# A note on regular rings.

To the memory of my beloved teacher Professor Tibor Szele.

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It is well known, that in a regular[1]) ring $R$ every left (right) ideal $I$ is idempotent (i. e. $I^2 = I$). The question, whether or not this is a characteristic property of the regular rings, seems to be of some interest. In this little note we give three theorems related to this problem. In Theorem 1 we characterize the regular rings by a condition for one-sided ideals, which in the commutative case is equivalent to the idempotency of the ideals in $R$. In Theorem 2 we solve an analogous problem, the result shows that the regular rings without nonzero nilpotent elements are characterized by the idempotency of their quasi-ideals. Applying this result and a theorem of A. KERTÉSZ we get Theorem 3, which is a criterion for decomposibility of rings into a direct sum of division rings. — I am indebted to A. KERTÉSZ for his valuable help.

The concept of the quasi-ideal was introduced by O. STEINFELD in [5]. A submodule $M$ of the ring $R$ is said to be a quasi-ideal if $RM \cap MR \subseteq M$. Elementary facts connected with this concept: a quasi-ideal is a subring, but not every subring is a quasi-ideal; the intersection of one-sided ideals is always a quasi-ideal; in the presence of a one-sided unity every quasi-ideal is intersection of one-sided ideals, etc. These and further results on quasi-ideals are to be found in [6] and [7].

In an arbitrary ring $R$

(1) $$JL \subseteq J \cap L$$

holds for any right-ideal $J$ and any left-ideal $L$ of $R$. Our first theorem characterizes the rings in which equality holds in (1) in every case. Namely, we prove the following

**Theorem 1.** *An arbitrary ring $R$ is regular if and only if*

(2) $$JL = J \cap L$$

*holds for every right-ideal $J$ and left-ideal $L$ of $R$.*

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[1]) In the sense of J. VON NEUMANN [4]. — Numbers in brackets refer to the bibliography at the end of this note.

First we assume (2) and show that $R$ is regular. Let $a$ be an arbitrary element of $R$. The right-ideal generated by $a$ is the set $[an+ar]$ of all elements of the form $an+ar$ ($n$ integer, $r\in R$). By (2)

$$[an+ar]=[an+ar]\cap R=[an+ar]R=aR$$

and so $a\in aR$. Analogously $a\in Ra$ and hence

$$a\in aR\cap Ra=aR^2a,$$

i. e. $a=axa$, $R$ is regular.

The converse statement is almost trivial. Let $R$ be a regular ring, by (1) we have only to show that any element $a$ of $J\cap L$ is in $JL$. From $a=axa$, $a\in J$, $xa\in L$ we conclude $a\in JL$.

In what follows we shall give a characterization of regular rings without nonzero nilpotent elements, in terms of quasi-ideals.

**Theorem 2.** *For an arbitrary ring $R$ the following conditions are equivalent:*

*α) $R$ is a regular ring without nonzero nilpotent elements;*

*β) every quasi-ideal of $R$ is idempotent;*

*γ) for every right-ideal $J$ and every left-ideal $L$ of $R$*

$$JL=J\cap L\subseteq LJ$$

*holds;*

*δ) $R$ is regular and isomorphic to a subdirect sum of division rings.*[2])

*α) implies β).* Let $M$ be a quasi-ideal of $R$ and $a$ an arbitrary element of $M$. Since $M$ is a subring, $M^2\subseteq M$ and so we have only to prove $M\subseteq M^2$, i. e. $a\in M^2$. By the regularity of $R$ we have $a=axa$. Here $xa$ is an idempotent and so, since $R$ has no nonzero nilpotents, $xa$ is in the center of $R$. Using also $MR^2M\subseteq MR\cap RM\subseteq M$ we get

$$a=(ax)a(xa)=(ax)(xa)a=(ax^2a)a\in MR^2M\cdot M\subseteq M^2,$$

qu. e. d.

*β) implies γ).* Let $M$ and $N$ denote quasi-ideals in $R$, then $M\cap N$ is also a quasi-ideal. By the idempotency of $M\cap N$ we have

$$M\cap N=(M\cap N)^2\subseteq MN\cap NM.$$

On the other hand

$$MN\cap NM\subseteq MR\cap RM\subseteq M$$

analogously $MN\cap NM\subseteq N$, and so $M\cap N=MN\cap NM$.

Now let $J$ be a right-ideal, $L$ a left-ideal in $R$. Since a one-sided ideal is always a quasi-ideal, we have $J\cap L=JL\cap LJ$, but $JL\subseteq J\cap L$ and so $JL=J\cap L\subseteq LJ$.

[2]) The equivalence of α) and δ) has been proved by A. Forsythe and N. H. McCoy in [2]; we prove it here only for completness' sake.

