangle\rangle, \ldots$ (An operation containing a variable on the left-hand side which does not figure on the right-hand side is a single valued transformation unless it contains another variable on the right-hand side not figuring on the left-hand side. E. g., the operation

$$\langle x, y\rangle \to x$$

transforms any complex which is an ordered pair into its first component, whereas the operation

$$x \to y$$

transforms any complex into any other complex or itself.)

Given some primary operations, we may obtain further ones as *derived operations*. E. g., given (4) and (5) as primary operations, we may transform any complex of the form $\langle\langle x, y\rangle, z\rangle$ by means of (4) into the complex $\langle z, \langle x, y\rangle\rangle$, i. e. $\langle z, x, y\rangle$ (taking $\langle x, y\rangle$ for x and z for y in (4)), which in turn may be transformed by means of (5) (taking z for x, x for y and y for z) into $\langle y, z, x\rangle$. Hence, we get

$$\langle\langle x, y\rangle, z\rangle \to \langle y, z, x\rangle$$

as a derived operation. One more application of the operation (5) (taking y for x, z for y and x for z) shows that we obtain also

$$\langle\langle x, y\rangle, z\rangle \to \langle x, y, z\rangle,$$

i. e.

$$\langle\langle x, y\rangle, z\rangle \to \langle x, \langle y, z\rangle\rangle$$

as a derived operation.

Generally, let be given a finite number of primary operations $\mathbf{E}_1 \rightarrow \mathbf{F}_1, \ldots, \mathbf{E}_k \rightarrow \mathbf{F}_k$ (where $\mathbf{E}_1, \mathbf{F}_1, \ldots, \mathbf{E}_k, \mathbf{F}_k$, as well as $\mathbf{E}, \mathbf{F}, \mathbf{G}, \mathbf{H}, \mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_n$ in the sequel, denote arbitrary expressions). Then, we call $\mathbf{G} \rightarrow \mathbf{H}$ a derived operation, provided for appropriate expressions $\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_n$, the first and last of which coincide with $\mathbf{G}$ and $\mathbf{H}$, respectively, each of the operations $\mathbf{G}_1 \rightarrow \mathbf{G}_2, \mathbf{G}_2 \rightarrow \mathbf{G}_3, \ldots, \mathbf{G}_{n-1} \rightarrow \mathbf{G}_n$ can be obtained form one of $\mathbf{E}_1 \rightarrow \mathbf{F}_1, \ldots, \mathbf{E}_k \rightarrow \mathbf{F}_k$ by replacing some of the variables by appropriate expressions, of course, all occurrences of the same variable in a given operation ($\mathbf{E}_1 \rightarrow \mathbf{F}_1, \ldots,$ or $\mathbf{E}_k \rightarrow \mathbf{F}_k$) by the same expression. (In the above example, we have $\mathbf{E}_1 = \langle x, y \rangle$, $\mathbf{F}_1 = \langle y, x \rangle$, $\mathbf{E}_2 = \langle x, y, z \rangle$, $\mathbf{F}_2 = \langle z, x, y \rangle$, $\mathbf{G} = \mathbf{G}_1 = \langle\langle x, y \rangle, z \rangle$, $\mathbf{G}_2 = \langle z, \langle x, y \rangle\rangle = \langle z, x, y \rangle$, $\mathbf{G}_3 = \langle y, z, x \rangle$, $\mathbf{G}_4 = \mathbf{H} = \langle x, y, z \rangle = \langle x, \langle y, z \rangle\rangle$; $\mathbf{G}_1 \rightarrow \mathbf{G}_2$ arises from $\mathbf{E}_1 \rightarrow \mathbf{F}_1$ by replacing the variables x and y by the expressions $\langle x, y \rangle$ and z, respectively, $\mathbf{G}_2 \rightarrow \mathbf{G}_3$ from $\mathbf{E}_2 \rightarrow \mathbf{F}_2$ by replacing the variables x, y and z by the expressions z, x and y, respectively, finally $\mathbf{G}_3 \rightarrow \mathbf{G}_4$ from $\mathbf{E}_2 \rightarrow \mathbf{F}_2$ by replacing the variables x, y and z by the expressions y, z and x, respectively.) It is advisable to consider every operation of the form $\mathbf{G} \rightarrow \mathbf{G}$ a derived operation irrespective of the primary operations; such an operation is called a *trivial derived operation.*

3. Given a finite number or an infinity of operations, we can ask for a system of primary operations, as simple as possible and eventually satisfying some additional conditions, such that each of the given operations can be obtained from them as a derived operation. This question is analogous to that of finding a system of generators, as simple as possible, for all permutations of given elements, or for all even permutations, or generelly, for a system of permutations forming a group (hence, apart notation, for an arbitrary finite group); and indeed, it contains these latter questions, suitably formulated, as particular cases.

The problem of the simplification of the axiom system of abstract set theory (see section **7** of this paper) leads to another question of this kind, namely to that of finding a system of primary operations, none of which containing a variable on the left-hand side which does not figure on the right-hand side of the same operation, such that each of the operations

$$x_p \rightarrow \langle x_1, \ldots, x_n \rangle \tag{6}$$

and

$$\langle x_p, x_q \rangle \rightarrow \langle x_1, \ldots, x_n \rangle \tag{7}$$

$(n = 1, 2, \ldots;\ p, q = 1, \ldots, n,\ p \neq q)$ can be obtained from them as a derived operation; if there are such systems of primary operations, we try to find one of them, as simple as possible, We show that the operations (3), (4) and (5) form a solution of this question;[7] i. e., we prove the

THEOREM. *Given* (3), (4) *and* (5) *as primary operations, every operation of the form* (6) *and* (7) ($n=1,2,\ldots;\ p,q=1,\ldots,n,\ p\neq q$) *can be obtained as a derived operation.*

PROOF. First we show that we can obtain

(8) $$\langle x,y\rangle\rightarrow\langle z,x,y\rangle,$$

(9) $$\langle x,y\rangle\rightarrow\langle x,z,y\rangle$$

and

(10) $$\langle x,y\rangle\rightarrow\langle x,y,z\rangle$$

as derived operations. Indeed, we get by (3)[8])

(8′) $$\langle x,y\rangle\rightarrow\langle z,\langle x,y\rangle\rangle,$$

i. e. (8) as a derived operation. Further, we get first by (8)[9])

(11) $$\langle y,x\rangle\rightarrow\langle z,y,x\rangle,$$

then by (5)

(12) $$\langle z,y,x\rangle\rightarrow\langle x,z,y\rangle$$

as derived operations; hence, by (4), (11) and (12),[10]) the operation (9) too. Finally, we get by (5)

(13) $$\langle z,x,y\rangle\rightarrow\langle y,z,x\rangle$$

and

(14) $$\langle y,z,x\rangle\rightarrow\langle x,y,z\rangle$$

as derived operations; hence, by (8), (13) and (14), the operation (10) too.

Now, in order to prove our theorem, let n, p and q be given natural numbers, $1\leqq p,q\leqq n,\ p\neq q$ (hence $n\geqq 2$). It suffices to show that (7) can be obtained as a derived operation. Indeed, we get by (3),

(15) $$x_p\rightarrow\langle x_q,x_p\rangle$$

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[7]) Probably this is the simplest system of primary operations meeting the above demands; we propose as a problem to prove this on the basis of a reasonable definition of simplicity. A somewhat more complicated system of primary operations, consisting of (3), (4), (5) and

(5′) $$\langle x,y,z\rangle\rightarrow\langle x,z,y\rangle,$$

has been used for the same purpose by GÖDEL (loc. cit. [2]), especially proof of Theorem M1, pp. 9—10; see also section 7 of the present paper).

