Publ. Math. Debrecen
52 / 1-2 (1998), 137–143

The sequentially-convex hull of certain compact operators

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Abstract. Necessary and sufficient conditions are given for the closed unit ball, positive face and positive quadrant in the space of compact linear operators between a pair of spaces of continuous functions to be the sequentially convex hulls of their extreme points.

1. Introduction

Let $C(X)$ and $C(Y)$ denote the continuous real- or complex-valued functions on the compact Hausdorff spaces X and Y and let $\mathcal{S}_c$ be the closed unit ball in the space $\mathcal{L}_c(C(X),C(Y))$ of compact linear operators from $C(X)$ to $C(Y)$ with its operator-norm topology. We denote by $\mathcal{E}$ the set of extreme points of $\mathcal{S}_c$.

The set $\mathcal{S}_c$ is among those investigated by Morris and Phelps in [3]. These are remarkable because although they are not compact, the conclusion of the Krein–Milman Theorem still holds for them when the pair (X,Y) satisfies certain very specific topological properties. Their result is that, excepting a special case, $\mathcal{S}_c$ is the operator-norm closed convex hull of $\mathcal{E}$ if and only if X is dispersed and Y is totally disconnected.

We consider here the question of when the set $\mathcal{S}_c$ satisfies the formally stronger conclusions of Choquet's Theorem: that each point is represented as the centroid of a probability measure on the extreme points.

Mathematics Subject Classification: Primary: 46A55, Secondary: 47B07, 47B38.
Key words and phrases: Compact linear operator, sequentially convex hull, totally disconnected space, dispersed space.

In fact we prove the stronger assertion that the same conditions on (X, Y) as in [3] are necessary and sufficient for each point of $\mathcal{S}_c$ to be the centroid of a countably supported measure on the extreme points.

In other words, we show that, excepting the same special case, the condition that X is dispersed and Y is totally disconnected is necessary and sufficient for $\mathcal{S}_c$ to be the sequentially-convex hull of $\mathcal{E}$ in the norm topology, where the *sequentially-convex hull* of $\mathcal{E}$ is defined to be the set

$$\text{s-co}\,\mathcal{E} = \left\{ \sum_{i=1}^{\infty} a_i E_i : a_i > 0,\ \sum_{i=1}^{\infty} a_i = 1,\ E_i \in \mathcal{E} \right\}.$$

The author is indebted to an anonymous referee for suggesting the use of the λ-property of Aron, Lohman and Suarez [1] and Theorem 1 of their paper in place of his original direct proof, which used the monotone convergence theorem to get a maximal representing measure.

The Aron–Lohman–Suarez Theorem works for the closed unit ball of $\mathcal{L}_c(C(X), C(Y))$ and its closed face K_1, but not for the positive part K_0 of the unit ball. We indicate how their theorem may be modified to show that our main result also holds with $\mathcal{S}_c$ replaced by K_0.

Our results develop that proved in [4] which covered the case where the scalars were real and X consisted of a single point.

2. Linear operators on $C(X)$ with X dispersed

Let $C(X)$ denote the space of all continuous real- or complex-valued functions on the compact Hausdorff space X, with the supremum norm, and let X_d be the set of point measures $\{\delta_x : x \in X\}$ taken with the discrete topology.

A compact space X is said to be *dispersed* if it contains no non-empty perfect subset and is *totally disconnected* if every open covering has an open refinement in which no two sets intersect.

We use the following characterisation of a dispersed space.

Proposition 2.1 ([5], [7]). *Let $C(X)$ be the space of all continuous real- or complex-valued functions on a compact Hausdorff space X. Then X is dispersed if and only if every non-zero element F of the dual space $C(X)^*$ may be written in the form $F = \sum_{i=1}^{\infty} c_i \delta_{x_i}$ where $c_1, c_2, \ldots$ are* *non-zero scalars, $x_1, x_2, \dots$ are points of X, the series converges in the norm topology and $\sum_{i=1}^{\infty} |c_i| < \infty$.*

When the points x_i are chosen to be distinct the expansion is unique and $\|F\| = \sum_{i=1}^{\infty} |c_i|$.

It is known from [3] that the extreme operators $\mathcal{E}$ may be identified with the set of all continuous functions g from Y to the set $\Gamma \times X_d$, where Γ is the unit circle. That is, for each $E \in \mathcal{E}$ there exist a continuous function α from X to Γ and a function β from Y to X with $Ef(y) = \alpha(y)\delta_{\beta(y)}(f)$ for all $f \in C(X)$.

Since Y is compact each β has finite range and the corresponding operator E is finite-dimensional.

We now show that the λ-condition of [1] holds for each point T of $\mathcal{S}_c$.

