Complementation of abelian normal subgroups.

In memoriam Tibor Szele.

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If N is a normal subgroup of the finite group G, then a subgroup C of G is called a complement of N if $NC=G$ and $N\cap C=1$. Our object is to give a sufficient condition for the existence and conjugacy of complements in case N is abelian. By theorems of GASCHÜTZ [1] and D. G. HIGMAN [2] it is enough to consider the case of an abelian p-group N; and the result in this case is:

Theorem 1. *Let N be an abelian normal p-subgroup of the finite group G. Suppose that G has a p'-subgroup X such that NX is normal in G, but N_0X is not, if N_0 is any normal subgroup of G properly contained in N. Then N is complemented in G, and any two complements of N are conjugate.*

(A p'-subgroup is one whose order is prime to p.)

We begin the proof with a well-known lemma from representation theory.

Lemma. *Suppose that the abelian p-group N admits the p'-group Ω as a group of operators. Then N is a direct product $S\times T$, where S is the subgroup of elements of N stable under Ω, and T admits any group Γ of operators on N which contains Ω as a normal subgroup.*

If n is the order of Ω, then n is prime to p, and so the endomorphism ring of N contains in its centre an inverse n^{-1} of the endomorphism $g\to g^n$. Thus the endomorphism ring contains the idempotent $\sigma=n^{-1}\Sigma\eta(g)$, where summation is over all elements of Ω, and $\eta(g)$ is the endomorphism of N induced by the element g of Ω. Thus $N=S\times T$, with $S=N^{\sigma}$, $T=N^{1-\sigma}$. Evidently S is the subgroup of elements of N stable under Ω; and if t belongs to Γ, transformation by t permutes the elements of Ω, so that $\sigma t=t\sigma$, and

$$T^t=N^{(1-\sigma)t}=N^{t(1-\sigma)}=N^{1-\sigma}=T.$$

Now let G, N, X be as in the theorem, and let C be the normaliser of X in G. Let g be any element of G, and put $Y = gXg^{-1}$. Because NX is normal in G, Y is contained in NX; and because its order is equal to that of X, and prime to that of N, it is a complement of N in NX. But N is abelian and of order prime to its index in NX, and we may therefore apply a theorem of SCHUR (cf. ZASSENHAUS [4] p. 132) to deduce that Y is conjugate to X in NX. Thus for some n in N and x in X, $gXg^{-1} = Y = nxXx^{-1}n^{-1} = nXn^{-1}$, so that $n^{-1}g$ belongs to C and g to NC. That is $NC = G$.

We now apply the Lemma, taking Γ to be G/N and Ω to be NX/N, both groups acting on N by transformation. The conclusion is that $N = S \times T$, where T is normal in G. and S is the centraliser of X in N. We observe that in fact $S = N \cap C$. That $S \subset N \cap C$ is obvious. Conversely, both $N \cap C$ and X are normal in C, and they have coprime orders. Thus they generate their direct product, and $N \cap C$ belongs to the centraliser of X. Hence TX is contained in the centraliser of S, and is therefore normal in $G = STC$. By the hypothesis of the theorem, this implies that $T = N$, so that $S = 1$. That is $N \cap C = 1$, and C is a complement of N.

Lastly, let D be another complement of N. Then $Y = NX \cap D$ is a complement of N in NX, and so, by an argument used already, is conjugate to X, say $Y = tXt^{-1}$. D is a subgroup of the normaliser of Y, and so $t^{-1}Dt$ is a subgroup of the normaliser of X, that is, of C. But $t^{-1}Dt$ is a complement of N, since D is, and so must be the whole of C. This proves the theorem.

We give next the corresponding condition when N is not necessarily a p-group.