[8]) I. e., (8′) arises from (3) by replacing some of the variables (in this case, x and y) by appropriate expressions (in this case, $\langle x,y\rangle$ and z, respectively). Similarly later on.

[9]) Here we treat (8) as if it were a primary operation instead of a derived one, and later on, we shall do the same with other derived operations too. This is allowed for by replacing in a derived operation some of the variables by arbitrary expressions, we get obviously a derived operation again; and together with $\mathbf{G}_1\rightarrow\mathbf{G}_2, \mathbf{G}_2\rightarrow\mathbf{G}_3,\ldots,\mathbf{G}_{n-1}\rightarrow\mathbf{G}_n$, the operation $\mathbf{G}_1\rightarrow\mathbf{G}_n$ is a derived operation.

[10]) I. e. the operations (4), (11), (12) and (9) are of the form $\mathbf{G}_1\rightarrow\mathbf{G}_2$, $\mathbf{G}_2\rightarrow\mathbf{G}_3,\ldots$, $\mathbf{G}_{n-1}\rightarrow\mathbf{G}_n$, and $\mathbf{G}_1\rightarrow\mathbf{G}_n$, respectively (in this case, $n=4$).

(where q may be chosed arbitrarily from the numbers $1, \dots, p-1, p+1, \dots, n$) and by (4)

(16) $$\langle x_q, x_p\rangle \to \langle x_p, x_q\rangle$$

as derived operations; hence, supposed the same for (7), we get by (15), (16) and (7) also the operation (6) as a derived operation.[11])

Also it suffices to consider the case $p<q$ in (7); for if $p>q$, we have $q<p$, hence, provided the case $p<q$ is settled already, we have

(17) $$\langle x_q, x_p\rangle \to \langle x_1, \dots, x_n\rangle$$

as a derived operation and, by (4), the same is true for the operation

(18) $$\langle x_p, x_q\rangle \to \langle x_q, x_p\rangle$$

and hence, by (18) and (17), for the operation (7) too.

Now, suppose $1 \leqq p < q \leqq n$. In case $q<n$ we get by (10)

$$\langle x_p, x_q\rangle \to \langle x_p, x_q, \langle x_{q+1}, \dots, x_n\rangle\rangle$$

i. e.

(19) $$\langle x_p, x_q\rangle \to \langle x_p, x_q, x_{q+1}, \dots, x_n\rangle$$

(see (2)) as a derived operation. In case $q=n$, (19) (which is of course to be read $\langle x_p, x_q\rangle \to \langle x_p, x_n\rangle$ in this case,) is a trivial derived operation. In case $p<q-1$ we get by (9)

$$\langle x_p, \langle x_q, x_{q+1}, \dots, x_n\rangle\rangle \to \langle x_n, x_{q-1}, \langle x_q, x_{q+1}, \dots, x_n\rangle\rangle,$$

i. e. (see (1) and (2))

(20_1) $$\langle x_p, x_q, x_{q+1}, \dots, x_n\rangle \to \langle x_p, x_{q-1}, x_q, x_{q+1}, \dots, x_n\rangle$$

further

$$\langle x_p, \langle x_{q-1}, x_q, x_{q+1}, \dots, x_n\rangle\rangle \to \langle x_p, x_{q-2}, \langle x_{q-1}, x_q, x_{q+1}, \dots, x_n\rangle\rangle,$$

i. e. (see (1) and (2))

(20_2) $$\langle x_p, x_{q-1}, x_q, x_{q+1}, \dots, x_n\rangle \to \langle x_p, x_{q-2}, x_{q-1}, x_q, x_{q+1}, \dots, x_n\rangle,$$

and so on, finally

(20_{q-p-1}) $$\langle x_p, x_{p+2}, x_{p+3}, \dots, x_q, x_{q+1}, \dots, x_n\rangle \to \\ \to \langle x_p, x_{p+1}, x_{p+2}, x_{p+3}, \dots, x_q, x_{q+1}, \dots, x_n\rangle$$

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[11]) Instead of using (16), we could, by interchanging p and q in (7), refer to

(7') $$\langle x_q, x_p\rangle \to \langle x_1, \dots, x_n\rangle$$

as a derived operation. Our argument fails in the case $n=1$, hence, $p=1$; howewer, in this case (6), i. e.

$$x_1 \to \langle x_1\rangle,$$

i. e. $x_1 \to x_1$ (see (1)) is a trivial derived operation. Of course, we could show (6) to be a derived operation also directly, by using an analogous (but somewhat more simple) method as for (7).

as derived operations; hence, by (19), (20_1), (20_2), ..., (20_{q-p-1}), the operation

(21) $$\langle x_p, x_q\rangle \rightarrow \langle x_p, x_{p+1}, \ldots, x_n\rangle$$

too. In case $p = q - 1$, hence $q = p + 1$, (21) is the same as the derived operation (19).

In case $p = 1$, the derived operation (21) is already the same as (7) On the other hand, in case $p > 1$, we get by (3)

$$\langle x_p, x_{p+1}, \ldots, x_n\rangle \rightarrow \langle x_{p-1}, \langle x_p, x_{p+1}, \ldots, x_n\rangle\rangle,$$

i. e. (see (1))

(22_1) $$\langle x_p, x_{p+1}, \ldots, x_n\rangle = \langle x_{p-1}, x_p, x_{p+1}, \ldots, x_n\rangle,$$

further

$$\langle x_{p-1}, x_p, x_{p+1}, \ldots, x_n\rangle \rightarrow \langle x_{p-2}, \langle x_{p-1}, x_p, x_{p+1}, \ldots, x_n\rangle\rangle,$$

i. e. (see (1))

(22_2) $$\langle x_{p-1}, x_p, x_{p+1}, \ldots, x_n\rangle \rightarrow \langle x_{p-2}, x_{p-1}, x_p, x_{p+1}, \ldots, x_n\rangle,$$

and so on, finally

(22_{p-1}) $$\langle x_2, x_3, \ldots, x_p, x_{p+1}, \ldots, x_n\rangle \rightarrow \langle x_1, x_2, x_3, \ldots, x_p, x_{p+1}, \ldots, x_n\rangle$$

as derived operations; hence, by (21), (22_1), (22_2), ..., (22_{p-1}), the same is true for the operation

$$\langle x_p, x_q\rangle \rightarrow \langle x_1, x_2, x_3, \ldots, x_p, x_{p+1}, \ldots, x_n\rangle,$$

i. e. (7) too, which finishes the proof.

4. The question to which the theorem of the preciding section gives an answer arose, as mentioned, from the problem of axiomatization of abstract set theory. As well known, the axiomatic foundation of set theory has been made necessary by the set theoretical paradoxes which show that the principle according to which to any property Φ there is a set containing those and only those elements which have the property Φ, generally used in the pre-axiomatic or naive set theory, leads to contradictory consequences. In the first system of axioms of set theory, due to ZERMELO,[12]) this principle is replaced by the "Axiom der Aussonderung", postulating, for every set s and every well-defined property ("definite Klassenaussage") Φ, the existence of a set s_Φ containing those and only those elements which, besides having the property Φ, belong to s. ZERMELO did not say exactly, what is to be meant by a well-defined property. However, in all cases where the axiom in question has been used in the proof of a theorem of set theory, the property Φ can be written as a formula, containing a free variable x for the set Φ is a property of, built up from propositions of the form $\mathbf{y} \in \mathbf{z}$ and $\mathbf{y} = \mathbf{z}$ by means of the logical operations (conjunction, disjunction, implication, equivalence and negation) and the quantifiers (universal or existential quan-

[12]) ERNST ZERMELO, Untersuchungen über die Grundlagen der Mengenlehre I, *Math. Ann.* **65** (1908), 261—281.

tifier); here instead of **y** and **z** any set variables may stand (viz. x, or variables bound by quantifiers, or free variables other than x, denoting sets on which the resulting set s_Φ depends). Hence, for the purpose of proving set-theoretic theorems, the definition, due to SKOLEM,[13]) of the notion of a well-defined property suffices, according to which a well-defined property is defined as a property which can be written as such a formula.

