On analytic half-groups of complex numbers.

To the memory of Professor Tibor Szele.

By MIKLÓS HOSSZÚ in Miskolc.

Let $F(x, y)$ be a univalent binary operation defined on a connected (not necessarily simply-connected and bounded) domain D of complex numbers ($x, y, F \in D$). D is said to form an analytic half-group with the operation $F(x, y)$, if $F(x, y)$ is differentiable and the associative law

$$F[F(x, y), z] = F[x, F(y, z)] \tag{1}$$

is satisfied for each $x, y, z \in D$. A KUWAGAKI [1] [1]) has proved that, under the supposition $F(0,0) = 0$ [2]), $F(x, y)$ belongs to one of the following four categories of functions:

$$F = x,\ F = y,\ F = x + y + xyG(x, y),\ F = xyH(x, y)$$

where G, H are holomorphic at $(0,0)$ and symmetric in x and y, and either $H(0,0) \neq 0$ or $H(x, y) \equiv 0$.

In this paper we shall treat the functional equation (1) without the restriction $F(0,0) = 0$. We shall denote the partial differential quotients of the function $F(x, y) = x \circ z$ by indices:

$$F_1(x, y) = \frac{\partial F}{\partial x},\ F_2(x, y) = \frac{\partial F}{\partial y}$$

and the set of left and right annullators by A_l resp. A_r, which we define here by:

$$A_l = \underset{x}{S}(x \circ y = \text{const. for all } y \in D),$$

$$A_r = \underset{x}{S}(y \circ x = \text{const. for all } y \in D).$$

We shall denote further the union of A_l and A_r by A:

$$A = A_l + A_r,$$

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1) The numbers in brackets refer to the Bibliography at the end of this paper.

2) A. KUWAGAKI has supposed only the existence of a number c (finite or infinite) with the property $F(c, c) = c$, and has shown that c can be carried into zero by a simple transformation.

further, the annullators (and at the same time zeros) of $F_1(x, y)$ and $F_2(x, y)$ by A_1 resp. A_2:

$$A_1 = \underset{x}{S}[F_1(x, y) = 0 \text{ for all } y \in D],$$

$$A_2 = \underset{y}{S}[F_2(x, y) = 0 \text{ for all } x \in D].$$

We prove the following:

Theorem. *Every analytic half-group D of complex numbers (where D is a domain) with the operation $F(x, y) = x \circ y$ is either locally isomorphic to the additive half-group everywhere in D except at the isolated points of the set $A + A_1 = A + A_2$ or the operation $F(x, y)$ is one of the following three degenerated operations:*

$$F = c, F = x, F = y, \tag{2}$$

i. e., every analytic function $F(x, y) = x \circ y$ satisfying the functional equation (1) is either one of the functions (2) or

$$f(x \circ y) = f(x) + f(y), \quad x, y \in D - A\,^{3)} \tag{3}$$

holds, where

$$f(x) = \int_{z_1}^{x_0 \circ x} \frac{F_2(x_0, z)}{F_1(x_0, z)} dz = \int_{z_2}^{x \circ y_0} \frac{F_1(z, y_0)}{F_2(z, y_0)} dz, \; z \in D - A \tag{4}$$

is an analytic function the derivative of which has on $D - A$ the set of zeros $A_1 - A = A_2 - A$ containing only isolated points. In (4) $x_0, y_0, z_1, z_2 \in D - A - A_1 = D - A - A_2$ are arbitrary constants and the function $f(x)$ is uniquely determined on $D - A$ up to a constant factor.