Proposition 2.2. *Let T be an operator in the closed unit ball $\mathcal{S}_c$ of $\mathcal{L}_c(C(X), C(Y))$ where X is a dispersed compact space and Y is a totally disconnected compact space. Then there exist $\lambda \in (0,1)$, $E \in \mathcal{E}$ and $S \in \mathcal{S}_c$ such that $T = \lambda E + (1-\lambda)S$.*

Proof. We may suppose that T is non-zero. By Theorem VI 7.1 of [2], since T is compact, it is represented by a continuous function τ from Y to the space $C(X)^*$ of regular Borel measures on X, taken with its norm topology. For all $s \in Y$ and $f \in C(X)$ $Tf(s) = \tau(s)(f)$ and $\|T\| = \sup_{s \in Y} \|\tau(s)\|$ holds.

For each $x \in X$ the set $U_x = \{F \in C(X)^* : |F(\{x\})| > 0\}$ is open. By Proposition 2.1, since X is dispersed, $\{U_x : x \in X\}$ is an open covering of $C(X)^* \setminus \{0\}$ and $\mathcal{V} = \{\tau^{-1}(U_x) : x \in X\} \cup \{V_0\}$ is an open covering of Y, where $V_0 = \{s \in Y : \|\tau(s)\| < \frac{1}{2}\|T\|\}$.

Since Y is totally disconnected and compact there is a finite partition $Y_0, \dots, Y_N$ of Y into open and closed subsets which refines $\mathcal{V}$ and where $Y_0 \supseteq \tau^{-1}(\{0\})$. Let $x_1, \dots, x_N$ be such that $\tau(Y_i) \subseteq U_{x_i}$ for $i = 1, \dots, N$. The continuous function $F \to |F(\{x_i\})|$ attains its infimum on the weakly compact set $\tau(Y_i)$ at a point $\tau(s_i)$ where $0 < |\tau(s_i)(\{x_i\})| = \epsilon_i \le |\tau(s)(\{x_i\})|$ for all $s \in Y_i$.

Now let $\lambda = \min\{\epsilon_1, \dots, \epsilon_N, \frac{1}{4}\|T\|\} > 0$. Define the function g from Y to $\Gamma \times X_d$ by

$$g(s) = \sum_{i=1}^{N} \frac{\tau(s)(\{x_i\})}{|\tau(s)(\{x_i\})|} \delta_{x_i} \chi_{Y_i}(s) + \delta_{x_0} \chi_{Y_0}(s). \tag{2.1}$$

When $s \in Y_0$ we have

$$\|\tau(s) - \lambda g(s)\| \leq \|\tau(s)\| + \lambda \leq \frac{3}{4}\|T\| \leq 1 - \lambda.$$

If $1 \leq i \leq N$ and $s \in Y_i$, then

$$\begin{aligned}\|\tau(s) - \lambda g(s)\| &= \|\tau(s)\| - |\tau(s)(\{x_i\})| + |\tau(s)(\{x_i\}) - \lambda g(s)(\{x_i\})| \\ &= \|\tau(s)\| - |\tau(s)(\{x_i\})| + \left|\tau(s)(\{x_i\}) - \lambda \frac{\tau(s)(\{x_i\})}{|\tau(s)(\{x_i\})|}\right| \\ &= \|\tau(s)\| - |\tau(s)(\{x_i\})| + |\tau(s)(\{x_i\})| - \lambda \\ &= \|\tau(s)\| - \lambda \leq 1 - \lambda.\end{aligned}$$

Now g represents an operator $E \in \mathcal{E}$ and we have proved that $\|T - \lambda E\| \leq 1 - \lambda$. If we define $S = \dfrac{T - \lambda E}{1 - \lambda}$, then $S \in \mathcal{S}_c$ and $T = \lambda E + (1 - \lambda)S$. □

Theorem 2.3. *Let X and Y be compact Hausdorff spaces with X dispersed and Y totally disconnected. Then $\mathcal{S}_c = \text{s-co}\,\mathcal{E}$.*

Proof. By Theorem 2.2, for each T of $\mathcal{S}_c$ there exist $\lambda \in (0, 1]$, $E \in \mathcal{E}$ and $S \in \mathcal{S}_c$ with

$$T = \lambda E + (1 - \lambda)S.$$

By Theorem 1 of [1] this is a necessary and sufficient condition for $\mathcal{S}_c = \text{s-co}\,\mathcal{E}$ to hold, which completes the proof. □

When X is a single point and the scalars are complex, $\mathcal{S}_c$ reduces to the closed unit ball in the complex space $C(Y)$. The need for a condition on the number of points in X in the following theorem arises from the result in [6] that this unit ball is the closed convex hull of its extreme points for every compact Hausdorff space Y.

Theorem 2.4. *Let X and Y be compact Hausdorff spaces and suppose that if the scalars are complex and Y contains at least two points then so does X. Then the following are equivalent.*

(a) X is dispersed and Y is totally disconnected,

(b) $\mathcal{S}_c = \overline{\text{co}}\,\mathcal{E}$,

(c) $\mathcal{S}_c = \text{s-co}\,\mathcal{E}$.