Using SKOLEM's definition, the *Axiom der Aussonderung* amounts to an *infinity* of purely set theoretic axioms (i. e. not containing notions different from the primitive notions of the ZERMELO set theory), viz. for each formula expressing a well-defined property a separate axiom.[14]) Hence the question arises if we can replace the *Axiom der Aussonderung* by a *finite number* of purely set theoretic axioms such that each particular case of it, belonging to a particular formula expressing a well-defined property, can be proved by means of these axioms. This questions has been answered in the affirmative by NEUMANN (loc. cit. [3])), however, by using primitive notions entirely different from those used by ZERMELO, viz. that of an *argument* ("I-Ding"), of a *function* ("II-Ding") and of the *value of a function for an argument* (instead of the notions of an *element*, of a *set* and of *belonging to a set as its element*, used by ZERMELO as primitive notions), and, moreover, the additional primitive notions of an *ordered pair* formed of two arguments as well as of two particular arguments, denoted by A and B, serving as *values of characteristic functions* taking over the part of sets (or classes). Later on, BERNAYS (loc. cit. [5])) has shown that the same effect can be reached without such a radical change of the primitive notions, however, by retaining NEUMANN's idea of allowing, besides of sets admitted also by ZERMELO, some more extensive multitudes or "classes" and avoiding paradoxes by refraining them from belonging to a set or class as its element or being an argument for which a function is defined. Unlike NEUMANN who defined classes ("Bereiche") as

[13]) THORALF SKOLEM, Einige Bemerkungen zur axiomatischen Begründung der Mengenlehre, *Wissenschaftliche Vorträge, gehalten auf dem fünften Kongreß der skandinavischen Mathematiker* (Helsingfors, 1922, published 1923), 217—232; see also the discussion between ZERMELO and SKOLEM on this question: ERNST ZERMELO, Über den Begriff der Definitheit in der Axiomatik, *Fund. Math.* **14** (1929), 339—344; TH. SKOLEM, Einige Bemerkungen zu der Abhandlung von E. ZERMELO: „Über die Definitheit in der Axiomatik", *ibidem*, **15** (1930), 337—341; ERNST ZERMELO, Über Grenzzahlen und Mengenbereiche. Neue Untersuchungen über die Grundlagen der Mengenlehre, *ibidem*, **16** (1930), 29—47, especially footnote [1]) on p. 30. Another definition of the notion of a well-defined property, using instead of the concept of a formula in the sense of mathematical logic that of a function, has been proposed by ADOLF FRAENKEL, Zu den Grundlagen der Cantor—Zermeloschen Mengenlehre, *Math. Ann.* **86** (1922), 230—237; see also the following paper and works of FRAENKEL: Untersuchungen über die Grundlagen der Mengenlehre, *Math. Z.* **22** (1925), 250—273; *Zehn Vorlesungen über die Grundlegung der Mengenlehre* (Leipzig and Berlin, 1927) and *Einleitung in die Mengenlehre* (Berlin, second edition, 1923, third edition, 1928).

[14]) The same is true if we use FRAENKEL's definition instead of SKOLEM's.

functions taking no value different from A and B, and sets ("Mengen") as classes which are at the same time arguments, hence regarded the notion of a class more general than that of a set, BERNAYS does not identify a set with the class having the same elements and he distinguishes two different relations of being an element of something, according as the latter is a set or a class. Hence, BERNAYS uses the notions of a *set*, of a *class* as well as of *being an element of a set* and of *being an element of a class* as primitive notions, confining, like FRAENKEL (loc. cit. [13]), the elements of a set or class to be sets. Finally, GÖDEL (loc. cit. [2]) modified the BERNAYS axiom system by avoiding the said reduplication of the sets as well as of the relation of being an element of a set or class and performing at the same time some minor changes on the system of axioms of BERNAYS. As primitive notions, that of a *set*, of a *class* and of *being an element of a class* (which, in particular, may be a set too) are used by GÖDEL.

5. In order to show, how the question treated in section **3** is connected with that of axiomatizing abstract set theory, we must survey partly the GÖDEL axiom system. We use lower case italics, except m, n, p, q which denote non-negative integers, as variables for sets and capital italics as those for classes; $X \in Y$ signifies, as usual, that X belongs to Y as one of its elements.

The four first axioms of the GÖDEL axiom system are as follows.

A1. Every set is a class.

A2. If $X \in Y$ then X is a set.

A3. Two classes having the same sets as elements are identical (extensionality axiom).

A4. For any sets x and y, there is a set z containing x and y but no other sets as elements (pairing axiom).

These axioms enable us to define for any sets x and y, different or identical, $\{x, y\}$ as the set, existing by A4 and unique by (A1 and) A3, which contains x and y but no other sets as elements. For $x = y$, the set $\{x, y\} = \{x, x\}$ contains x as its only element; this set is denoted by $\{x\}$ too. Further, we can define the ordered pair $\langle x, y\rangle$ formed of the (different or identical) sets x and y in this order as follows:[15]

$$\langle x, y\rangle = \{\{x\}, \{x, y\}\}.$$

Further, the sequences $\langle x\rangle, \langle x, y, z\rangle, \langle x, y, z, u\rangle, \ldots$ can be defined by (1).[16]

[15]) This definition is due to C. KURATOWSKI, Sur le notion de l'ordre dans la théorie des ensembles, *Fund. Math.* **2** (1921), 161—171, especially p. 171. By means of axioms A1 A3 and A4 one proves readily that $\langle x, y\rangle$ is uniquely determined by x and y and conversely, it determines uniquely its first component x and second component y.

[16]) In axiomatic proofs, we are not allowed to use (1) as a general definition of the sequence $\langle x_1, \ldots, x_n\rangle$ of n sets $x_1, \ldots, x_n$ for a variable natural number n until we do not define the general concept of a natural number by means of the primitive notions and

The subsequent group B of axioms serves to replace the *Axiom der Aussonderung* of ZERMELO, or rather the principle of naive set theory, mentioned above, according to which to each property there is a set containing those and only those elements which have the given property. Now, the elements (of a set or a class) are in the GÖDEL set theory (like in the FRAENKEL and BERNAYS set theories) necessarily *sets*, and it is appropriate to confine ourselves to well-defined properties in a sense suggested by SKOLEM's definition mentioned above, i. e. to properties of a set x representable by a formula Φ which can be built up from propositions of the form $\mathbf{y} \in \mathbf{z}$ and $\mathbf{y} = \mathbf{z}$, where instead of $\mathbf{y}$ and $\mathbf{z}$, any set *or class* variables may stand, by means of the logical operations as well as the quantifiers, confining however the latters to those binding *set variables* only. Moreover, in order to avoid paradoxes, we do not postulate the existence of a *set* but only that of a *class* containing those and only those sets x as elements which have the given property, i. e. for which the formula Φ (containing x as a free variable) holds.[17])

Let us call a formula built up from propositions of the form $\mathbf{y} \in \mathbf{z}$ and $\mathbf{y} = \mathbf{z}$ with set or class variables $\mathbf{y}$ and $\mathbf{z}$ by means of the logical operations as well as quantifiers binding set variables only an *elementary formula.*[18]) We denote arbitrary elementary formulae by Greek capitals, followed eventu-

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justify inductive definition by means of the axioms of set theory (and even afterwards, (1) requires some re-formulation), but only as a definition of $\langle x_1 \rangle$ and as an abbreviation for a definition of $\langle x_1, \ldots, x_n \rangle$ for some numerically given n (e. g., for $n = 5$, as an abbreviation of the definition

$$\langle x_1, x_2, x_3 \rangle = \langle x_1, \langle x_2, x_3 \rangle \rangle,$$
$$\langle x_1, x_2, x_3, x_4 \rangle = \langle x_1, \langle x_2, x_3, x_4 \rangle \rangle,$$
$$\langle x_1, x_2, x_3, x_4, x_5 \rangle = \langle x_1, \langle x_2, x_3, x_4, x_5 \rangle \rangle$$

of $\langle x_1, x_2, x_3, x_4, x_5 \rangle$). For the axiomatic proof of a *given* theorem of set theory, this will always do; of course, an indefinite development of set theory based on the GÖDEL axiom system will require an indefinite growth of this n. On the other hand, in considerations *about* the GÖDEL axiom system or, more generally, about the axiomatization of set theory, we can freely use inductive definition, hence, we can state theorems concerning sequences in general.