We shall make use of the following

Lemma. *If the analytic function $F(x, y) = x \circ y$ satisfies (1), then, using the above notations,*

I. *$A = D$ is equivalent to $F = c$ or $F = x$ or $F = y$;*
$A \subset D$ implies that $A + A_1 + A_2$ has no point of accumulation and vice versa;

II. *$x \in A_l$ implies $x \circ y \in A_l$ and $y \in A_r$ implies $x \circ y \in A^r$;*
$x \circ y \in A$ implies $x \in A$ or $y \in A$;
$A \subset D$ implies $A_l = A_r$;

III. *$x \circ y \in A_1$ for an $y \in D - A$ implies $x \in A_1$;*
$x \circ y \in A_2$ for an $x \in D - A$ implies $y \in A_2$;

IV. *$A \subset D$ implies $F_1(x, y) \neq 0$ for every $x \in D - A_1, y \in D - A$ and $F_2(x, y) \neq 0$ for every $x \in D - A, y \in D - A_2$.*

[3]) $D - A$ consists of the points contained in D but not in A.

PROOF OF THE LEMMA.

Let $A = D$, then A_l or A_r contains at least one point of accumulation. Since $F(x, y)$ is analytic, this involves

$$F_1(x, y) \equiv 0 \quad \text{or} \quad F_2(x, y) \equiv 0,$$

consequently $F(x, y)$ is independent from x or independent from y. In the case $F(x, y) = f(x)$ we have by (1)

$$f[f(x)] = f(x).$$

From this it obviously follows that the analytic function $f(x)$ is identically constant or $f(x) = x$, hence $F = c$ or $F = x$. We conclude $F = y$ or $F = c$ similarly for the case where $F(x, y)$ does not depend on x. Thus one implication of (I_1) is proved, the other is obvious.

(I_2) is obvious by the analyticity of $F(x, y)$ and by (I_1). (I_2) states that $A + A_1 + A_2$ contains only isolated points, if (and only if) $A \subset D$.

In order to prove (II) we consider

$$(x \circ y) \circ z = x \circ (y \circ z). \tag{1}$$

If $x \in A_l$, then $x \circ (y \circ z)$ does not depend on z, hence $(x \circ y) \circ z$ is also constant. Thus $x \circ y \in A_l$. A similar conclusion can be drawn for $y \in A_r$. Conversely, if $x \circ y \in A$, e. g. if $x \circ y \in A_l$, then $(x \circ y) \circ z$ is independent from z, hence also $x \circ (y \circ z)$. Thus by the analyticity it is easy to see that $x \in A_l$ or $y \in A_l$ and similarly $x \in A_r$ or $y \in A_r$, if $x \circ y \in A_r$.

Now, let y be, for example, a left annullator: $y \in A_l$. Then $x \circ (y \circ z)$ and also $(x \circ y) \circ z$ is independent from z, hence $x \circ y \in A_l$ for all $x \in D$. Since $x \circ y$ is analytic and $A \subset D$, this involves, by (I_2), that $x \circ y$ is constant, i. e. $y \in A_r$. Thus $A \subset D$ implies $A_l \subset A_r$ and similarly also $A_r \subset A_l$. This proves (II).

To prove (III), we derive (1) with respect to x:

$$F_1[F(x, y), z]\, F_1(x, y) = F_1[x, F(y, z)]. \tag{5}$$

$x \circ y \in A_1$ implies that $F_1[F(x, y), z]$ vanishes identically for all $z \in D$, hence, by (5), also $F_1[x, F(y, z)] = 0$. If $y \in D - A$, then $y \circ z$ depends on z, hence $F_1(x, t) = 0$ holds for all $t = y \circ z$, moreover, by the analyticity, also for every $t \in D$. Thus $x \in A_1$. The rest of (III) can be proved similarly, since (1) makes no distinction between the first and second variable of $F(x, y)$.

Finally, we prove (IV) by showing that $F_1(a, b) = 0$ for a $b \in D - A$ implies that $a \in A_1$, i. e., that $F_1(a, y) = 0$ holds for all $y \in D$. Indeed, from (5) it follows for $x = a, y = b$ that $F_1[a, F(b, z)]$ vanishes for all $z \in D$. So $F_1(a, t) = 0$ holds for all $t = F(b, z)$, hence for all $t \in D$, since $F(x, y)$ is analytic and $b \notin A$. One can prove the remaining part of (IV) similarly.

