# Boundaries of domains of attraction

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## Introduction

If $f(z)$ is an entire or rational function, the iterates $f_n(z)$ $(n=1, 2, \dots)$ are defined inductively by

$$f_1(z)=f(z) \quad \text{and} \quad f_{n+1}(z)=f(f_n(z))$$

and are entire or rational respectively according as $f(z)$ is.

A first order fixpoint of $f(z)$ is a zero of $f(z)-z=0$. The *multiplier* of the fixpoint $\alpha$ is $f'(\alpha)$. A fixpoint $\alpha$ is said to be *attractive* or *non-attractive* according as $|f'(\alpha)|<1$ or $|f'(\alpha)|\geqq 1$ respectively.

The main object of study in the global iteration theory is (see e.g. [2]) the subset $\mathfrak{F}$ of the plane where the sequence $\{f_n(z)\}$ is not normal. This definition implies that the complement $C\mathfrak{F}$ of is open and hence $\mathfrak{F}$ is closed. If $f$ is non-bilinear then $\mathfrak{F}$ is perfect and non-empty.

In [2] is proved,

Result A. *If the set $\mathfrak{F}$ has an interior point, then it is identical with the extended plane.*

From the definition of attractive fixpoints follows

Result B. *The set of attractive fixpoints belongs to $C\mathfrak{F}$.*

Thus with an attractive fixpoint $\alpha$ there is an associated domain of normality of $\{f_n(z)\}$ in which it can be easily shown that $\lim_{n\to\infty} f_n(z)\to\alpha$. For such a domain we introduce the following definition.

Definition 1. The immediate domain of attraction $D_\alpha$ of a first order attractive fixpoint $\alpha$ is the maximal domain of normality of $\{f_n(z)\}$ which contains $\alpha$. In $D_\alpha$ we have $\lim_{n\to\infty} f_n(z)=\alpha$.

FATOU [2] proved that if $f(z)$ is a rational function and $\alpha$ a first order attractive fixpoint of $f(z)$, then there always exists a first order fixpoint on the boundary $\partial D_\alpha$ of $D_\alpha$. Since $\partial D_\alpha$ is in $\mathfrak{F}$ this fixpoint is non-attractive. Further it is shown that this non-attractive fixpoint is an accessible boundary point of $D_\alpha$. A simple example is provided by $f(z)=z^2$ which has the attractive fixpoint $\alpha=0$ for which $D_\alpha$ is $\{|z|<1\}$, $\mathfrak{F}$ is $|z|=1$ and 1 is a non-attractive fixpoint.

When $f(z)$ is entire transcendental Fatou's result is no longer generally true. In fact an entire transcendental function may not have any non-attractive fixpoint of first order at all, provided the rate of growth of the function is not too small. Indeed WHITTINGTON [3] has proved the following theorem [3, p. 534].

Theorem C. *For every $t<0$ there exists an entire function of order $\frac{1}{2}$ and type $t$ for which there is no non-attractive fixpoint of first order and infinity of first order attractive fixpoints.*

In this paper we prove

Theorem D. *Let $f(z)$ be entire transcendental and $\alpha$ be a first order attractive fixpoint of $f(z)$ such that $D_\alpha$ is bounded. Then there exists a first order fixpoint on the boundary $\partial D_\alpha$ of $D_\alpha$, which is accessible from the interior of $D_\alpha$.*

While for a rational function it is clear that the immediate domain of attraction of an attractive fixpoint is bounded, an entire function may have unbounded domains of attraction. Theorem D shows that if all the first order fixpoints of an entire transcendental function are attractive the the immediate domains of attraction of each of these fixpoints must be unbounded. This is the case e.g. for Whittington's examples.