*$\gamma$) implies $\delta$).* Let $R$ be a subdirect sum of the subdirectly irreducible rings $R_1, \ldots, R_\nu, \ldots$.[3]) Since every $R_\nu$ is a homomorphic image of $R$, all the $R_\nu$'s have property $\gamma$). Thus, by Theorem 1, $R$ and all the $R_\nu$'s are regular. Suppose, that one of the $R_\nu$'s has divisors of zero, i. e. for some nonzero elements $a$ and $b$ of $R_\nu$ we have $ab=0$. Then by $\gamma$)

$$bR_\nu \cdot R_\nu a = bR_\nu \cap R_\nu a \subseteqq R_\nu a \cdot bR_\nu = 0$$

and so

$$R_\nu bR_\nu \cap R_\nu aR_\nu = R_\nu bR_\nu \cdot R_\nu aR_\nu = 0$$

where, by $a=(ax)a(xa) \in R_\nu aR_\nu$ and $b=(by)b(yb) \in R_\nu bR_\nu$, none of these ideals is 0. This contradicts to the supposition that $R_\nu$ is subdirectly irreducible, and so we have that all the $R_\nu$'s are regular rings without divisors of zero, i. e. all the $R_\nu$'s are division rings.

That $\delta$) *implies* $\alpha$) does not need a proof.

**Corollary.** *A commutative ring R is regular if and only if every ideal is idempotent in R.*

This follows immediately from Theorem 2, since in a commutative ring every quasi-ideal is an ideal, and a commutative regular ring can have no nonzero nilpotents. Our statement is also a simple corollary of Theorem 1, since (2) in the case $J=L$ implies the idempotency of the ideals, and the proof of the converse statement is analogous to that of $\beta$) implies $\gamma$) in Theorem 2.

As an application of Theorem 2 we have

**Theorem 3.** *A ring R is a direct sum*[4]) *of division rings if and only if R satisfies the descending chain condition for principal (two-sided) ideals and every quasi-ideal of R is idempotent.*[5])

The proof is based on the following (not yet published) theorem of A. KERTÉSZ:

Let $\varphi$ be a property defined for simple rings. A ring $R$ is a direct sum of simple rings with property $\varphi$ if and only if $R$ satisfies the descending chain condition for principal ideals and contains a system of its maximal ideals $M_\nu$, the intersection of which is 0 and for which all the factor rings $R/M_\nu$ have property $\varphi$.

Since a direct sum $R$ of division rings is a regular ring without nonzero nilpotent elements, by Theorem 2 every quasi-ideal of $R$ is idempotent;

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[3]) According to a theorem of G. BIRKHOFF [1] such a representation always exists.

[4]) By a direct sum we always mean a two-sided (ring-theoretical) discrete direct sum, but we use the term "subdirect sum" in the usual sense, i. e. for certain subrings of the complete direct sum.

[5]) Another criterion is given by A. GERTSCHIKOFF [3].

moreover it is evident, that $R$ satisfies the descending chain condition for principal ideals. Conversely, if $R$ satisfies the conditions of Theorem 3, then by Theorem 2 it is a subdirect sum of division rings, i. e. there exists a system of maximal ideals $M_\nu$ in $R$, the intersection of which is 0 and for which every factor ring $R/M_\nu$ is a division ring. So we have by the above theorem of A. KERTÉSZ (in our case property $\varphi$ characterizes the division-rings) that $R$ is a direct sum of division rings.

REMARK. I am indebted to Professor L. FUCHS who has kindly directed my attention to the fact that

*A ring $R$ is a finite direct sum of division rings if and only if it has no nilpotent quasiideals and satisfies the descending chain condition for quasi-ideals.*

This is an immediate consequence of the WEDDERBURN—ARTIN structure theorem, and also of GERTSCHIKOFF's theorem [3].

## Bibliography.

[1] G. BIRKHOFF, Subdirect unions in universal algebra, *Bull. Amer. Math. Soc.* **50** (1944) 764—768.

[2] A. FORSYTHE and N. H. McCOY, On the commutativity of certain rings, *Bull. Amer. Math. Soc.* **52** (1946), 523—526.

[3] A. GERTSCHIKOFF, Über Ringe, die in eine Summe von Körpern zerlegbar sind, *Mat. Sbornik N. S.* **7** (1940), 591—597 (Russian with German summary).

[4] J. VON NEUMANN, On regular rings, *Proc. Nat. Acad. Sci. U. S. A.* **22** (1936), 707—713.

[5] O. STEINFELD, On ideal-quotients and prime ideals, *Acta Math. Acad. Sci. Hungar.* **4** (1953), 289—298.

[6] O. STEINFELD, Gyűrűk és félcsoportok ideálelméletéhez, *Kandidátusi disszertáció* (1955).

[7] O. STEINFELD, Bemerkung zu einer Arbeit von T. Szele, *Acta Math. Acad. Sci. Hungar.* **6** (1955), 479—484.

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