PROOF OF THE THEOREM. We shall treat only the case where $F(x, y)$ is not one of functions (2), or, what is the same by (I_1), the case where $A \subset D$ holds.

Differentiating (1) with respect to y, we get

$$F_1[F(x,y),z]\,F_2(x,y)=F_2[x,F(y,z)]\,F_1(y,z). \tag{6}$$

Let us keep $x=x_0\in D-A-A_1$ constant; such a value exists according to (I_2). Then, with the notation

$$g'(t)=\frac{F_2(x_0,t)}{F_1(x_0,t)},\; t\in D-A, \tag{7}$$

from (5) and (6)

$$g'(y)=g'[F(y,z)]\,F_1(y,z)$$

follows for all $y,z\in D-A$, since, by (5), (IV) and (II_2), we have

$$F_1[F(x_0,y),z]\neq 0,\; x_0\in D-A-A_1$$

for all $y,z\in D-A$. Taking into account (IV), $g'(t)$ is an analytic function different from zero for all $t\in D-A-A_2$. Let us choose a simply-connected domain in $D-A$ which contains the points $y, z, y\circ z$ and z_0 fixed for the time being. By integrating in this domain over a Jordan curve departing from z_0, we obtain

$$g(y\circ s)=g(y)+f(z),$$

where also $f(z)$ is an analytic function with non-zero derivative on $D-A-A_2$, since

$$f'(z)=g'[F(y,z)]F_2(y,z)=g'[F(x_0,z)]F_2(x_0,z)\neq 0,\; z\in D-A-A_2. \tag{8}$$

By (I_2) and (II_2) the domain containing the curve of integration can be choosen so that it contains the arbitrarily fixed points $x, y, z, x\circ y, y\circ z\in D-A$. Substituting both sides of the equation (1) into $g(t)$, we have

$$g(x)+g(y)+f(z)=g(x)+f(y\circ z)$$

and this proves (3).

We get the explicit form (4) of $f(x)$ from (8) and (7):

$$f(x)=\int\limits_{z_0}^{x}\frac{F_2[x_0,F(x_0,z)]}{F_1[x_0,F(x_0,z)]}\,F_2(x_0,z)\,dz=\int\limits_{z_1}^{x_0\circ x}\frac{F_2(x_0,z)}{F_1(x_0,z)}dz,\; z\in D-A.$$

Here $z_1=x_0\circ z_0\in D-A-A_1$ holds by (IV) since $x_0\notin A_1$ and $z_0\in D-A$. The second part of (4) can be obtained similarly. One sees by (IV) and (III) that

$$f'(x)=\frac{F_2[x_0,F(x_0,x)]}{F_1[x_0,F(x_0,x)]}\,F_2(x_0,x)=\frac{F_1[F(x,y_0),y_0]}{F_2[F(x,y_0),y_0]}\,F_1(x,y_0)=0$$

holds on $D-A$ if (and only if) $x\in A_1$ but at the same time also if (and only if) $x\in A_2$. This implies

$$A_1-A=A_2-A,$$
$$A+A_1=A+A_2.$$

Finally, we verify that $f(x)$ is uniquely determined on $D-A$ up to a constant factor. Let $f(x)$ and $h(x)$ be two functions with the property

$$f(x\circ y)=f(x)+f(y),\; h(x\circ y)=h(x)+h(y),\; x,y\in D-A.$$

Keeping $x, y \in D-A-A_1$ constant, there exists a neighbourhood of $x \circ y$ in which $f(x)$ and $h(x)$ have non-zero derivatives. Therefore $f(x)$ and $h(x)$ are invertable in a neighbourhood of $x \circ y$ and here