[17]) The rest of GÖDEL's axioms — besides of the axiom of choice as well as an axiom to the effect of excluding the existence of sets without an "innermost kernel", e. g. of a set which is the only element of itself — serve to assure that some classes are sets, roughly speaking, those admitted as sets in the ZERMELO set theory. These axioms imply among others that the intersection (i. e. the class of common elements) of a class and a set is a set again. Hence, provided there is a class containing those and only those sets as elements for which the formula Φ holds, we can prove also, for every set s, the existence of a *set* containing the elements of s satisfying the formula Φ and no other elements; i. e. we have a theorem (more exactly, for each elementary formula Φ, a separate theorem) in the GÖDEL set theory having the same effect as the *Axiom der Aussonderung* in the ZERMELO set theory.

[18]) We use this term instead of Gödel's *primitive proposition function ;* howewer, GÖDEL excludes propositions of the form $\mathbf{y} = \mathbf{z}$ (which will be shown to do not matter).

ally by some set variables in parentheses and divided by commas, e. g. $\Phi(x_1, \ldots, x_n)$. Such a denotation does not imply that the elementary formula in question contains necessarily each of the free variables $x_1, \ldots, x_n$, nor that it does not contain any other free variables; it serves only to emphasize that we regard the elementary formula in question as a (propositional) function of $x_1, \ldots, x_n$ which eventually does not depend really on some of these variables.[19]) It is clear from which is said above that the axioms of group **B** have to be chosen such that for each elementary formula $\Phi(x)$, the folloving proposition can be proved by means of them (together with the axioms A1 to A4): *there is a class containing those and only those sets x for which $\Phi(x)$ holds.*[20])

The fact that an elementary function containing a free set variable x may be derived from another containing additional free set variables by means of quantifiers binding them, suggests that we should require more generally that for each elementary formula $\Phi(x_1, \ldots, x_n)$, the following proposition can be proved by means of the axioms A1 to A4 as well as those of the group B: *there is a class containing among the sequences of sets of the form $\langle x_1, \ldots, x_n \rangle$ those and only those for which $\Phi(x_1, \ldots, x_n)$ holds.*[21])

[19]) Of course, it is allowed to denote an elementary formula by $\Phi(x_1, \ldots, x_n)$ irrespective of the order in which the variables (eventualy) occur in it.

[20]) Note that the italicized proposition is, for each elementary formula $\Phi(x)$, a set theoretical proposition, i. e., a proposition containing no other notions than those defined by means of the primitive notions of (the GÖDEL) set theory (using standard logical devices of definition). On the contrary, the proposition stating that *for each elementary formula $\Phi(x)$, the proposition italicized above is a theorem of the* GÖDEL *set theory, provable by means of the axioms* A1 *to* A4 *and* B is not a set theoretical proposition for it contains the notion of an elementary formula which has not been defined by means of the primitive notions of set theory. The letter proposition states a fact *about set theoretical propositions,* viz. that all of them having a specified form belong to the theorems provable by means of the GÖDEL axioms (or rather a part of them). Such a proposition is called sometimes a *metaproposition* or, if proposed to be proved, a *metatheorem*. To prove a metatheorem, methods as e. g. inductive proof are of course available even if it states provability of propositions by means of axioms which do not do for justification of inductive proof.

[21]) See GÖDEL, loc. cit. [2]), p. 8, General Existence Theorem M1. — For the italicized proposition as well as for that stated about it, the same remark applies as in the preceding footnote for its particular case $n = 1$. It can be shown that the above requirement concerning elementary formulae $\Phi(x_1, \ldots, x_n)$ is not really more general as its above particular case. Indeed, we can construct an elementary formula $\Omega(x, x_1, \ldots, x_n)$ stating that $x = \langle x_1, \ldots, x_n \rangle$ as we shall see (also for more general expressions instead of $\langle x_1, \ldots, x_n \rangle$) in section **6**; and the class containing those and only those sets x for which the elementary formula $(Ex_1) \cdots (Ex_n) (\Omega(x, x_1, \ldots, x_n) \,\&\, \Phi(x_1, \ldots, x_n))$ holds, contains obviously those and only those sequences $\langle x_1, \ldots, x_n \rangle$ for which we have $\Phi(x_1, \ldots, x_n)$. — As here, we shall use also in the sequel the symbol & for conjuction and $(E\mathbf{x})$ as existential quantifier (binding the set variable $\mathbf{x}$); and, in addition, the symbols $\vee$ for disjunction, $\rightarrow$ for implication, $\longleftrightarrow$ for equivalence, $\sim$ for negation and $(\mathbf{x})$ as universal quantifier (bilding the set variable $\mathbf{x}$). The fact that we use the sign $\rightarrow$ for operations in the sense of section **2** too will not cause any misunderstanding.

In order to satisfy this requirement, it suffices to satisfy it in the case where $\Phi(x_1, \ldots, x_n)$ does not contain any part of the form $\mathbf{y} \in \mathbf{z}$ unless $\mathbf{y}$ is one of the free set variables $x_1, \ldots, x_n$ of $\Phi(x_1, \ldots, x_n)$ or of the set variables bound by means of a quantifier at every place where it occurs in $\Phi(x_1, \ldots, x_n)$, further, $\mathbf{z}$ is a (class or set) variable different from $\mathbf{y}$, moreover $\Phi(x_1, \ldots, x_n)$ does not contain any part of the form $\mathbf{y} = \mathbf{z}$. Indeed, on account of A2, any proposition $\mathbf{y} \in \mathbf{z}$ which does not meet the said conditions can be replaced by $(E\mathbf{u})(\mathbf{u} = \mathbf{y} \,\&\, \mathbf{u} \in \mathbf{z})$ with a set variable $\mathbf{u}$ which is different from $\mathbf{z}$ and does not occur in $\Phi(x_1, \ldots, x_n)$ as a free variable and afterwards, on account of A3 and A1, any proposition $\mathbf{y} = \mathbf{z}$ with set or class variables $\mathbf{y}$ and $\mathbf{z}$ by $(\mathbf{u})(\mathbf{u} \in \mathbf{y} \leftrightarrow \mathbf{u} \in \mathbf{z})$, $\mathbf{u}$ being a set variable different from $\mathbf{y}$ and $\mathbf{z}$ and not occuring in $\Phi(x_1, \ldots, x_n)$ as a free variable. Also, we may confine ourselves to the case where $\Phi(x_1, \ldots, x_n)$ does not contain any disjunction, implication or equivalence nor any universal quantifier, for as well known, these logical operations can be expressed by means of conjuctions and negations only and a universal quantifier can be expressed by means of negations and existential quantifiers.