Proof. By Theorem 2.3, (a) implies (c). That (c) implies (b) is immediate. If (b) holds then (a) follows from Theorem 4.6 of [3]. □

Let K_0 and K_1 denote respectively the set of non-negative operators in $\mathcal{S}_c$ and its subset consisting of the non-negative compact operators satisfying the condition $T1 = 1$, where 1 is the unit function.

Corollary 2.5. *For real scalars, if X is dispersed and Y is totally disconnected then K_1 is the sequentially-convex hull of its extreme points.*

Proof. When X is dispersed, Y is totally disconnected and the scalars are real, Theorem 2.3 and the fact that K_1 is a closed face of $\mathcal{S}_c$ show that $K_1 = \text{s-co}\,\text{ext}\,K_1$. □

Proposition 2.6. *Let the scalars be real. Then X is dispersed and Y is totally disconnected if and only if K_0 is the sequentially-convex hull of its extreme points.*

Proof. Let X be dispersed, Y be totally disconnected and let $T \geq 0$ be in $\mathcal{S}_c$. In the proof of Theorem 2.2, we may replace equation (2.1) by

$$g(s) = \sum_{i=1}^{N} \frac{\tau(s)(\{x_i\})}{|\tau(s)(\{x_i\})|} \delta_{x_i} \chi_{Y_i}(s) + 0\chi_{Y_0}(s) \tag{2.2}$$

where 0 is the origin in $C(X)^*$.

As proved in [6], the operator E represented by g is an extreme point of K_0 because its range is in $X_d \cup \{0\}$. Also the operator $T - \lambda E$ is non-negative.

This shows that K_0 satisfies the following version of the λ-property: For each point T of K_0 there exist $\lambda \in (0, 1]$, $E \in \text{ext}\,K_0$ and $S \in K_0$ with

$$T = \lambda E + (1 - \lambda)S.$$

We now observe that the proof of Theorem 1 in [1] holds for K_0, the non-negative part of the closed unit ball, provided their function $\lambda(T)$ is redefined as

$$\lambda(T) = \sup\{\lambda \in (0, 1] : T = \lambda E + (1 - \lambda)S,\ E \in \text{ext}\,K_0,\ T \in K_0\}.$$

Remark 2.2 of [1] then shows that $K_0 = \text{s-co}\,\text{ext}\,K_0$.

Conversely, let X and Y be compact spaces with $K_0 = \text{s-co}\,\text{ext}\,K_0$. Let y_1 and y_2 belong to the same connected component of Y and let $h \in C(Y)$ separate y_1 and y_2 with $0 \leq h(y) \leq 1$ for all $y \in Y$. Let x be fixed in X.

The linear operator defined on $C(X)$ by $Tf(y) = f(x)h(y)$ lies in K_0. Now $T = \sum_{i=1}^{\infty} a_i E_i$ where $a_i > 0$, $\sum_{i=1}^{\infty} a_i = \|T\|$ and $E_i \in \operatorname{ext} K_0$. There exist continuous functions β_i from Y to $X_d \cup \{0\}$ in $C(X)^*$ with $E_i f(y) = \beta_i(y)(f)$ for all $y \in Y$ and all $f \in C(X)$. Since $X_d \cup \{0\}$ is discrete, $\beta_i(y_1) = \beta_i(y_2)$ for all $i \geq 1$.

We apply the operator T to a function $f_0 \in C(X)$ with $f_0(x) = 1$. Now

$$h(y_1) = f_0(x)h(y_1) = Tf_0(y_1) = \sum_{i=1}^{\infty} a_i E_i f_0(y_1) = \sum_{i=1}^{\infty} a_i \beta_i(y_1)(f_0)$$
$$= \sum_{i=1}^{\infty} a_i \beta_i(y_2)(f_0) = \sum_{i=1}^{\infty} a_i E_i f_0(y_2) = Tf_0(y_2) = f_0(x)h(y_2) = h(y_2).$$

But this contradicts the fact that h separates y_1 from y_2, so each component of Y consists of a single point and Y is totally disconnected.

Let $U^+ = \{F \in C(X)^* : F \geq 0, \|F\| \leq 1\}$. It follows from $K_0 = \text{s-co}\operatorname{ext} K_0$ that $C(Y, U^+) \subseteq C(Y, \text{s-co}(X_d \cup \{0\}))$ and $U^+ \subseteq \text{s-co}(X_d \cup \{0\})$. Since $P(X)$, the set of regular probability measures on X, is a face of U^+, $P(X) = \text{s-co}\, X_d$ in the norm topology and using the Riesz decomposition of measures, each $F \in C(X)^*$ is the norm convergent sum of a series of scalar multiples of point evaluations. Proposition 2.1 then asserts that X is dispersed. $\square$

We note that Corollary 2.5 may also be proved directly by the first part of the above argument with the final term in (2.2) omitted altogether.

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(Received February 14, 1997)