There are indeed examples of entire transcendental functions having an attractive fixpoint $\alpha$ for which $D_\alpha$ is bounded. BAKER [1] has constructed an entire function $g(z)$ for which $g(0)=0$, $g'(0)=0$, so that 0 is an attractive fixpoint and in its immediate domain of attraction $g_n(z)\to 0$. However, there are annuli $R<|z|<R'$ in which $g_n(z)\to\infty$, and so $D_0$ lies completely inside $|z|<R$.

## Proof of theorem *D*

We have $f(\alpha)=\alpha$, $|f'(\alpha)|<1$, $f(D_\alpha)\subset D_\alpha$, $D_\alpha$ bounded.

Take the brach or branches of $f_{-1}(z)$ such that $f_{-1}(\alpha)=\alpha$. Under analytic continuation within $D_\alpha$, these branches always yield values $w=f_{-1}(z)$ lying in $D_\alpha$, since $\mathfrak{F}$ and $C\mathfrak{F}$ are completely invariant in the sense that $z\in\mathfrak{F}$ implies $f(z)\in\mathfrak{F}$ and $z\in C\mathfrak{F}$ implies $f(z)\in C\mathfrak{F}$, also $f(z)\in\mathfrak{F}$ whenever $z\in\mathfrak{F}$ and $f(z)\in C\mathfrak{F}$ whenever $z\in C\mathfrak{F}$, cf. [2].

Throughout this proof let $\hat{f}_{-1}(z)$ be the inverse of the restriction of $f(z)$ to $D_\alpha$.

Under the continuation of a branch of $\hat{f}_{-1}(z)$ along a path $\gamma$ in $D_\alpha$ we never encounter any transcendental singularity, for this would imply that the values of $|\hat{f}_{-1}(z)|\to\infty$ as $z$ traverses a part of $\gamma\subset D_\alpha$; this in turn would imply that $D_\alpha(\supseteqq\hat{f}_{-1}(\gamma))$ is unbounded.

An algebraic singularity $\beta$ of a branch of $\hat{f}_{-1}(z)$ in $D_\alpha$ corresponds to a zero $b_1$ of $f'(z)$: $f'(b_1)=0$, $f(b_1)=\beta$ and $f_{-1}(\beta)=b_1$ for the appropriate branch of $\hat{f}_{-1}$. Since $\beta\in D_\alpha$ we see that $b_1\in D_\alpha$. Now $D_\alpha$ is bounded and so contains only finitely many zeros $b_1, b_2, \ldots, b_n$, of $f(z)$ in $D_\alpha$. Thus $\hat{f}_{-1}(z)$ is a function with a finite number $n$ of branches in $D_\alpha$. Moreover $f_m(b_j)\to\alpha$ as $m\to\infty$, for each $j$.

Thus the set $S$ of singularities of all $\hat{f}_{-m}(z)$ ($=$ inverse of the restriction of $f_m(z)$ to $D_\alpha$) is precisely the set of all $f_m(b_j)$ ($m=1, 2, \ldots; j=1, 2, \ldots, n$) and this is a countable subset of $D$ with just one limit point at $\alpha$. The set $S$ is thus closed and bounded and hence it is compact.

Thus there exists a $\xi \in D_\alpha$, $\xi \neq \alpha$, $\xi \notin S$. Then if we pick any branch $p(z)$ of $\hat{f}_{-1}(\xi)=\xi_1$, we have $\xi_1 \notin S$, $\xi_1 \neq \alpha$. We can join $\xi$ to $\xi_1$ in $D_\alpha - S$, say by a polygonal arc $l$ which will have positive distance from the compact set $S$. We can therefore enclose $l$ in a domain $\delta(\subset D_\alpha)$ which does not meet $S$ (e.g. the domain $\delta$ may be all points distant less than $\varepsilon$ from $l$, for all small $\varepsilon > 0$).