Hence, in order to satisfy the above requirement for each elementary formula $\Phi(x_1, \ldots, x_n)$, it suffices (1) to satisfy it in the particular case where $\Phi(x_1, \ldots, x_n)$ is either of the form $x_p \in x_q$ with $1 \leqq p, q \leqq n$ and $p \neq q$ or of the form $x_p \in \mathbf{y}$, with $1 \leqq p \leqq n$, $\mathbf{y}$ being either a class variable or a set variable different from $x_1, \ldots, x_n$; further, (2) to ensure that provided it is satisfied for an elementary formula $\Phi(x_1, \ldots, x_n)$, the same is true for the elementary formula $\sim \Phi(x_1, \ldots, x_n)$ and provided it is satisfied for the elementary formulae $\Phi(x_1, \ldots, x_n)$ and $\Psi(x_1, \ldots, x_n)$, the same is true for the elementary formula $\Phi(x_1, \ldots, x_n) \,\&\, \Psi(x_1, \ldots, x_n)$; finally, (3) to ensure that provided it is satisfied for the elementary formula $\Phi(y, x_1, \ldots, x_n)$ with a set variable y different from $x_1, \ldots, x_n$, the same is true for the elementary formula $(Ey)\,\Phi(y, x_1, \ldots, x_n)$.[22])

As to (2) and (3), they have been met by GÖDEL (like by BERNAYS) by means of postulating the following "logical construction axioms":[23])

B3. For any class A, there is a class B containing those and only those sets x as elements which do not belong to A (axiom of the complementary class).

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[22]) Here, of course, instead of y any other set variable (different from $x_1, \ldots, x_n$) might stand; hence, in order to be consequent, we should use $\mathbf{y}$ instead of y (a bold-face letter denoting, as above, indifferently any of the set variables). Note that here we use the fact that in an elementery formula any free set variable can be treated as the "first" one (in our case, the variable to be bound by the next quantifier); see footnote [19]).

[23]) We enumerate them in the order we need them, retaining their notation used by GÖDEL. The denomination "logical construction axioms" ("logische Konstruktionsaxiome") has been used by NEUMANN (loc. cit. [3])) for axioms of his system analogous to the above axioms of the GÖDEL system.

B2. For any classes A and B, there is a class C containing those and only those sets x as elements which belong at the same time to A and to B (axiom of the intersection).

B4. For any class A, there is a class B containing those and only those sets x which are the second components of at least one ordered pair belonging to A (axiom of the domain[24])).

Indeed, provided there is a class A containing among the sequences of the form $\langle x_1, \ldots, x_n \rangle$ those and only those for which $\Phi(x_1, \ldots, x_n)$ holds and a class B containing those and only those sets x for which $x \in A$ does not hold, the class B contains a sequence of the form $\langle x_1, \ldots, x_n \rangle$ if and only if $\langle x_1, \ldots, x_n \rangle \in A$ does not hold, i. e. if $\Phi(x_1, \ldots, x_n)$ does not hold, that is, if $\sim \Phi(x_1, \ldots, x_n)$ holds. Further, provided there are classes A and B containing among the sequences of the form $\langle x_1, \ldots, x_n \rangle$ those and only those for which $\Phi(x_1, \ldots, x_n)$ and $\Psi(x_1, \ldots, x_n)$ holds, respectively, and a class C containing those and only those sets x for which both $x \in A$ and $x \in B$ hold, the class C contains a sequence of the form $\langle x_1, \ldots, x_n \rangle$ if and only if both $\langle x_1, \ldots, x_n \rangle \in A$ and $\langle x_1, \ldots, x_n \rangle \in B$ hold, i. e. if both $\Phi(x_1, \ldots, x_n)$ and $\Psi(x_1, \ldots, x_n)$ hold, that is, if $\Phi(x_1, \ldots, x_n) \,\&\, \Psi(x_1, \ldots, x_n)$ holds. Finally, provided there is a class A containing among the sequences of the form $\langle y, x_1, \ldots, x_n \rangle$ those and only those for which $\Phi(y, x_1, \ldots, x_n)$ holds and a class B containing those and only those sets x for which, for at least one set y, the ordered pair $\langle y, x \rangle$ is contained in A, the class B contains a sequence of the form $\langle x_1, \ldots, x_n \rangle$ if and only if for at least one set y $\langle y, x_1, \ldots, x_n \rangle = \langle y, \langle x_1, \ldots, x_n \rangle \rangle \in A$, i. e. if there is a set y for which $\Phi(y, x_1, \ldots, x_n)$ holds, that is, if $(Ey)\, \Phi(y, x_1, \ldots, x_n)$ holds.

As to (1), the above requirement in the simplest case, viz. that in which $n = p = 1$, $\Phi(x_1)$ is $x_1 \in \mathbf{y}$, is satisfied automatically by the class $\mathbf{y}$, whereas in the case in which $n = 2$, $p = 1$, $q = 2$, $\Phi(x_1, x_2)$ is $x_1 \in x_2$, the above requirement has been formulated as an axiom by GÖDEL, viz.

B1. There is a class containing among the ordered pairs $\langle x, y \rangle$ those and only those for which $x \in y$.

However, we have to meet not only the case in which $n = 2$, $p = 2$, $q = 1$, $\Phi(x_1, x_2)$ is $x_2 \in x_1$, but also the cases $n > 2$ (and $n = 2$, $\Phi(x_1, x_2)$ is $x_1 \in \mathbf{y}$ or $x_2 \in \mathbf{y}$) too.[25]) This obligation requires further axioms; we shall dis-

[24]) In the BERNAYS—GÖDEL set theory, functions are treated as classes of ordered pairs, the function F being indentified with the class of the pairs $\langle F(x), x \rangle$ (in BERNAYS, $\langle x, F(x) \rangle$), x running trough the domain of F. If A is a function in this sense, its domain B satisfies the requirement of B4.

[25]) The necessity of settling the case $n > 2$ is clear from the fact that, e. g., we cannot prove the existence a class containing a sequence of the form $\langle x_1, x_2, x_3 \rangle$ if and only if $x_1 \in x_2 \,\&\, x_2 \in x_3$ holds by means of axiom B2 unless we first prove the existence of a class A and a class B such that $\langle x_1, x_2, x_3 \rangle \in A$ if and only if $x_1 \in x_2$ and $\langle x_1, x_2, x_3 \rangle \in B$ if and only if $x_2 \in x_3$.

cuss the question, how they are to be chosen in order to get an axiom system as simple as possible.

6. To each operation $\mathbf{G} \rightarrow \mathbf{H}$ ($\mathbf{G}$ and $\mathbf{H}$ expressions) in the sense of section **2**, considering the variables occuring in it as set variables, we can attach the proposition of set theory stating that for any class A there is a class B such that for any sets denoted by the variables figuring in at least one of the expressions $\mathbf{G}$ and $\mathbf{H}$, we have $\mathbf{H} \in B$ if and only if $\mathbf{G} \in A$. E. g., to the operations (3), (4) and (5), the following propositions B5, B6 and B7, differing only in the notation of the set variables from Gödels's axioms denoted thus, have been attached, respectively:

B5. For any class A, there is a class B such that for any sets x and y, the ordered pair $\langle y, x \rangle$ is contained in B if and only if $x \in A$.

B6. For any class A, there is a class B such that for any sets x and y, the ordered pair $\langle y, x \rangle$ is contained in B if and only if $\langle x, y \rangle \in A$.

B7. For any class A, there is a class B such that for any sets x, y and z, the sequence (ordered triple) $\langle z, x, y \rangle$ is contained in B if and only if $\langle x, y, z \rangle \in A$.