Theorem 1a. *Let N be an abelian normal subgroup of the finite group G. Suppose that, for each prime p, there is a p'-subgroup X of G such that NX is normal in G, but not N_0X for any normal subgroup N_0 of G, properly contained in N and of p-power index in N. Then N is complemented in G, and any two complements of N are conjugate.*

The condition is, of course, trivially satisfied, by $X = 1$, for any prime which does not divide the order of N, since there are then no subgroups N_0. If the condition is satisfied, and $N = P \times Q$, where P is a p-group and Q a p'-group, then it permits the application of Theorem 1 to show that N/Q is complemented in G/Q, and that any two complements of G/Q are conjugate. This being true for all primes p, the theorem follows from the results of GASCHÜTZ [1] and D. G. HIGMAN [2].

Next, suppose that G and N satisfy the conditions of Theorem 1, for some subgroup X, and let M/N be the greatest normal p'-subgroup of G/N. By the theorem of SCHUR, N has a complement Y in M, which we may suppose contains X. Then the conditions of Theorem 1 still hold if we replace

X by Y. For first, $NY = M$ is normal in G by assumption. But secondly, if, for any normal subgroup N_0 of G properly contained in N, N_0Y is normal, so is its intersection N_0X with NX. If H is any group containing both G and N as normal subgroups, NY/N, as a characteristic subgroup of G/N is normal in H/N, and so NY is normal in H. But if N_0 is a normal subgroup of H properly contained in N, N_0Y is not normal in H, since it is not even normal in G. Thus N is complemented in H, and two complements of N in H are conjugate. A similar argument applies to Theorem 1a, and we may therefore state:

Theorem 2. *If G and N satisfy the hypotheses either of Theorem* 1 *or of Theorem* 1a, *then N is complemented in any group H which contains both G and N as normal subgroups, and any two complements of N in H are conjugate.*

As examples of these theorems we consider the following construction. Let E be an equivalence relation defined on the rational primes. Call a group G *E-decomposable* if it is a direct product $G_1 \times G_2 \times \ldots \times G_r$, where two primes dividing the order of the same G_i are equivalent under E. Thus if E is the relation of equality, E-decomposable is the same as nilpotent. If G is a finite group we denote by $K(G)$ the intersection of all normal subgroups of G whose factor groups are E-decomposable. $K(G)$ is evitently a characteristic subgroup of G; and since direct products and subgroups of E-decomposable groups are again E-decomposable, $G/K(G)$ is E-decomposable, but G/L is not if L is a normal subgroup of G properly contained in $K(G)$.

We shall show that if $N = K(G)$ is abelian, the conditions of Theorem 1a are satisfied. If p is a prime, we can write $N = P \times Q$, where P is a p-group and Q is a p'-group, because N is abelian; and we can write $G/N = A/N \times B/N$, where the order of A/N is divisible only by the primes equivalent to p, and the order of B/N only by the remaining primes, because G/N is E-decomposable. P is a characteristic subgroup of N and so a normal subgroup of G, and its order is prime to its index in B. By the theorem of SCHUR, P has a complement X in B. Then X is a p'-group, and $B = PX = NX$ is normal. Suppose, if possible, that N_0 is a normal subgroup of G, properly contained and of p-power index in N, such that N_0X is normal. Then N_0 is of p-power index in N and of prime-to-p index in N_0X, and so is the intersection of these two groups. Thus $N_0X/N_0 = N_0X/N \cap N_0X \cong NX/N = B/N$, which is an E-decomposable group whose order is divisible by no prime equivalent to p. But the order of A/N_0 is divisible only by primes equivalent to p. Hence these two normal subgroups of G/N_0 generate their direct product, and this product is E-decomposable. Since $G = AB = ANX = AX$, this product is the whole of G/N_0, which is not E-decomposable. Thus our assumption that such a group as N_0 exists leads to a contradiction, which shows that N satisfies the condition of Theorem 1a.

By Theorem 2, $K(G)$, if abelian, is complemented in any group containing G as a normal subgroup. In particular we have:

Theorem 3. *If $K^r(G)$ is abelian for any integer r it is complemented, and any two complements are conjugate.*

The special case when $r=1$ and E is equality was proved by SCHENKMAN [3] by quite different methods.

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