Now continue $p(z)$ along $l$ to $\xi_1$. $p$ maps $l$ onto $l_1$, joining $\xi_1$ to $\xi_2$, say. Also $p$ extends to a single valued branch of $\hat{f}_{-1}$ in $\delta$ (by the Monodromy Theorem) and maps $\delta$ one to one onto $\delta_1$, another simply connected which contains $l_1$ within it and which does not meet $S$. Similarly by further continuation we get a chain of points $\xi, \xi_1, \xi_2, \ldots$, and curves $l, l_1, l_2, \ldots$, contained in simply connected domains $\delta, \delta_1, \delta_2, \ldots$, and such that $l_n$ joins $\xi_n$ to $\xi_{n+1}$, $p(z)$ continues analytically along $l + l_1 + \cdots$, and is regular in each $\delta_n$, since no $\delta_n$ meets $S$.

Let $p_n$ denote the $n$th continuation of $p$ by the above process. Then $p_n$ maps $\delta_n$ one to one onto $\delta_{n+1}$. This is equivalent to saying that $f(z)$ maps $\delta_{n+1}$ one to one onto $\delta_n$. The compound map $q_n = p_n \cdot p_{n-1} \ldots p_1 \cdot p$ is a branch of $\hat{f}_{-(n+1)}(z)$, regular in $\delta$ mapping $\delta$ onto $\delta_{n+1} \subset D_\alpha$.

Consider the branches $q_n$. These are regular and bounded in $\delta$ and hence normal in $\delta$. We shall show that the limit function $\psi(z)$ of any convergent subsequence $q_{n_k}$ is a constant on the boundary $\partial D_\alpha$ of $D_\alpha$. For $q_{n_k}(\delta) \subset D_\alpha$. Hence $\psi(z) \in \bar{D}_\alpha$. However for any compact $A \subset D_\alpha$, $f_n(A) \to \alpha$ as $n \to \infty$ and hence for large $n$, $f_n(A) \cap \cap \delta = \emptyset$, since $\delta$ has positive distance from $\alpha$. Thus for large $n_k$, $q_{n_k}(\delta)$ does not meet $A$. Hence $\psi(z)$ must be on the boundary of $\bar{D}_\alpha (= \partial D_\alpha)$. But $\partial D_\alpha$ contains no open set since $\partial D_\alpha \subset \mathfrak{F}$ and $\mathfrak{F}$ contains no open set unless $\mathfrak{F}$ is identically equal to the extended plane (c.f. result $A$.) which is not the case under the assumption of our theorem. Hence $\psi(z)$ must be a constant. Suppose now $\lim\limits_{k \to \infty} q_{n_k}(z) \to \lambda$, $z \in \delta$, i.e.

$$\lim_{k \to \infty} q_{n_k}(\delta) = \lim_{k \to \infty} \delta_{n_k+1} = \lambda.$$

Then $\delta_{n_k+2} \to \lambda$ since $\delta_{n_k+1} \cap \delta_{n_k+2} \neq \emptyset$, and $\operatorname{diam} \delta_{n_k+2} \to 0$. But

$$f(\delta_{n_k+2}) = \delta_{n_k+1}, \text{ and hence when } k \to \infty$$

$$f(\lambda) = \lambda$$

Thus $\lambda \in \mathfrak{F}$ is a first order fixpoint of $f(z)$ which must lie on the boundary $\partial D_\alpha$ of $D_\alpha$.

It remains now to show that $\lambda$ is an accessible boundary point of $D_\alpha$. FATOU [2] only outlined the proof, but we shall give a detailed proof here.

We define $L = \limsup \delta_n$ as the set of those points $t$ for which there exists a sequence of integers $N_q$ and points $z_q \in \delta_{N_q}$ such that $N_q \to \infty$, $z_q \to t$ (see e.g. [4], p. 10).

Then $L \subset \partial D_\alpha$, because each $t$ is a limit function of a sequence of $\{q_{n_k}\}$ and we have already shown that this limit must lie on $\partial D_\alpha$. Furthermore any $t \in L$ is afirst order fixpoint of $f(z)$ by the arguments in the proof of the first part of the theorem. It follows from the definition that $L$ is closed. Further $L$ is bounded, since $L \subset \partial D_\alpha$ and $D_\alpha$ is bounded. Hence $L$ is compact. We now show that $L$ is connected.