Moreover, to the operations (6) and (7) the following propositions have been attached, respectively:

$\mathbf{P}_{np}$. For any class A, there is a class B such that for any sets $x_1, \ldots, x_n$, the sequence $\langle x_1, \ldots, x_n \rangle$ is contained in B if and only if $x_p \in A$.

$\mathbf{P}_{npq}$. For any class A there is a class B such that for any sets $x_1, \ldots, x_n$, the sequence $\langle x_1, \ldots, x_n \rangle$ is contained in B if and only if $\langle x_p, x_q \rangle \in A$.

In general, we call a proposition which has been attached to an operation in the above sense a *combinatorial proposition*. Thus, B5, B6, B7 and $\mathbf{P}_{np}, \mathbf{P}_{npq}$ are combinatorial propositions and so is the proposition "for any class A, there is a class B such that for any sets x and y, we have $y \in B$ if and only if $x \in A$", attached to the operation $x \rightarrow y$. Whereas B5, B6, B7, $\mathbf{P}_{np}$ and $\mathbf{P}_{npq}$ are reasonable propositions of set theory, this does not hold for the last proposition for it would imply that every class which is not empty contains all sets. Indeed, if there is a class A which is not empty and does not contain all sets, let x and x' be sets satisfying $x \in A$ and $x' \notin A$. The proposition between inverted commas would imply the existence of a class B such that, for all sets y, we would have $y \in B$ by $x \in A$ and $y \notin B$ by $x' \notin A$, which is impossible.

More generally, a combinatorial proposition attached to an operation $\mathbf{G} \rightarrow \mathbf{H}$ such that the expression $\mathbf{G}$ contains a variable x not contained in $\mathbf{H}$ is unreasonable for it would imply a contradiction to the axioms A1 to A4 supplemented by the proposition stating the existence of two sets, different from each other.[26]) Indeed, let be $x_1, \dots, x_n$ the variables, other than x, occurring in $\mathbf{G}$ or in $\mathbf{H}$; in order to emphasize the dependence of $\mathbf{G}$ and $\mathbf{H}$ on $x, x_1, \dots, x_n$ and $x_1, \dots, x_n$, respectively, let us write them in the form $\mathbf{G}(x, x_1, \dots, x_n)$ and $\mathbf{H}(x_1, \dots, x_n)$, respectively. Then, the combinatorial proposition attached to the operation $\mathbf{G}(x, x_1, \dots, x_n) \rightarrow \mathbf{H}(x_1, \dots, x_n)$ states the existence, for any class A, of a class B such that, for any sets $x, x_1, \dots, x_n$, we have $\mathbf{H}(x_1, \dots, x_n) \in B$ if and only if $\mathbf{G}(x, x_1, \dots, x_n) \in A$. Now, let us consider $x_1, \dots, x_n$ as fixed sets; as a consequence of the property of ordered pairs, mentioned in footnote [15]), provable by means of the axioms A1, A3 and A4, one proves readily that for different sets x and x', we have $\mathbf{G}(x, x_1, \dots, x_n) \neq \mathbf{G}(x', x_1, \dots, x_n)$.[27]) Now, let us choose two sets x and x', different from each other, and let us take as class A the set $\{\mathbf{G}(x, x_1, \dots, x_n)\}$.[28]) Then, the combinatorial proposition in question implies the existence of a class B such that we have $\mathbf{H}(x_1, \dots, x_n) \in B$ by $\mathbf{G}(x, x_1, \dots, x_n) \in A$ and, at the same time, $\mathbf{H}(x_1, \dots, x_n) \notin B$ by $\mathbf{G}(x', x_1, \dots, x_n) \notin A$, thus, a contradiction.

On the contrary, in the case where the expression $\mathbf{G}$ does not contain any variable not contained in $\mathbf{H}$, the combinatorial proposition attached to the operation $\mathbf{G} \rightarrow \mathbf{H}$ is a reasonable proposition of set theory, viz. a consequence of a proposition of the form: there is a class containing a set x if and only if $\Phi(x)$ holds, $\Phi(x)$ being an elementary formula determined by $\mathbf{G}$ and $\mathbf{H}$.

To show this, first we prove that for any expression $\mathbf{E}(x_1, \dots, x_n)$, containing no other variables than $x_1, \dots, x_n$, we can construct an elementary formula $\Phi(x, x_1, \dots, x_n)$ which holds for sets $x, x_1, \dots, x_n$ if and only if $x = \mathbf{E}(x_1, \dots, x_n)$. This is obvious in the case $\mathbf{E}(x_1, \dots, x_n)$ is one of the variables $x_1, \dots, x_n$, e. g. x_p, for $x = x_p$ is an elementary formula. Supposing we can construct for two expressions $\mathbf{E}(x_1, \dots, x_n)$ and $\mathbf{F}(x_1, \dots, x_n)$ the elementary formulae $\Phi(x, x_1, \dots, x_n)$ and $\Psi(x, x_1, \dots, x_n)$ such that we have $\Phi(x, x_1, \dots, x_n)$ if and only if $x = \mathbf{E}(x_1, \dots, x_n)$ and $\Psi(x, x_1, \dots, x_n)$ if and only if $x = \mathbf{F}(x_1, \dots, x_n)$, the same is true for the expression $\langle \mathbf{E}(x_1, \dots, x_n), \mathbf{F}(x_1, \dots, x_n) \rangle$. Indeed, we have

$$x = \langle \mathbf{E}(x_1, \dots, x_n), \mathbf{F}(x_1, \dots, x_n) \rangle = \\ = \{\{\mathbf{E}(x_1, \dots, x_n)\}, \{\mathbf{E}(x_1, \dots, x_n), \mathbf{F}(x_1, \dots, x_n)\}\}$$

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[26]) This proposition does not follow from GÖDEL's axioms belonging to the groups A and B; however, it is of course a theorem of set theory (to be proved by using the rest of the axioms too).

[27]) In order to show this in general, i. e. for an arbitrary expression $\mathbf{G}(x, x_1, \dots, x_n)$, we have to use induction with respect to the order of this expression (to be defined analogously to that of a complex, see footnote [5])). To show the same for a *given* expression $\mathbf{G}(x, x_1, \dots, x_n)$, the order of $\mathbf{G}(x, x_1, \dots, x_n)$ being a *given* natural number, no induction is needed.

[28]) Note here we use axioms A4 (in order to construct the set $\{\mathbf{G}(x, x_1, \dots, x_n)\}$) and A1 (in order to show that it is a class).

if and only if there are sets y, z, u and v such that $y = \mathbf{E}(x_1, \ldots, x_n)$, $z = \mathbf{F}(x_1, \ldots, x_n)$, $u = \{y\}$, $v = \{y, z\}$ and $x = \{u, v\}$ hold; i. e. if $\Phi(y, x_1, \ldots, x_n)$, $\Psi(z, x_1, \ldots, x_n)$, further, $y \in u$, $y \in v$, $z \in v$, $u \in x$, $v \in x$ hold, finally, there is no set w such that $w \in u$ but $w \neq y$, or $w \in v$ but $w \neq y$ and $w \neq z$, or $w \in x$ but $w \neq u$ and $w \neq v$ hold. That is, we have $x = \langle \mathbf{E}(x_1, \ldots, x_n), \mathbf{F}(x_1, \ldots, x_n) \rangle$ if and only if

$$(Ey)(Ez)(Eu)(Ev)(\Phi(y, x_1, \ldots, x_n) \,\&\, \Psi(z, x_1, \ldots, x_n) \,\&$$
$$\& \, y \in u \,\&\, y \in v \,\&\, z \in v \,\&\, u \in x \,\&\, v \in x \,\&\, \sim(Ew)((w \in u \,\&\, \sim(w = y)) \vee$$
$$\vee (w \in v \,\&\, \sim(w = y) \,\&\, \sim(w = z)) \vee (w \in x \,\&\, \sim(w = u) \,\&\, \sim(w = v))))$$

holds; and this is an elementary formula indeed. Hence our assertion has been proved, for each expression arises from variables by (iterated) forming of ordered pairs.[29])