$$x \circ y = f_{x\circ y}^{-1}[f(x)+f(y)] = h_{x\circ y}^{-1}[h(x)+h(y)]$$

holds, where the index $x \circ y$ reminds of the inverse function being defined only in a neighbourhood of $x \circ y$. We use the notations h_x^{-1} and h_y^{-1} in the same sense. We find

$$f[h_x^{-1}(X)]+f[h_y^{-1}(Y)] = f[h_{x\circ y}^{-1}(X+Y)].$$

Denoting here

$$\varphi(z) = f[h_x^{-1}(z)],\ \psi(z) = f[h_y^{-1}(z)],\ \chi(z) = f[h_{x\circ y}^{-1}(z)],$$

we get a generalized functional equation of CAUCHY's type:

$$\varphi(X)+\psi(Y) = \chi(X+Y).$$

It is easy to see, e. g. by differentiating with respect to X, that

$$\varphi(X) = cX+k,$$

or, what is the same,

$$f(z) = ch(z)+k$$

holds in a neighbourhood of x, consequently, in the whole domain $D-A$, the analytic continuation being uniquely determined.

But

$$cf(x \circ y)+k = cf(x)+k+cf(y)+k$$

holds only if $k=0$, and this completes the proof of our theorem.

Corrollary. *Every Lie-group of complex numbers is abelian.* [4])

PROOF. Let e be the identity of the group operation $F(x, y)$, then

$$F_1(x, e) = F_2(e, y) = 1.$$

Hence, there exists a neighbourhood of e on which (3) holds with $f'(z) \neq 0$. Consequently, there exists a neighbourhood H of e such that $w=f(z)$ has a uniquely determined inverse function $z=f^{-1}(w)$ for all $z \in H$. Let us consider the set K with $K^2 \subseteq H$. Such a set K, containing all elements x, y for which $F(x, y) \in H$ holds, obviously exists. Thus, taking (3) into account, we have

$$F(x, y) = f^{-1}[f(x)+f(y)] = F(y, x),\ x, y \in K. \tag{9}$$

A theorem of O. SCHREIER [3] states that every continuous and connected group is uniquely determined by an arbitrary little neighbourhood of the identity, i. e., all elements of the group can be represented by a finite number of elements contained in an arbitrarily little but fixed neighbourhood of

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[4]) A similar theorem is stated for formal Lie-groups over an arbitrary field with zero characteristic by M. LAZARD [2].

the identity. Hence, by (1) and (9), we conclude

$$x \circ y = \prod_{i=1}^{n} {}^{\circ}\, x_i \circ \prod_{i=1}^{m} {}^{\circ}\, y_i = \left(\prod_{i=1}^{n-1} {}^{\circ}\, x_i\right) \circ x_n \circ y_1 \circ \prod_{i=2}^{m} {}^{\circ}\, y_i =$$

$$= \left(\prod_{i=1}^{n-1} {}^{\circ}\, x_i\right) \circ y_1 \circ x_n \circ \prod_{i=2}^{m} {}^{\circ}\, y_i = \cdots = \prod_{i=1}^{m} {}^{\circ}\, y_i \circ \prod_{i=1}^{n} {}^{\circ}\, x_i = y \circ x$$

and this proves the corollary.

Remarks. 1. In the case where $A \subset D$, A^2 contains at most one point c; namely, by (I_2) $a_i \circ a_k = c$ is constant for any fixed $a_i \in A$ resp. for any fixed $a_k \in A$. This c has the property $c \circ c = c$.

2. L. KOVÁCS has shown by a symple example that the analyticity is a necessary condition for proving (II).

3. The method is applicable also in solving the generalized functional equation of transformation:

$$F[F(x, y), z] = F[x, G(y, z)].$$

Here we have the formulae

$$f[F(x, y)] = f(x) + g(y), \quad g[G(x, y)] = g(x) + g(y)$$

similar to (3).

4. The corollary stated above is an easy consequence of the Lie-theory, but the present proof is more elementary and does not make use of any apparatus beyond the well-known theorem of SCHREIER on continuous groups.

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