Suppose $L$ is not connected. Then there exist closed nonempty sets $L_1$ and $L_2$ such that $L \subset L_1 \cup L_2$, $L_1 \cap L_2 = \emptyset$ and the distance between $L_1$ and $L_2$ is

$$\varrho(L_1, L_2) = 4\eta \text{ (say)} \quad \text{where} \quad \eta > 0,$$

Now for some $N_0$, $\delta_n$ lies in an $\eta$-neighbourhood of $L$ for all $n > N_0$. Otherwise there exist $n_1, n_2, \ldots$ tending to infinity and $\xi_{n_1} \in \delta_{n_1}$, $\xi_{n_2} \in \delta_{n_2}$, $\varrho(\xi_{n_i}, L) \geqq \eta$, and $\xi_{n_i}$ will have a point of accumulation (since they are inside the compact set $D_\alpha$) $t$ and $t \notin L$. This is against the definition of $L$.

Take $\bar{l}_1 \in L_1$ and $\bar{l}_2 \in L_2$. Then there exists a $\delta_{n_1}$, $n_1 > N_0$ which meets an $\eta$-neighbourhood of $\bar{l}_1$ and similarly there exists a $\delta_{n_2}$, $n_2 > n_1$ which meets an $\eta$-neighbourhood of $\bar{l}_2$.

Consider the chain

$$C = \delta_{n_1} \cup \delta_{n_1+1} \cup \ldots \cup \delta_{n_2},$$

where $\delta_{n_1}$ meets an $\eta$-neighbourhood of $\bar{l}_1$ and $\delta_{n_2}$ meets an $\eta$-neighbourhood of $\bar{l}_2$.

Now $C$ must be connected since $\delta_n \cap \delta_{n+1} \neq \emptyset$. Also $C$ lies in an $\eta$-neighbourhood of $L$ i.e. of $L_1$ and $L_2$. Given $\mu_1 \in \delta_{n_1} \subset L_1' = \eta$-neighbourhood of $L_1$ and $\mu_2 \in \delta_{n_2} \subset L_2' = \eta$-neighbourhood of $L_2$ there exists a polygon of sides less than $\eta$ with vertices at $\mu_1 = \mu_1^1, \mu_1^2, \ldots, \mu_1^n = \mu_2$ all lying in $C$. Let $\mu_1^r$ be the last $\mu_1$ in $L_1'$. Then $r \neq n$ since $\mu_2 = \mu_1^n$ lies n $L_2'$. Also $\mu_1^{r+1} \in L_2'$. Now $\varrho(\mu_1^r, \mu_1^{r+1}) < \eta$

i.e. $$\varrho(L_1, L_2) < \varrho(L_1, \mu_1^r) + \varrho(\mu_1^r, \mu_1^{r+1}) + \varrho(\mu_1^{r+1}, \mu_2) < 3\eta$$

which is a contradiction. We have now proved that $L$ is connected and compact. Hence it is a continuum.

We now show that $L$ is in fact a single point. If $L$ is not a single point then we shall have a continuum of first order fixpoints of $f(z)$ lying on the finite part of the plane i.e. $f(z) - z = 0$ shall have infinity of solutions in a bounded area. This is impossible unless $f(z) \equiv z$. Thus $L$ must reduce to a single point. This means that the whole sequence $q_n(z_0)$ for $z_0 \in \delta$ must tend uniformly to the single frontier point $\lambda$ i.e. the chain $C = \delta_1 \cup \delta_2 \cup \ldots$, which is connected tends uniformly to the frontier point $\lambda$. Hence $\lambda$ is an accessible frontier (boundary) point of $D_\alpha$, accessible from the interior of $D_\alpha$, along the curve $l + l_1 + l_2 + \ldots$.

The proof of the theorem is now complete.

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