Now, let be $\mathbf{G}$ and $\mathbf{H}$, or, more explicitely, $\mathbf{G}(x_1, \ldots, x_m)$ and $\mathbf{H}(x_1, \ldots, x_n)$, two expressions such that no variable does occur in $\mathbf{G}$ unless it occurs in $\mathbf{H}$ too. Contrarily to our above usage, we suppose that all variables occurring in $\mathbf{G}$ or $\mathbf{H}$ have been displayed explicitely; hence, we have $m \leqq n$. The combinatorial proposition attached to the operation $\mathbf{G}(x_1, \ldots, x_m) \to \to \mathbf{H}(x_1, \ldots, x_n)$ states, for any class A, the existence of a class B such that, for any sets $x_1, \ldots, x_n$, we have $\mathbf{H}(x_1, \ldots, x_n) \in B$ if and only if $\mathbf{G}(x_1, \ldots, x_m) \in A$. Denote $\Phi(x, x_1, \ldots, x_n)$ and $\Psi(x, x_1, \ldots, x_m)$ elementary formulae such that we have $\Phi(x, x_1, \ldots, x_n)$ if and only if $x = \mathbf{H}(x_1, \ldots, x_n)$ and $\Psi(x, x_1, \ldots, x_m)$ if and only if $x = \mathbf{G}(x_1, \ldots, x_m)$; and let us require that the class B should contain those and only those sets x for which, for appropriate sets $x_1, \ldots, x_n$, we have $x = \mathbf{H}(x_1, \ldots, x_n)$ and $\mathbf{G}(x_1, \ldots, x_m) \in A$. If we prove the existence of a class B satisfying this demand, we are ready, for, as easily proved, the sets $x_1, \ldots, x_n$ are uniquely determined by the set $x = \mathbf{H}(x_1, \ldots, x_n)$,[30]) hence, if $x = \mathbf{H}(x_1, \ldots, x_n)$, there cannot be sets $x'_1, \ldots, x'_n$ such that $x = \mathbf{H}(x'_1, \ldots, x'_n)$ and $\mathbf{G}(x'_1, \ldots, x'_m) \in A$, unless we have $\mathbf{G}(x_1, \ldots, x_m) \in A$.[31])

Now, the above requirement about the class B can be stated as follows: we have to have $x \in B$ if and only if for appropriate sets $x_1, \ldots, x_n$ and y we have $x = \mathbf{H}(x_1, \ldots, x_n)$, $y = \mathbf{G}(x_1, \ldots, x_m)$ and $y \in A$, i. e. if there are

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[29]) Here again we need induction in order to prove our assertion for arbitrary expressions $\mathbf{E}(x_1, \ldots, x_n)$ but not in order to prove it for a given expression $\mathbf{E}(x_1, \ldots, x_n)$.

[30]) Here we use that each of the variables $x_1, \ldots, x_n$ occurs really in $\mathbf{H}(x_1, \ldots, x_n)$. In this case, the unicity of $x_1, \ldots, x_n$ with $\mathbf{H}(x_1, \ldots, x_n) = x$ for given x is an immediate consequence of the fact that both components of a given ordered pair are uniquely determined and that given $\mathbf{H}(x_1, \ldots, x_n)$, we can get each of the sets $x_1, \ldots, x_n$ by iterated forming of one of the components of an ordered pair. For a formal proof of the unicity of the sets $x_1, \ldots, x_n$ with given $\mathbf{H}(x_1, \ldots, x_n)$ we need or do not need induction according as we want to prove it for arbitrary expressions $\mathbf{H}(x_1, \ldots, x_n)$ or for a given expression $\mathbf{H}(x_1, \ldots, x_n)$ only.

[31]) Here we use $m \leqq n$ for in case $m > n$, $x_{n+1}, \ldots, x_m$ would not be determined by $\mathbf{H}(x_1, \ldots, x_n)$.

sets $x_1, \ldots, x_n$ and y such that we have $\Phi(x, x_1, \ldots, x_n)$, $\Psi(y, x_1, \ldots, x_m)$ and $y \in A$. This condition can be written in the form

$$(Ex_1)\cdots(Ex_n)(Ey)(\Phi(x, x_1, \ldots, x_n) \,\&\, \Psi(y, x_1, \ldots, x_m) \,\&\, y \in A)$$

which is an elementary formula indeed; hence, the combinatorial proposition attached to the operation $\mathbf{G}(x_1, \ldots, x_m) \rightarrow \mathbf{H}(x_1, \ldots, x_n)$ is a consequence of a proposition which we wish to get fulfilled in set theory.

Thus, the following definition has been justified: we call the combinatorial proposition attached to an operation $\mathbf{G} \rightarrow \mathbf{H}$ a *false* or a *true combinatorial proposition* according as there is or there is not a variable occurring in the expression $\mathbf{G}$ but not in $\mathbf{H}$. E. g., B5, B6 and B7 are true combinatorial propositions and the same holds for $\mathbf{P}_{np}$ and $\mathbf{P}_{npq}$ for $n = 1, 2, \ldots$, $1 \leqq p, q \leqq n$ and $p \neq q$ but not for $p > n$, say.

7. Returning to the requirement (1) of section **5** to the effect that if $\Phi(x_1, \ldots, x_n)$ is either $x_p \in x_q$ with $1 \leqq p, q \leqq n$ and $p \neq q$ or $x_p \in \mathbf{y}$ with a class or set variable $\mathbf{y}$ different from $x_1, \ldots. x_n$ and $1 \leqq p \leqq n$, there is a class containing, for any sets $x_1, \ldots, x_n$, the sequence $\langle x_1, \ldots, x_n \rangle$ if and only if $\Phi(x_1, \ldots, x_n)$ holds, we see at once that it is the propositions $\mathbf{P}_{np}$ and $\mathbf{P}_{npq}$ that we are needing. Indeed, in the case $\Phi(x_1, \ldots, x_n)$ is $x_p \in x_q$, by axiom B1 a class A exists such that we have $\langle x_p, x_q \rangle \in A$ if and only if $x_p \in x_q$; and in the case $\Phi(x_1, \ldots, x_n)$ is $x_p \in \mathbf{y}$, $A = \mathbf{y}$ is a class such that we have $x_p \in A$ if and only if $x_p \in \mathbf{y}$. Hence, if propositions $\mathbf{P}_{npq}$ and $\mathbf{P}_{np}$ are available, we may infer the existence of a class B such that, for any sets $x_1, \ldots, x_n$, we have $\langle x_1, \ldots, x_n \rangle \in B$ if and only if $\langle x_p, x_q \rangle \in A$, that is, $x_p \in x_q$, and if and only if $x_p \in A$, that is, $x_p \in \mathbf{y}$, respectively, i. e., if and only if $\Phi(x_1, \ldots, x_n)$ holds.

Now, $\mathbf{P}_{np}$ and $\mathbf{P}_{npq}$ represent an infinity of propositions, therefore, we cannot subjoin them to the axioms A1 to A4 and B1 to B4 lest we should have an infinity of axioms. Hence, the question arises if there is a finite number of true combinatorial propositions implying the propositions $\mathbf{P}_{np}$ and $\mathbf{P}_{npq}$ for $n = 1, 2, \ldots$, $1 \leqq p, q \leqq n$, $p \neq q$. In the affirmative case, we can reach our object by subjoining these true combinatorial propositions to the axioms A1 to A4 and B1 to B4; of course, we try to choose them as simple as possible.

This question reduces to that solved in section **3** by the following remark. *Given any primary operations* $\mathbf{E}_1 \rightarrow \mathbf{F}_1, \ldots, \mathbf{E}_k \rightarrow \mathbf{F}_k$, *denote* $\mathbf{P}_1, \ldots, \mathbf{P}_k$ *the combinatorial propositions attached to them, respectively. If an operation* $\mathbf{G} \rightarrow \mathbf{H}$ *can be obtained as a derived operation, the combinatorial proposition* $\mathbf{P}$ *attached to it is a consequence*[32]) *of the propositions* $\mathbf{P}_1, \ldots, \mathbf{P}_k$.

[32]) Consequence is meant here in the sense of the restricted predicate calculus. As a matter of fact, if universal propositions are expressed by means of free variables rather then universal quantifiers, the only rules of inference to be used are the rule of substitu-

Indeed, let be $\mathbf{G}_1, \ldots, \mathbf{G}_n$ expressions such that $\mathbf{G}_1$ is $\mathbf{G}$, $\mathbf{G}_n$ is $\mathbf{H}$ and each of the operations $\mathbf{G}_1 \rightarrow \mathbf{G}_2, \ldots, \mathbf{G}_{n-1} \rightarrow \mathbf{G}_n$ arises from one of $\mathbf{E}_1 \rightarrow \rightarrow \mathbf{F}_1, \ldots, \mathbf{E}_k \rightarrow \mathbf{F}_k$ by means of replacing some of the variables by appropriate expressions. Then the corresponding combinatorial propositions $\mathbf{P}'_1, \ldots, \mathbf{P}'_n$, attached to the operations $\mathbf{G}_1 \rightarrow \mathbf{G}_2, \ldots, \mathbf{G}_{n-1} \rightarrow \mathbf{G}_n$, respectively, arise by the same replacements from the propositions $\mathbf{P}_1, \ldots, \mathbf{P}_k$, hence, each of them is a consequence of the propositions $\mathbf{P}_1, \ldots, \mathbf{P}_k$ (a particular case of one of them indeed). On the other hand, the proposition $\mathbf{P}$ is a consequence of the propositions $\mathbf{P}'_1, \ldots, \mathbf{P}'_n$. Indeed, let be $x_1, \ldots, x_m$ the variables figuring in at least one of the expressions $\mathbf{G}_1, \ldots, \mathbf{G}_n$. Then by $\mathbf{P}'_1$, for each class $A = A_1$, there is a class A_2 such that, for any sets $x_1, \ldots, x_m$, we have $\mathbf{G}_2 \in A_2$ if and only if $\mathbf{G}_1 \in A_1$; similarly, by $\mathbf{P}'_2$, there is a class A_3 such that, for any sets $x_1, \ldots, x_m$, we have $\mathbf{G}_3 \in A_3$ if and only if $\mathbf{G}_2 \in A_2$, and so on; finally, by $\mathbf{P}'_n$, there is a class $B = A_n$ such that, for any sets $x_1, \ldots, x_m$, we have $\mathbf{G}_n \in A_n$ if and only if $\mathbf{G}_{n-1} \in A_{n-1}$. Hence, for any sets $x_1, \ldots, x_m$, we have $\mathbf{H} \in B$ if and only if $\mathbf{G} \in A$, thus, the proposition $\mathbf{P}$ holds as a consequence of $\mathbf{P}_1, \ldots, \mathbf{P}_k$.

Owing to the theorem proved in section **3**, we get the result that axioms B5, B6 and B7, together with A1 to A4 and B1 to B4, suffice to have the propositions italicized in section **5**, for each elementary formula $\Phi(x)$ and $\Phi(x_1, \ldots, x_n)$, respectively, as consequences. For the same purpose, GÖDEL uses (loc. cit. [2])), besides the axioms B5, B6 and B7, the following axiom too:

B8. For any class A, there is a class B such that for any sets x, y and z, the ordered triple $\langle x, y, z \rangle$ is contained in B if and only if $\langle x, z, y \rangle \in A$.

This axiom is the combinatorial proposition attached to the operation $\langle x, z, y \rangle \rightarrow \langle x, y, z \rangle$, or, apart from notation, (5′) (see footnote [7])). Indeed, GÖDEL's procedure corresponds to the fact that given (3), (4), (5) as well as (5′) as primary operations, we can obtain every operation of the form (6) and (7) ($n = 1, 2, \ldots,\ 1 \leqq p, q \leqq n,\ p \neq q$) as a derived operation.

By what has been shown, we see not only that axiom B8 is superfluous for the purpose it has to serve but also that it is a consequence of the axioms A1 to A4 as well as B1 to B7.[33]) Indeed, it has been shown in section **6** that each true combinatorial proposition is a consequence of a pro-

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tion (of expressions for free variables), as well as a form of chain inference leading from propositions of the form $(A_1)(EA_2)(\Phi_1(A_1) \longleftrightarrow \Phi_2(A_2))$, $(A_2)(EA_3)(\Phi_2(A_2) \longleftrightarrow \Phi_3(A_3)), \ldots$, $(A_{n-1})(EA_n)(\Phi_{n-1}(A_{n-1}) \longleftrightarrow \Phi_n(A_n))$ to the proposition $(A_1)(EA_n)(\Phi_1(A_1) \longleftrightarrow \Phi_n(A_n))$, $\Phi_1, \ldots, \Phi_n$ being propositional functions.

[33]) GÖDEL remarked (loc. cit. [2]), p. 7, footnote [5])) that BERNAYS assumes a further axiom requiring the existence of the class of all sets of the form $\{x\}$, which allows B7 and B8 to be replaced by one axiom. From this remark it seems probable that GÖDEL guessed axiom B8 to be independent on his system of axioms. The same seems to hold for MARKOV too, who proved that axiom B6 is a consequence of axioms B4, B5 and B8 without remarking that on the other hand, B8 is a consequence of axioms A1 to A4 and B1 to B7; see A. Марков, О зависимости аксиомы B6 от других аксиом BERNAYS'a—

position of the form: there is a class such that, for any set x, we have $x \in A$ if and only if $\Phi(x)$ holds, $\Phi(x)$ being an elementary formula; in particular, this holds for axiom B8 too for it is a true combinatorial proposition. On the other hand, we have seen that each proposition of the last form is a consequence of the axioms A1 to A4 and B1 to B7; hence, the same holds for axiom B8.[34])

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Gödel'я, Известия Академии Наук СССР (сер. мат.) **12** (1948), 569—570. (Of course, from the result of this paper Markov's, together with that of the present paper, it does not follow that axioms B6 and B8 can be dispensed with at the same time, for to prove B6, Markov needs axiom B8 and to prove B8, we need axiom B6). — On the other hand, already the above remark Gödel's, together with that of Bernays' (loc. cit. [3]), Part VII, p. 94) according to which the axiom of his system referred to (stating the existence of the class of all sets of the form $\{x\}$) can be proved by means of the rest of his axioms made probable that the same holds for axiom B8 in the Gödel system.

[34]) For the proof of axiom B8 by means of the axioms A1 to A4 and B1 to B7, a *particular* proposition of the above form (with a *particular* elementary formula $\Phi(x)$) is needed only, hence, no inductive proof is necessary. Of course, we could prove B8 by means of the same axioms using the remark that B8 can be transformed directly to the form: there is a class B such that, for any sets x, y and z, we have $\langle x, y\ z\rangle \in B$ if and only if $\Phi(x, y, z)$ holds, $\Phi(x, y, z)$ denoting a *particular* elementary proposition containing, besides the free set variables x, y, z, the class variable A.